



















NLE  
ed.gov

THE

# *Young Mathematician's Guide.*

Being a PLAIN and EASIE

## INTRODUCTION

TO THE

# Mathematicks.

IN FIVE PARTS.

VIZ.

- I. Arithmetick, Vulgar, and Decimal, with all the Useful Rules; And a General Method of Extracting the Roots of all Single Powers.
- II. Algebra, or Arithmetick in Species; wherein the Method of Raising and Resolving *Æquations* is rendered Easie; and Illustrated with Variety of Examples, and Numerical Questions. Also the whole Business of Interest and Annuities &c. perform'd by the Pen.
- III. The Elements of Geometry, Contracted, and Analytically Demonstrated; With a New and Easie Method of finding the Circle's Periphery and Area to any assigned Exactness. by one *Æquation* only; Also a New way of making Sines and Tangents.
- IV. Conic-Sections, wherein the Chief Properties, &c. of the Ellipsis, Parabola, and Hyperbola, are clearly demonstrated.
- V. The Arithmetick of Infinites explain'd, and render'd Easie; with its Application to Superficial, and Solid Geometry.

With an APPENDIX of Practical Gauging.

The Sixth Edition, carefully Corrected; and New Tables of Compound Interest at Five per Cent. Calculated, and Added by the Author,

JOHN WARD.

LONDON Printed: And Dublin Reprinted, by and for SAMUEL FULLER, at the Globe and Scales in Meath-Street, 1731.

*The*

$$\begin{array}{r}
 6 \quad 4 \quad 6 \quad 5 \quad 3 \quad 2 \quad 9 \quad 682 \\
 \underline{36} \\
 120 \quad 1653 \\
 \underline{1024} \\
 1362 \quad 3920 \\
 \quad \quad 77 \quad 4 \\
 \quad \quad 119 \text{ Remainder}
 \end{array}$$

Root

$$\begin{array}{r}
 17 \quad 6 \quad 5 \quad 4 \quad 3 \quad 9 \quad 96 \\
 \underline{13296} \\
 23 \quad 76 \\
 \quad \quad 69 \\
 262 \quad 754 \\
 \underline{524} \\
 2649 \quad 23039 \\
 \quad \quad 21184 \\
 26566 \quad 105596 \\
 \quad \quad 10296
 \end{array}$$

Root



---

TO THE HONOURABLE

SIR *RICHARD GROSVENOR*  
of *Eaton*, in the County *Palatine* of  
*Chester*, Baronet.

SIR,

WHEN requested by some Booksellers in *London*, to Revise and Prepare this Treatise for a New Impression, and once resolved to Answer their Demands; I was not long considering at whose Feet to lay it.

My Memory may indeed be impair'd by Age, Misfortunes and Accidents; Nay, I am sensible it is so: But it must be entirely lost, when I am forgetful of the great Obligations, I lie under to Sir *Richard Grosvenor*.

Your Hospitality and Generosity, make you stand unenvied in the Abundance of Fortune. Any Upstart may contrive to spend a great Estate: But it is a Felicity almost peculiar to great Birth to become One.

Were I now to describe Liberality without Profuseness; Steadiness in Principles, without any private View; Candor and Affability, good Nature join'd to sound Judgment, and a Serenity of Temper, which your Enemies will always find the Companion of true Courage; And then pronounce that you are possessed of all these good Qualities in as high a Degree as most Men living; No Gentleman that knows you well, would think I flatter'd you.

## *The DEDICATION.*

Sir, Give me Leave to say, I Honour your Character, and Love your Person; My Expressions are uncourtly, my Stile unpolish'd, and therefore more proper to be prefix'd to a Work, wherein the Matters related are indeed clad in a plain and homely Drefs; but they are True, and designed to propagate Mathematical Learning amongst such as desire to be introduced into that sort of Knowledge; And I am extremely pleas'd they are permitted to be sent into the World under your Protection.

That you may long Live, to promote the Good of your Country, and that City in whose Interest you have so heartily engag'd your Self; And that you may ever succeed in your own private Affairs; And live to enjoy all the Blessings that attend a quiet prudent Life, Is the earnest Prayer of,

*Honoured S I R,*

*Your most Obliged, Humble,*

*and Obedient Servant,*

*Thos. Ward*  
*his Book August*  
*the 6<sup>th</sup> 1815*

J. WARD.

To



---

# The P R E F A C E.

---

## To the R E A D E R.

**I** Think it needless (and almost endless) to run over all the Usefulness, and Advantages of Mathematicks in General; shall therefore only touch upon those Two Admirable Sciences Arithmetick, and Geometry; which are indeed the Two grand Pillars (or rather the Foundations) upon which all other parts of Mathematical Learning depend.

As to the Usefulness of Arithmetick, 'Tis well known that no Business, Commerce, Trade, or Imployment whatsoever, even from the Merchant to the Shop-keeper, &c. can be manag'd and carry'd on, without the Assistance of Numbers.

And as to the Usefulness of Geometry, 'Tis as certain, that no Curious Art, or Mechanick-Work, can either be invented, improved or performed, without its assisting Principles, tho' perhaps the Artist, or Workman, has but little (nay scarce any) Knowledge in Geometry.

Then, as to the Advantages that arise from both these Noble Sciences, when duly join'd together, to assist each other, and then apply'd to Practice (according as occasion requires) will readily be granted by all who consider the vast Advantages that accrue to Mankind from the Business of Navigation only. As also from that of Surveying and dividing of Lands betwixt Party and Party. Besides the great Pleasure and Use there is from Time-keepers, as Dials, Clocks, and Watches, &c. All these, and a great many more very useful Arts, (too many to be enumerated here) wholly depend upon the aforesaid Sciences.

And therefore 'tis no Wonder, that in all Ages so many Ingenious, and learned Persons have imploy'd themselves in writing upon the Subject of Mathematicks; but then most of those Authors seem to presuppose that their Readers had made some Progress in that sort of Learning before they attempted to peruse those Books, which are generally large Volumes, written in such abstruse Terms that young Learners were really affraid of looking into those Studies.

These



---

# The P R E F A C E.

---

*These Considerations first put me (many Years ago) upon the Thoughts of endeavouring to compose such a plain, and familiar Introduction to the Mathematicks, as might Encourage those that were willing (to spend some Time that way) to venture and proceed on with Chearfulness; Tho' perhaps they were wholly ignorant of its first Rudiments. Therefore I began with their first Elements or Principles.*

*That is, I began with an Unit in Arithmetick, and a Point in Geometry; And from these Foundations proceeded gradually on, leading the young Learner Step by Step with all the Plainness I can, &c.*

*And for that Reason I published this Treatise (Anno 1707) by the Title of the Young Mathematician's Guide; which has answer'd the Title so well, that I believe I may truly say (without Vanity) this Treatise hath prov'd a very helpful Guide to near Five Thousand Persons; and perhaps most of them such as would never have look'd into the Mathematicks at all, but for it.*

*And not only so, but it hath been very well receiv'd amongst the Learned, and (I've been often told) so well Approv'd on at the Universities, in England, Scotland, and Ireland, that it's Order'd to be publickly read to their Pupils, &c.*

*The Title Page gives a short Account of the several Parts treated of, with the Corrections and Additions that are made to this Sixth Edition, which I shall not enlarge upon, but leave the Book to speak for it self; and if it be not able to give Satisfaction to the Reader, I'm sure, all I can here say in its behalf will never recommend it: But this may be truly said, That whoever reads it over, will find more in it than the Title doth promise, or perhaps he expects: 'Tis true indeed, the Dress is but Plain and Homely, it being wholly intended to Instruct, and not to Amuse or Puzzle the young Learner with hard Words, and obscure Terms: However, in this I shall always have the Satisfaction; That I've sincerely aim'd at what's useful, tho' in one of the meanest Ways; 'Tis Honour enough for me to be accounted as one of the Under-Labourers in Clearing the Ground a little, and removing some of the Rubbish that lay in the Way to this Sort of Knowledge. How well I have perform'd That, must be left to proper Judges.*

*To be brief; As I am not sensible of any Fundamental Error in this Treatise; so I will not pretend to say, it is not without Imperfections, (Humanum est errare) which I hope the Reader will excuse, and pass them over with the like Candor and Good Will that it was compos'd for his Use; by his real Well-wisher,*

J. W A R D.

THE

# The CONTENTS.

## Arithmetick. Part I.

Pages:

**P***Ræcognita*, Concerning the proper Subjects, or Business of Mathematicks, &c. 1

Chap. I. Concerning the several Parts of *Arithmetick*, and of such Characters as are used in this Treatise. 3

Chap. II. Concerning the principal Rules in *Arithmetick*, and how they are perform'd in whole Numbers. 5

Chap. III. Concerning *Addition*, *Subtraction*, and *Reduction* of Numbers that are of different Denominations. 31

Chap. IV. Of *Vulgar Fractions*, with all their various Rules. 48

Chap. V. Of *Decimal Fractions* or Parts, with all the useful Rules, and Contractions, &c. 57

Chap. VI. Of continued Proportion, both *Arithmetical* and *Geometrical*; And how to vary the Order of things. 72

Chap. VII. Of *Disjunct Proportion*, or the *Golden Rule*, both *Direct*, *Reciprocal* or *Inverse*, and *Compound*. 85

Chap. VIII. The Rules of *Fellowship*, *Bartering* and *Exchanging of Coins*. 99

Chap. IX. Of *Alligation* or *Mixing* of Things, with all its Varieties or Cases. 110

Chap. X. Concerning the *Specific Gravities* of Metals, &c. 117

Chap. XI. *Evolution* or *Extracting* the Roots of all Single Powers, how High soever they are, by one General Method. 123

## Algebra. Part II.

Chap. I. The Method of *Noting down Quantities*, and *Tracing* of the Steps used in bringing them to an *Æquation*. 143

Chap. II. The Six Principal Rules of *Algebraic Arithmetick*, in whole Quantities. 147

Chap. III. Of *Algebraic Fractions*, or *Broken Quantities*. 163

Chap. IV. Of *Surds*, or *Irrational Quantities*. 171

Chap. V. Concerning the Nature of *Æquations*, and how to prepare them for a *Solution*, &c. 175

Chap.



# The CONTENTS.

Chap. VI. Of <i>Proportional Quantities</i> , both <i>Arithmetical</i> and <i>Geometrical Continued</i> ; Also of <i>Musical Proportion</i> .	184
Chap. VII. Of <i>Proportional Quantities Disjunct</i> , both <i>Simple</i> , <i>Duplicate</i> , and <i>Triplicate</i> ; And how to turn <i>Equations</i> into <i>Analogies</i> , &c.	190
Chap. VIII. Of <i>Substitution</i> ; And <i>Resolving Quadratic Equations</i> .	194
Chap. IX. Of <i>Analysis</i> , or the <i>Method of Resolving Problems</i> , Exemplified by <i>Forty Numerical Questions</i> .	202
Chap. X. The <i>Solution of all Kind of Adjected Equations</i> in <i>Numbers</i> .	234
Chap. XI. Of <i>Simple Interest</i> and <i>Annuities</i> in all their various <i>Cases</i> .	245
Chap. XII. Of <i>Compound Interest</i> , and <i>Annuities</i> both for <i>Years</i> and <i>Lives</i> ; And of <i>Purchasing Free-hold Estates</i> .	253

## Geometry. Part III.

Chap. I. Of <i>Geometrical Definitions</i> and <i>Axioms</i> , &c.	283
Chap. II. The <i>First Rudiments</i> or <i>Leading Problems</i> in <i>Geometry</i> .	292
Chap. III. A <i>Collection of the most useful Theorems</i> in <i>Plain Geometry</i> , <i>Analytically Demonstrated</i> .	300
Chap. IV. The <i>Algebraical Solution of Twenty Easie Problems</i> in <i>Plain Geometry</i> ; which does in Part shew the <i>Use of the Last Theorems</i> .	320
Chap. V. <i>Practical Problems</i> and <i>Rules</i> , for finding the <i>Area's</i> of <i>Right lin'd Superficies</i> <i>Demonstrated</i> .	338
Chap. VI. A <i>New and Easie Method</i> of finding the <i>Circle's Periphery</i> , and <i>Area</i> , to any assigned <i>Exactness</i> ; by the <i>Solution of One Equation</i> only. Also a <i>New Way</i> of making <i>Natural Sines</i> and <i>Tangents a priori</i> .	347

## Conic-Sections. Part IV.

Chap. I. <i>Definition of a Cone</i> . and all its <i>Sections</i> , &c.	361
Chap. II. Concerning the <i>Chief Properties</i> of the <i>Ellipsis</i> . &c.	367
Chap. III. Concerning the <i>Chief Properties</i> of the <i>Parabola</i> .	380
Chap. IV. Concerning the <i>Chief Properties</i> of the <i>Hyperbola</i> .	386

## Arithmetick of Infinites. Part V.

The <i>Arithmetick of Infinites</i> explain'd. and rendered <i>easie</i> ; with its <i>Application</i> to <i>Geometry</i> . in <i>Demonstrating the Superficial</i> , and <i>Solid Contents</i> of <i>Circular</i> and <i>Elliptical Figures</i> &c.	397
--	-----

## An Appendix of Practical Gaging.

Wherein all the <i>Chief Rules</i> and <i>Problems Useful</i> in <i>Gaging</i> , are applied to <i>Practice</i> , &c.	433
---	-----

A N

# INTRODUCTION

T O T H E

## Mathematicks.

---

P A R T I.

---

*P R Æ C O G N I T A.*

**T**HE Business of *Mathematicks* in all its Parts both *Theory* and *Practice*, is only to search out and determine the true Quantity; either of *Matter*, *Space*, or *Motion*, according as Occasion requires.

By *Quantity* of *Matter* is here meant the *Magnitude* or Bigness of any visible Thing, whose *Length*, *Breadth*, and *Thicknes*s may either be measured, or estimated.

By *Quantity* of *Space* is meant the Distance of one Thing from another.

And by *Quantity* of *Motion* is meant the Swiftnes of any Thing moving from one Place to another.

The Consideration of these according as they may be proposed, are the *Subjects* of the *Mathematicks*, but chiefly that of *Matter*.

Now the Consideration of *Matter*, with respect to its *Quantity*, *Form* and *Position*, which may either be *Natural*, *Accidental*, or *Designed*, will admit of infinite Varieties; but all the Varieties that are yet known, or indeed possible to be conceived, are wholly comprised under the due Consideration of these two, *Magnitude* and *Number*, which are the proper Subjects of *Geometry*, *Arithmetick* and *Algebra*. All other Parts of the *Mathematicks* being only the Branches of these three Sciences, or rather their Application or particular Cases.



GEOMETRY is a *Science* by which we search out and come to know either the whole *Magnitude* or some Part of any proposed *Quantity*; and is to be obtained by comparing it with another known *Quantity* of the same kind, which will always be one of these, *viz.* a *Line* (or *Length* only) a *Surface*, (that is, *Length* and *Breadth*) or a *Solid* (which hath *Length*, *Breadth* and *Depth*, or *Thickness*) Nature admitting of no other *Dimensions* but these three.

ARITHMETICK is a *Science* by which we come to know what *Number* of *Quantities* there are (either real or imaginary) of any *Kind*, contained in another *Quantity* of the same *Kind*: Now this *Consideration* is very different from that of *Geometry*, which is only to find out true and proper *Answers* to all such *Questions* as demand, how *Long*, how *Broad*, how *Big*, &c. But when we are to consider either of more *Quantities* than one, or how often one *Quantity* is contained in another, then we have recourse to *Arithmetick*, which is to find out true and proper *Answers* to all such *Questions* as demand how *Many*, what *Number*, or *Multitude* of *Quantities* there are. To be brief the Subject of *Geometry* is that of *Quantity*, with respect to its *Magnitude* only; and the Subject of *Arithmetick* is *Quantities* with respect to their *Number* only.

ALGEBRA is a *Science* by which the most abstruse or difficult *Problems* either in *Arithmetick* or *Geometry* are resolved and demonstrated, that is, it equally interferes with them both; and therefore it is promiscuously named, being sometimes called *Specious Arithmetick*, as by *Harriot*, *Vieta*, and *Doctor Wallis*, &c. And sometimes it is called *Modern Geometry*, particularly the ingenious and great Mathematician, Mr. *Edmond Halley*, Savilian Professor of *Geometry* in the University of *Oxford*, giving this following Instance of the Excellence of our *Modern Algebra*, writes thus,

‘ The Excellence of the *Modern Geometry* (saith he) is in nothing  
 ‘ more evident, than in those full and *Adequate Solutions* it gives to  
 ‘ *Problems*: Representing all the possible Cases at one View, and  
 ‘ in one general *Theorem* many times comprehending whole *Sciences*;  
 ‘ which deduced at length into *Propositions* and demonstrated after  
 ‘ the Manner of the *Ancients*, might well become the Subjects of  
 ‘ large *Treatises*: For whatsoever *Theorem* solves the most compli-  
 ‘ cated *Problem* of the Kind, does with a due *Reduction* reach all  
 ‘ the subordinate Cases. Of which he gives a notable Instance in  
 the *Doctrine* of *Dioptricks* for finding the *Foci* of *Optick Glasses* univer-  
 sally, (vid. *Philosophical Transactions*, Numb. 205.)

Thus you have a short and general Account of the proper Subjects of those three noble and useful *Sciences*, *Arithmetick*, *Geometry* and *Algebra*. I shall now proceed to give a particular Account of each, and first of *Arithmetick*, which is the *Basis* or *Founda-*  
 tion



dition of all *Arts*, both *Mathematick* and *Mechanick*; and therefore it ought to be well understood before the rest are medled withal.

CH A P. I.

*Concerning the several Parts of Arithmetick, with the Definition of such Characters as are used in this Treatise.*

**A**RITHMETICK, or the Art of Numbering, is fitly divided into three distinct Parts, two of which are properly called Natural; and the third Artificial.

The First being the most plain and easiest, is commonly called Vulgar Arithmetick in whole Numbers; because every Unit or Integer concerned in it, represents one whole Quantity of some Species or Thing proposed.

The Second is that which supposes an Unit (and consequently the Quantity or Thing represented by that Unit) to be broken or divided into equal Parts (either even, or uneven) and considers of them either as pure Parts, *viz.* each less than an Unit, or else of Parts and Integers intermixt. And is usually called the Doctrine of Vulgar Fractions.

The Third, or Artificial Part, is called Decimal Arithmetick; being an Artificial Invention of managing Fractions or broken Numbers, by a much more commodious and easy Way than that of Vulgar Fractions: For the several Operations performed in Decimals, differ but little from those in whole Numbers; and therefore it is now become of general Use, especially in Geometrical Computations.

A R I T H M E T I C K (in all its Parts) is performed by the various ordering and disposing of ten *Arabick* Characters or Numeral Figures (which by some are called Digits)

*viz.*  $\left\{ \begin{array}{l} \text{One} \text{ Two} \text{ Three} \text{ Four} \text{ Five} \text{ Six} \text{ Seven} \text{ Eight} \text{ Nine} \text{ Cypher.} \\ \text{I} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 0 \end{array} \right.$

The Use of the Characters is said to be first introduced into *England* near six hundred Years ago, *viz.* about the Year 1130. vide Doctor *Wallis's* Algebra, Page 12.

The first of these Characters is called Unity, and represents one of any Kind of Species or Quantity. As one World, one Star, one Man, &c.

*Viz.* Unity is that by which every Thing that is, is called one (*Euclid. 7. Def. 1.*) and is the Beginning of all Numbers. That is to say, Number is a Multitude of Units. *Euclid. 7. Def. 2.*

For, one more one, makes two; and one, more one, more one makes three, &c. Which is the first and chief *Postulate*, or rather *Axiom* to Arithmetick.

*Viz.*  $\left\{ \begin{array}{l} \text{That } 1+1=2. \quad 1+1+1=3. \quad 1+1+1+1=4. \\ 1+1+1+1+1=5. \quad \text{And so on to } 9. \end{array} \right.$

Nine of these Figures were thus composed of Units, and differently form'd to represent so many Units put together into one Sum, as was intended each should denote: Nine being the greatest Number of Units that was then thought convenient to be expressed by one single Character; the last of the Ten is only a Cypher, or (as some phrase it) a Nothing, because of it self it signifies nothing; for if never so many Cyphers be added to, or subtracted from any Number, they can neither increase nor diminish that Number; but yet as a Cypher (or Cyphers) may be placed, the other Figures will become of different Values from what they were before, as will appear further on.

For the more convenient Ordering of the aforesaid Numeral Figures, according to the several Varieties that happen in Computations; I do advise the young Learner to acquaint himself with the Signification of the following Algebraick Signs or Characters; which he will find of excellent Use, as being a much shorter, better and more significant Way of denoting what is to be done (in most Operations) than can otherwise be expressed in Words at length.

### Significations.

Signs Names.

$\left. \begin{array}{l} + \\ + \end{array} \right\} \left\{ \begin{array}{l} \text{Plus or} \\ \text{more.} \end{array} \right. \left\{ \begin{array}{l} \text{The Sign of Addition; As } 8 + 7 \text{ is 8 more 7,} \\ \text{and signifies that the Numbers 8 and 7 are to be} \\ \text{added into one Sum. The like is to be under-} \\ \text{stood when several Numbers are connected toge-} \\ \text{ther with the Sign } +. \\ \text{As } 34 + 22 + 9 + 45, \text{ \&c. denotes these are all to} \\ \text{be added into one Sum.} \end{array} \right.$

The



$-$  } { Minus or less. } The Sign of *Subtraction*; As  $9 - 6$  is 9 less 6, and signifies that 6 is to be taken from 9, that so their Difference may be found.

$\times$  } { Into or with. } The Sign of *Multiplication*; As  $9 \times 6$ , is 9 into 6, and signifies that 9 is to be multiplied into or with 6.

$\div$  } { By. } The Sign of *Division*; As  $8 \div 2$ , is 8 by 2, and signifies that 8 is to be divided by 2, also thus  $2 \overline{) 8}$  (4. or thus  $\frac{8}{2}$ , each signifying the same Thing, to wit, 8 divided by 2.

$=$  } { Equal. } The Sign of *Equality* or *Equation*, viz. whenever this Sign  $=$  is placed betwixt Numbers (or Quantities) it denotes them to be equal, as  $9 = 9$ , or  $9 + 6 = 15$ , or  $9 - 6 = 3$ , &c. That is, 9 is equal to 9, or 9 more 6, is equal to 15, and 9 less 6, is equal to 3, &c.

$:$  } { So is. } The Sign of *Proportion*, or that commonly called the *Golden Rule*, or *Rule of Three*, and  $:$  is always placed betwixt the two middle Terms or Numbers in Proportion. Thus,  $2 : 8 :: 6 : 24$ . To be read thus; As 2, is to 8, so is 6, to 24.

These Signs and their Significations, being perfectly learnt, will help to shorten the Work.

## CHAP. II.

*Concerning the principal Rules in Arithmetick, and how they are performed in whole Numbers.*

THE Rules by which Numerical Operations are perform'd in all the Parts of Arithmetick, are many and various, several of them being form'd and rais'd as Occasion requires, when applied to Practice, yet they are all comprehended within the due Consideration of these Six, viz. NUMERATION (or NOTATION

NOTATION, ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION, and EVOLUTION, or Extraction of Roots.

Sect. I. Of Numeration, or Notation.

NUMERATION or Notation, teacheth to read or express the true Value of any Number when writ down; and consequently to write down any proposed Number according to its true Value when it is named: And this consisteth of two Parts;

1. The due Order of placing down Figures.
  2. The true Valuing of each Figure in its Place.
- Both which are plainly exhibited in the following Table.

*Handwritten note on left margin: Multiplication against the 6th 10th & 18th Digits*

Hundreds of Thousands of Millions.	Tens of Thousands of Millions.	Thousands of Millions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units	} And is the first Place in counting.
6	7	8	9	8	7	6	5	4	3	2	1	
Period of Thousands of Millions.			Period of Millions.			Period of Thousands.			Period of Units.			

By this Numeration Table it's apparent that the Order of Places is reckoned from the Right-Hand towards the Left; the first Place of any Number being always that which is the outmost Figure to the Right-Hand; and whatever Figure stands in that Place doth only signify its own simple Value, viz. so many Units as that Figure represents.

The second Place is that of Tens, any Figure standing in that Place signifieth so many Tens as that Figure represents Units.

The



The third Place is Hundreds, the fourth Place Thousands, &c. That is, each Place towards the Left-Hand is ten times the Value of that next it towards the Right.

For Instance, suppose 759 were proposed to be read or pronounced according to the Value of each Figure as they now stand. The first Figure in this Sum is 9, because it stands in the Place of Units, and therefore signifies but its own simple Value, to wit 9 Units, or Nine. The second Figure 5 stands in the Place of Tens, and therefore signifies five Tens or Fifty. The Figure 7 stands in the third Place, or Place of Hundreds, and therefore it signifies seven Hundred, and the whole Sum is to be read or pronounced thus, Seven Hundred Fifty Nine.

Note, Although the Figure 7, stands in the third Place (according to the Order of Numbering) yet when the whole Sum comes to be read it is first pronounced, the reading of Numbers being perform'd like that of Letters or Words, always beginning with the outmost Figure towards the Left-Hand, and so many Figures as are placed together without any Point, Comma, Line, or other Note of Distinction between them, are all but one Sum, and must be read as such.

For Example, 763596 is but one intire Sum or Number, notwithstanding it consists of six Places of Figures, and is thus read; Seven Hundred Sixty Three Thousand Five Hundred Ninety Six.

The like is to be observed in reading or expressing the true Value of any Sum or Rank of Numbers consisting of Seven, Eight, Nine, or more Places of Figures, each Figure being to be valued according to its Distance from the Place of Unity: As in the foregoing Table.

Now such Values may as well arise by *Cyphers*, as by other Figures; for Instance, 6 standing by it self, represents but six Units: But if a Cypher be annext to it thus, 60, then it becomes Sixty; for the Cypher possessing the Place of Units, hath thereby removed the 6 into the Place of Tens; and another Cypher more would make it 600, Six Hundred, &c.

Whence it may be noted, that altho' a Cypher of it self signify nothing (as hath been said before) yet being placed on the Right-Hand of any Figure. it augments the Value of that Figure by advancing it into a higher Place than otherwise it would have been, had not the Cypher been there.

Take one Example more in Numeration, if you please, that in the Table viz. 678987654321. which is, according as is there signified,

*Six Hundred Seventy Eight Thousand Millions,*

*Nine Hundred Eighty Seven Millions,*

*Six Hundred Fifty Four Thousand,*

*Three Hundred Twenty One Units.*

Of any proposed Species or Quantities whatsoever.

And

And here it may be observed, that every third Figure from the Place of Units, bears the Name of Hundreds; which shews that if any great Sum be parted, or rather distinguished into Periods, of three Figures in each Period (as in the foregoing Table) it will be of good Use to help the young Learner in the easier valuing and expressing that Sum.

## Sect. 2. Of Addition:

### *Postulate or Petition.*

*That any given Number may be increased or made more, by putting another Number to it.*

**ADDITION** is that Rule by which several Numbers are collected and put together, that so their Sum or total Amount may be known.

In this Rule two Things being carefully observed, the Work will be easily performed.

1. The first is the true placing of the Numbers, so as that each Figure may stand directly underneath those Figures of the same Value, *viz.* Place Units under Units, Tens under Tens, and Hundreds under Hundreds, &c.

Then underneath the lowest Rank (always) draw a Line to separate the given Numbers from their Sum when it's found.

*Example.* If these Numbers 54327, and 2651, were given to be added together, they must be placed

$$\text{Thus, } \begin{array}{r} 54327 \\ 2651 \\ \hline \end{array}$$

2. The second Thing to be observed is the due collecting or adding together each Row of Figures that stand over one another of the same Value: And that is thus performed.

### **R U L E.**

Always begin your Addition at the Place of Units, and add together all the Figures that stand in that Place, and if their Sum be under Ten, set it down below the Line underneath its own Place; but if their Sum be more than Ten, you must set down only the Overplus, or odd Figure above the Ten (or Tens) and so many Tens as the Sum of those Units amount to, you must carry to



to the Place of Tens ; Adding them and all the Figures that stand in the Place of Tens together, in the same Manner as those of the Units were added ; then proceed in the same Order to the Place of Hundreds, and so on to each Place until all is done.

The Sum arising from those Additions will be the total Amount required.

## Example 1.

Let it be required to find the Sum of the aforesaid Numbers.

viz.  $\left\{ \begin{array}{r} 54327 \\ 2651 \end{array} \right.$

---

56978 the Sum required.

Beginning at the Place of Units, I say 1 and 7 is 8, which being less than 10, I set it down (according to the Rule) underneath its own Place of Units; and then proceed to the Place of Tens, saying 5 and 2 is 7, which being less than 10, I set it down underneath its own Place of Tens, and proceed to do the like at the Place of Hundreds, and then at Thousands, setting each of their Sums underneath their own respective Places: Lastly, Because there is not any Figure in the lower Rank to be added to the Figure 5, which stands in the Place of Ten Thousands, in the upper Rank, I therefore bring down the said 5 to the rest, placing it underneath its own Place, and then I find that  $54327 + 2651 = 56978$ , the true Sum required.

## Example 2.

Suppose it were required to find the Sum of these Numbers,  $3578 + 496 + 742 + 184 + 95$ . These being placed, as before directed, will stand as in the Margin. Then beginning (as before) at the Place of Units, say 5 and 4 is 9, and 2 is 11, and 6 is 17, and 8 is 25, set down the 5 Units underneath its own Place of Units, and carry the 20, or two Tens, to the Place of Tens (at which Place they are only 2) saying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39, set down the 9 underneath its own Place of Tens, and carry the 30, or three Tens (which indeed is 300) to the Place of Hundreds, at which Place they are but 3, saying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 5 is 20, here because there is no Figure overplus (as before) I set down a Cypher underneath the Place of Hundreds, and carry the 2 Tens (or rather the 2000) to the Place of Thousands, saying

3578  
496  
742  
184  
95  
—  
5095

C

(as

(as before,) 2 I carry and 3 is 5, which being the last, I set it down underneath its own Place, and all is finished. And find the Sum or total Amount to be  $5095 = 3578 + 496 + 742 + 184 + 95$ .

If this Example be well considered, it will be sufficient to shew the usual Method of *Addition* in *Whole Numbers*; but to make all plain and clear, I shall shew the young Learner the Reason of carrying the Tens from one Degree or Row of Figures, to the next Superior Degree, which is done purely to save Trouble, and prevent the using of more Figures than are really necessary, as will appear by the following Method of adding together the same Numbers of the last Example.

Thus, Add together each single Row of Figures by it self; as if there were no more but that one Row, setting down the Sum underneath its own Place.

3	5	7	8
4	9	6	
7	4	2	
1	8	4	
1	9	5	

The Sum of the Row of Units, is  
 The Sum of the Row of Tens, is  
 The Sum of the Row of Hund. is  
 The three Thousand brought down

1	2	5	
3	7	0	
1	7	0	
3	0	0	

Add

The Sum or total Amount as before, is 5 0 9 5

From hence I presume it will be easy to conceive the true Reason of carrying the aforesaid Tens; and also that Cyphers do not augment or increase the Sum in Addition. (See Page 4.)

I might have here inserted a Lineal Demonstration of this Rule of Addition; but I thought it would rather puzzle than improve a young Learner, especially in this Place; besides the Reason of it is sufficiently evident from that Natural Truth of *the Whole being equal to all its Parts taken together*. Euclid. I. Axiom 19.

That is, the Numbers which are proposed to be added together are by that *Axiom* understood to be the several Parts, and their Sum or total Amount found by Addition is understood to be the Whole,

And from thence is deduced the Method of proving the Truth of any Operation in Addition, viz. By parting or separating the given Numbers into two Parcels (or more, according to the largeness of it) and then Adding up each Parcel by it self: For if those particular Sums so found, be Added into one Sum, and that Sum prove equal or the same with the total Sum first found,



found, then all is right; if not, care must be taken to discover and correct the Error.

*Example.*

Add {	5647	}	The Sum of these Parts is,	12952
	3289			
	4016			
{	2900	}	The Sum of these, is	9513
	5007			
	1606			

The total Sum of  
all these Parts } 22465

The Sum of each }  
Parcel put together } 22465

## Sect. 3. Of Subtraction.

*Postulate or Petition.*

*That any Number may be Diminished, or made Less, by taking another Number from it.*

SUBTRACTION is that Rule by which one Number is deducted or taken out of another, that so the Remainder, Difference, or Excess may be known.

As 6 taken out of 9, there remains 3. This 3 is also the Difference betwixt 6 and 9, or it is the Excess of 9 above 6.

Therefore the Number (or Sum) out of which Subtraction is required to be made, must be greater than (or at least equal to) the Subtrahend or Number to be Subtracted.

Note, *This Rule is the Converse or direct Contrary to Addition.*

And here the same Caution that was given in Addition; of placing Figures directly under those of the same Value, viz. Units under Units, Tens under Tens, and Hundreds under Hundreds, &c. must be carefully observed; also underneath the lowest Rank there must be drawn a Line (as before in Addition) to separate the given Numbers from their Difference when it is found.

Then having placed the lesser Number under the greater, the Operation may be thus performed.

R U L E.

Begin at that Right-Hand Figure or Place of Units (as in Addition) and take or subtract the lower Figure in that Place from the Figure that stands over it, setting down the Remainder



or Difference underneath its own Place, if the two Figures chance to be equal, set down a Cypher. But if the upper Figure be less than the lower Figure, then you must add 10 to the upper Figure, or mentally call it 10 more than it is, and from that Sum subtract the lower Figure, setting down the Remainder (as before directed.) Now because the 10 thus added, was suppos'd to be borrow'd from the next superior Place (*viz.* of Tens) in the upper Figures, therefore you must either call the upper Figure in that Place from whence the 10 was borrow'd, one less than really it is, or else (which is all one, and most usual) you must call the lower Figure in that Place one more than it really is, and then proceed to Subtraction in that Place, as in the Former; and so gradually on from one Row of Figures to another until all be done.

*Example 1.*

Let it be required to find the Difference between 6785, and 4572. That is, let 4572 be subtracted from 6785.

These Numbers being placed down, as before directed, will stand

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{r} 6785 \\ 4572 \\ \hline 2213 \end{array} \right. \end{array}$$

Beginning at the Place of Units, take 2 from 5 and there will remain 3 which must be set down underneath its own Place, and then proceed to the Place of Tens, taking 7 from 8, there will remain 1, to be set down underneath its own Place; again at the Place of Hundreds take 5 from 7, and there remains 2, which set down, as before; lastly, take 4 from 6 and there will remain 2, which being set down underneath its own Place, the Work is finished, and the Difference so found will be  $2213 = 6785 - 4572$ , as was required.

*Example 2.*

The Difference between 5849, and 7496 is required.

Having placed the Numbers as in the Margin, begin at the Place of Units (as before) and say 9 from 6 cannot be, but 9 from 16 and there remains 7, to be set down under its own Place, next proceed to the Place of Tens, where you must now pay the 10 that was borrowed to make the 6, 16, by accounting the upper Figure 9 in that Place one less than it is, saying, 4 from 8 and there remains 4, or else (which is the most practised) say 11 borrowed and 4 is 5 from

$$\begin{array}{r} 7496 \\ 5849 \\ \hline 1647 \end{array}$$

from 9 and there Remains 4, to be set down under it's own Place (as before); again at the Place of Hundreds, say 8 from 4 that cannot be, but 8 from 14 there will remain 6 to be set down; and here I have borrowed 10 (as before) which must be paid in the same Manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, saying, 5 from 6 there remains 1, or else by saying 1 borrowed and 5 is 6 from 7 and there remains 1, which being set down under its own Place all is done, and the Difference required will be  $1647 - 7496 = 5849$ .

## Example 3.

From 830476  
Take 741068

Remains 89408

By this Example you may perceive that Cyphers in the Subtrahend, *viz.* in the Numbers to be subtracted, do not diminish the Number from whence Subtraction is made. See Page 4.

These three Examples, I presume, may be sufficient to shew the young Learner the Method of Subtracting whole Numbers; as for the Reason thereof it is the same with that of Addition, Page 10, *viz.* of the Whole being equal to all its Parts taken together.

That is, in this Rule the Number from which Subtraction is required to be made, is understood to be the Whole, and the Subtrahend or Number to be subtracted, is suppos'd to be a Part of that Whole, consequently if that Part be taken from the Whole the Remainder will be the other Part.

From hence is deduced the common Method of proving Subtraction, by adding together the Subtrahend and the Remainder. For if the Sum of those Two which are here called Parts, be equal to the Number from whence Subtraction was made (which is here called the whole) then the Work is right; if not, care must be taken to discover and correct the Error.

## Example.

From 59435  
Take 47608

Proof  $\left\{ \begin{array}{l} 11827 \\ 59435 \end{array} \right\}$  Add

The Sum which is equal to the Number from whence Subtraction was made.

Or



Or from the abovesaid Reason, it will be easy to conceive how to prove the Truth of Subtraction by Subtraction.

For if from	59435	being here the Whole,
there be taken	47608	as Part of that Whole,
there will remain	11827	the other Part (as before)
And if from	59435	the Whole, there be subtracted
the last Part, viz.	11827	

there will remain 47608 the first Part, or Number which was required to be first subtracted.

From	75643
Take	9000

---

66643

Remains

From	7000000
Take	986432

---

6013568

### Sect. 4. Of Multiplication.

**MULTIPLICATION** is a Rule by which any given Number may be speedily increased, according to any proposed Number of Times.

That is, One Number is said to multiply another, when the Number multiplied is so often added to it self, as there are Units in the Number multiplying; and another Number is produced. (*Euclid. 7. Def. 15.*)

To perform *Multiplication* there are required two given Numbers, called *Factors*.

The First is that Number which is to be multiplied, and is generally put the greatest of the two Numbers, commonly call'd the *Multiplicand*.

The other is that Number by which the First is to be multiplied, and is usually called the *Multiplicator* or *Multiplier*; and this denotes the Number of Times that the *Multiplicand* is required to be added to it self. For so many Units as are contained in the *Multiplier*, so many times will the *Multiplicand* be really added to it self, (as per *Euclid* above.) And from thence will arise a third Number, called the *Product*. But in *Geometrical Operations* it is called the *Rectangle* or *Plane*.

For Instance; Suppose it were required to increase 6 four Times, That is, to multiply 6 into or with 4, these two Numbers are to be set (or placed) down as in *Addition* or *Subtraction*.

Thus.



Thus  $\left\{ \begin{array}{l} 6 \text{ Multiplicand,} \\ 4 \text{ Multiplier,} \end{array} \right\}$  or Factors.

Product 24 viz. 4 times 6 is 24, as plainly appears by Addition, viz. By setting down 6 four times, and then adding them together into one Sum

$$\begin{array}{r|l} 1 & 6 \\ 2 & 6 \\ 3 & 6 \\ 4 & 6 \\ \hline & 24 \end{array} \left. \vphantom{\begin{array}{r|l} 1 & 6 \\ 2 & 6 \\ 3 & 6 \\ 4 & 6 \end{array}} \right\} \text{Add}$$

From hence it is evident that Multiplication is only a concise or compendious Way of adding any given Number to it self, so often as any Number of Times may be proposed.

Before any Operation can be readily perform'd in Multiplication, the several Products of the single Figures, one into another must be perfectly learn'd by Heart, viz. That 2 times 2 is 4, that 3 times 3 is 9, that 3 times 6 is 18, &c. According as they are expressed in the following Table. Wherein I have omitted multiplying with 2, it being so very easy that any one may do it.

Multiplication Table.

3 × 3 = 9	4 × 4 = 16	5 × 5 = 25	6 × 6 = 36	7 × 7 = 49	8 × 8 = 64
3 × 4 = 12	4 × 5 = 20	5 × 6 = 30	6 × 7 = 42	7 × 8 = 56	8 × 9 = 72
3 × 5 = 15	4 × 6 = 24	5 × 7 = 35	6 × 8 = 48	7 × 9 = 63	9 × 9 = 81
3 × 6 = 18	4 × 7 = 28	5 × 8 = 40	6 × 9 = 54		
3 × 7 = 21	4 × 8 = 32	5 × 9 = 45			
3 × 8 = 24	4 × 9 = 36				
3 × 9 = 27					

I think it needless to give any Explanation of this Table; for if the Signs and their Significations be well understood, (vide Page 5) it must needs be easy. Only this may be noted, that  $4 \times 3 = 3 \times 4$ , or  $7 \times 5 = 5 \times 7$ , &c.

That is, 3 times 4, is the same with 4 times 3, or 5 times 7, is the same with 7 times 5, &c. The like must be understood of all the rest in the Table.

And when all these single Products are so perfectly learn'd by Heart, as to be said without pausing; you may then proceed (but not till then) to the Business of Multiplication; which will be found very easy, if the following Rule (and Examples) be carefully observed.

## R U L E.

Always begin with that Figure which stands in the Units Place of the Multiplier; and, with it multiply the Figure which stands

in

in the Units Place of the Multiplicand; if their Product be less than Ten, set it down underneath its own Place of Units, and proceed to the next Figure of the Multiplicand. But if their Product be above Ten (or Tens) then set down the Overplus only (or odd Figure, as in Addition) and bear (or carry) the said Ten or Tens in mind until you have multiplied the next Figure of the Multiplicand, with the same Figure of the Multiplier; then to their Product add the Ten or Tens carried in Mind, setting down the Overplus of their Sum above the Tens, as before: And so proceed on in the very same Manner, until all the Figures of the Multiplicand are multiplied with that Figure of the Multiplier.

*Example 1.*

Suppose it were required to multiply 3213 into or with 3.

3213 Multiplicand, }  
3 Multiplier, } or Factors.

Product 9639

Beginning at the Units Place, say, 3 times 3 is 9, which because it is less than Ten, set it down underneath its own Place, and proceed to the next Place of Tens; saying 3 times 1 is 3, which set down underneath its own Place, then to the next Place, viz. of Hundreds, saying 3 times 2 is 6, which set down, as before; Lastly, at the Place of Thousands, say 3 times 3 is 9, which being set down underneath its own Place, the Operation is finished; and the true Product is  $9639 = 3213 \times 3$ , as was required.

*Example 2.*

Let it be required to multiply 8569 into 8. Set down these Numbers as before,

Thus } 8569  
8

68552

Beginning at the Units Place say, 8 times 9 is 72, set down the 2 underneath its own Place of Units, and bear the 70, or 7 Tens in mind, and proceed to the next Figure of the Multiplicand (at which Place the 7 Tens are only 7) saying 8 times 6 is 48, and the 7 carried in mind is 55, set down the odd 5 underneath its own Place of Tens, and carry the 50 (which is really 500) to the next Place (viz. of Hundreds) at which Place it is only 5, where say, 8 times 5 is 40; and the 5 carried in mind is 45, set down the 5 underneath its own Place; and carry the 40 or 4 Tens (which is really 4000) to the next Place, viz. of Thousands,



Thousands, saying, 8 times 8 is 64, and 4 carried in mind is 68. (Now this being the last Place or Figure to be multiplied) set down the whole Product 68, and the Work is done.

So that,  $8569 \times 8 = 68552$ , the Product required.

Now the Reason of this and all other the like Operations, may be easily conceived from this which follows,

$  \begin{array}{r}  8\ 5\ 6\ 9 \\  \times 8 \\  \hline  68\ 552  \end{array}  $	$\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\}$	<p>The same Factors as before,</p> <p>Here 8 times 9 is but 72, as before, because the 9 stands in the Units Place.</p> <p>Now here it is not really 8 times 6=48, but it is 8 times 60=480, because the 6 stands in the Place of Tens.</p> <p>And here it is not 8 times 5=40, but it is really 8 times 500=4000, because the 5 stands in the Place of Hundreds.</p> <p>Lastly, because the 8 in the Multiplicand stands in the Place of Thousands, it is therefore, 8 times 8000=64000, and not 8 times 8=64.</p> <p>The Sum of the particular Products, which gives the true Product, as before.</p>
--	--	---

By what hath been already said, with a little Consideration had to the Examples: I presume the Learner may easily understand how to multiply whole Numbers with any single Figure. And when it is requir'd to multiply with more than one; Then so many Figures as there are in the Multiplier, so many particular Products there must be.

That is, all the Figures of the Multiplicand must be multiplied with every single Figure of the Multiplier as if there were but one single Figure; and the Sum of all those particular Products, will be the true Product required; but in those Operations great Care must be taken in setting down the particular Products, (which arise by each multiplying Figure) in their proper Places. Which will be easily done if the following Directions be carefully observ'd.

*Viz.*  $\left\{ \begin{array}{l} \text{Always place the first Figure (or Cypher) of every particular Product, directly underneath the multiplying Figure. Or thus:} \end{array} \right.$

The first Figure (or Cypher) of the second particular Product must stand directly under the second Figure (or Place) of the first Product; and the first Figure (or Cypher) of the third particular

D



particular Product, must stand directly underneath the third Figure of the first Product: And so on until all is done.

Now the Reason of placing the first Figure of every particular Product in this Order, will be very obvious to any one that considers the last Example; wherein the Cyphers are only set down to shew the true distance of the first Figure in each particular Product from the Units Place. And altho' it is not usual to set down Cyphers in this Manner; yet they are always suppos'd to be there: That is, their Places are always left void, as in the two following Examples; wherein I have placed Points instead of Cyphers.

*Example 3.*

Let it be required to multiply 78094, into or with 7563.

78094	} Factors.		
7563			
<hr/>			
234282	The First particular Product with		3
468564.	The Second particular Product with		60
390470..	The Third particular Product with		500
546658...	The Fourth particular Product with		7000
<hr/>			
590624922	The Total, or true Product required.		

*Example 4.*

Suppose it be required to multiply 57498 into 60008.

57498	
60008	
<hr/>	
459984	The Product with 8
344988.....	The Product with 60000
<hr/>	

3450339984 = 57498 × 60008 as was required.

Here you may observe, that I pass over the Cyphers, and only take care of placing the first Product of the last Figure, viz. of 60000 according to the foregoing Directions.

When there is a Cypher or Cyphers, to the Right-hand either of the Multiplicand, or Multiplier, or to both; in that Case multiply the Figures as before; neglecting the Cyphers until the particular Products are added together; then to their Sum annex so many Cyphers as are in either or both the Factors. As in these.

*Example 5.*

Example 5.

Example 6.

Example 7.

9538  
4600

87600  
79

785000  
56900

57228  
38152

7884  
6132

7065  
4710

43874800

6920400

3925

44666500000

Take a few Examples without their Work at large.

75649X579=43800771  
687000X356=244572000  
530674X45007=23884044718  
7901375X30000=237041250000  
537084000X590700=317255518800000  
102030405X504030201=51426405540261405  
987654321X123456789=121932631112635269

Note, If it be required to multiply any Number with 10, 100, 1000, 10000 &c. it is only annexing the Cyphers of the Multiplier to the Figures of the Multiplicand, and the Work is done.

Thus  $\left\{ \begin{array}{l} 578 \times 10 = 5780 \\ 578 \times 100 = 57800 \end{array} \right.$  .  $\left\{ \begin{array}{l} 578 \times 1000 = 578000 \\ 578 \times 10000 = 5780000 \end{array} \right.$  , &c.

These Examples (being well understood) are sufficient to instruct the Learner, in all the Varieties that can happen in Multiplying of whole Numbers, according to the Method generally practised: However it may not be amiss to shew here how Multiplication may be performed (with many Figures) by Addition only.

Example.

Let it be required to multiply 879654 into 79863.

In Order to perform this (or any other Operation of this Kind) by Addition only; you must make a Tariffa or small Table of the given Multiplicand, in this Manner:

First, Make a small Column, and in it place gradually downward the nine single Figures; viz. 1. 2. 3. 4. 5. &c.

D. 2

Then



Then against the Figure 1, set down the Multiplicand (which in this Example is 879654) and against the Figure 2, set down the Double of the Multiplicand, found by adding it to it self; To this Double add the Multiplicand, setting down their Sum against the Figure 3. And so proceed on by a continued Addition until there be ten Times the Multiplicand in the Table, which, if the Work is true, will be the Multiplicand it self with a Cypher to the Right-hand of it (as in the annexed Table) this being done, it will be easy to conceive, that the Figures in the small Column of the Table, do respectively represent those of the Multiplier; And that the Numbers against any of those Figures in the small Column, will be the true Product of the Multiplicand agreeing to any Figure of the Multiplier; as plainly appears by the Work of this Example.

1	879654
2	1759308
3	2638962
4	3518616
5	4398270
6	5277924
7	6157578
8	7037232
9	7916886
10	8796540

Then  $\begin{array}{r} 879654 \\ 79863 \end{array}$  } The Factors as before.

Against 3, in the Table, is	2638962	= 879654 × 3
Against 6, is	5277924	= 879654 × 60
Against 8, is	7037232	= 879654 × 800
Against 9, is	7916886	= 879654 × 9000
Against 7, is	6157578	= 879654 × 70000

The Product required  $70251807402 = 879654 \times 79863$

*Note,* This Method of Tabulating the Multiplicand, is both easy and certain; being neither subject to Errors, nor burdenson to the Memory, and therefore in large Calculations it may be found very useful. But for common Practice the usual Method (as in Page 18, &c.) is best, and to be preferr'd before this.

Most Masters that teach (and several Authors that write of) Arithmetick, do teach to prove the Truth of Multiplication, by casting away all the Nines that are contain'd in both the Factors, and their Product; but because that Method is very erroneous, as might be easily shew'd; I shall therefore omit inserting it, and leave the Proof of Multiplication to the next Section. wherein (I presume) the Reason and Proof, both of it, and Division, will plainly appear.



## Sect. 5. Of Division.

**Division** is a *Rule* by which one *Number* may be speedily *subtracted* from another, so many times as it is contained therein.

That is, It speedily discovers how often one Number is contained (or may be found) in another: And to perform that, there are required two Numbers to be given.

1. The one of them is that Number which is proposed to be divided, and is called the *Dividend*.

2. The other is that Number by which the said *Dividend* is to be divided, and is called the *Divisor*.

And by comparing these Two, *viz.* the *Dividend* and the *Divisor* together, there will arise a third Number, called the *Quotient*; which shews how often the *Divisor* is contained in the *Dividend*, or into what Number of equal Parts the *Dividend* is then divided. Therefore,

*Division* is by *Euclid* fitly term'd the measuring of one Number by another, *viz.* one Number is said to measure another by that Number, which when it multiplies, or is multiplied by it, it produceth. *Euclid. 7. Def. 23.*

And, if a Number measuring another multiply that Number by which it measureth, or be multiplied by it, it produceth the Number which it measureth. *Euclid. 7. Axiom 9.*

That is to say, If that Number which divides another, (called the *Divisor*) be multiplied with the Number which is produced by *Division* (called the *Quotient*) their Product will be the Number divided or *Dividend*. Whence it follows that *Division* and *Multiplication* are the *Converse* or direct Contrary one to another (as *Subtraction* is to *Addition*) and do mutually prove the Truth of each others Operations.

I shall therefore make Choice of the foregoing Examples in *Multiplication*, in order (as I presume) to render the Business of *Division* more plain and easy.

*First*, Let it be required to find how often 6 is contained in 24. That is, to divide 24 by 6.

N. B. Always place down the given Numbers in this Order; First set down the *Divisor*, and to the Right-hand of it draw a crooked Line; then set down the *Dividend*, and to the Right of it draw another crooked Line, in which must be placed the *Quotient* Figure, or Figures as they are found.

*Dividend*

Dividend

Thus, Divisor 6) 24 (4 the Quotient.

Here I consider how many times 6 there is in 24, and find it 4, viz. 4 times 6 is 24, therefore 4 is the true Quotient or Answer required.

This is apparent by Subtraction, as in the Margin, where 24 the Dividend being set down, and from it 6, the Divisor is continually subtracted so often as it can be, which is just 4 times. Therefore 4 is the true Quotient or Answer required.

Compare this with the  
Example, Page 15.

	24
1	6
—	—
	18
2	6
—	—
	12
3	6
—	—
	6
4	6
—	—
	0

## COROLLARY.

From hence it is evident; that *Division* is but a concise or compendious Method of subtracting one Number from another so often as it can be found therein; for if the Divisor be continually subtracted from the Dividend, accounting an Unit (or 1) for each time it is subtracted (as above) the Sum of those Units will be the Quotient.

All Operations in *Division* do begin contrary to those of *Multiplication*, viz. at the first Figure to the Left-hand, or that of the highest Value, and decrease the Dividend by a repeated Subtraction of each Product arising from the Divisor when multiplied into the Quotient Figure. And the only Difficulty in *Division* of whole Numbers (or indeed of any Numbers) lies in making Choice of such a Quotient Figure, as is neither too big, nor too little; and that may be easily obtained by observing the following Rule, which hath two Cases.

## RULE.

*Case 1.* As often as the first Figure of the Divisor is taken from the first Figure of the Dividend: So often must the second Figure of the Divisor be taken from the second Figure of the Dividend, when it is joined with what remains of the First. And as often must the third Figure of the Divisor be taken from the third Figure of the Dividend, &c.

But if the first Figure of the Divisor cannot be taken from the first Figure of the Dividend. Then,

*Case*



*Case 2.* So often as the first Figure of the Divisor, is taken from the two first Figures of the Dividend, so often must the second Figure of the Divisor be taken from the third Figure of the Dividend, when it is join'd with what remain'd of the Second: And so often must the third Figure of the Divisor be taken from the fourth Figure of the Dividend, &c.

That is, the Quotient Figure must be such, as being multiplied into the Divisor, will produce a Product equal to such a Part of the Dividend as is then taken for that Operation: But if such a Product cannot be exactly found, then the next less must be taken, and ordered, as in the following Examples: Of which let that in *Page 16* be the first, wherein there was given 8569 the Multiplicand, and 8 the Multiplier. To find the Product 68552. Let us here suppose the said Product 68552, and 8 the Multiplier, both given; thence to find the Multiplicand. That is, Let it be required to divide 68552 by 8.

<i>Dividend</i>		
Divisor	8) 68552 (	<i>Quotient when found.</i>

According to the *Rule, Case 1.* I compare 8 the Divisor with 6 the first Figure of the Dividend. and finding I cannot take it from that; I then consider (by *Case 2.*) how often 8 can be taken from 68, the two first Figures of the Dividend, and find it may be taken 8 times; for 8 times 8 is 64, being the greatest Product of 8 (into any Figure) that can be taken from 68. I therefore place 8 in the Quotient, and with it multiply 8 the Divisor, setting down their Product underneath the said two first Figures of the Dividend, subtracting it from them, and then the Work will stand

Thus	8) 68552 (8
	64
	-----
	4

In order to a second Operation, I make a Point under the next Figure of the Dividend, *viz.* under the 5, and bring it down underneath its own Place to the Remainder 4, which will by that Means become 45. Then I consider how many times 8 can be taken from 45, and find it may be 5 times; for 5 times 8 is 40, I therefore place 5 in the Quotient, and with it multiply 8 the Divisor, setting down and subtracting their Product, as before. Then the Work will stand

Thus

$$\begin{array}{r}
 \text{Thus } 8 \overline{) 68552} \text{ (85} \\
 \underline{64} \phantom{00} \\
 45 \\
 \underline{40} \\
 5
 \end{array}$$

For a third Operation, I make a Point under the next Figure of the Dividend *viz.* under the 5, and bring it down, as before, proceeding in all Respects, as before; and then the Work will stand

$$\begin{array}{r}
 \text{Thus } 8 \overline{) 68552} \text{ (856} \\
 \underline{64} \phantom{00} \\
 45 \\
 \underline{40} \\
 55 \\
 \underline{48} \\
 7
 \end{array}$$

Lastly, I point and bring down the 2, *viz.* the last Figure of the Dividend to the Remainder 7, which will then become 72, and proceeding as in the other Operations, I find that 8, the Divisor, can be taken just 9 times from 72, and the Work is finished, and will stand

$$\begin{array}{r}
 \text{Thus } 8 \overline{) 68552} \text{ (8569} \\
 \underline{64} \phantom{00} \\
 45 \\
 \underline{40} \\
 55 \\
 \underline{48} \\
 72 \\
 \underline{72} \\
 (0)
 \end{array}$$

The true Quotient is found to be 8569, being exactly the eighth Part of 68552, or the Multiplicand of the proposed Example of Multiplication. As was required.

The Reason of these Operations will be very plain to any one that will a little consider of it as follows.

*Divisor*



Divisor 8) 6 8 5 5 2 (8000. The First Quotient or Figure.

Subtract 

6	4	0	0	0
---	---	---	---	---

 } This Product of the Divisor into the Quotient is 64000. viz. 8 Times 8000. the Quotient Figure being always of the same Value or Degree with that Figure under which the Unit's Place of its Product stands.

Divisor 8) 4 5 5 2 (500. The second Quotient Figure.

Subtract 

4	0	0	0
---	---	---	---

 } And here the Product is 4000. viz. 8 times 500. not 8 times 5, &c.

Divisor 8) 5 5 2 (60. The third Quotient Figure.

Subtract 

4	8	0
---	---	---

 } Also here the Product is 480. viz. 8 times 60. for the Reasons abovesaid.

Divisor 8) 7 2 (9. The fourth Quotient Figure.

Subtract 

7	2
---	---

 } Now here the Product is but 72. viz. 9 times 8. because the 9 stands in the Place of Units.

Remains (00) Now the Sum of all the several Quotients, viz.  $8000 \times 500 \times 60 \times 9 = 8569$ . as before.

If the Process of this Example be well considered and compared with that of Multiplication, Page 17. it will evidently appear to be only the Converse of that; for the particular Products are alike in both, only that which is *last* there, is *first* here; there they are *added*, here they are *subtracted*. So that whoever understands the true Reason of the one, must needs understand the Reason of the other, and then Division will become very easie, although the Divisor consist of several Places of Figures.

### Example.

Let it be required to divide 590624922 by 7563.

Dividend

Divisor 7563) 590624922 (

'Tis plain at Sight, that 7563 the Divisor, cannot be taken from 5906. the like Number of Figures in the Dividend.

Therefore, by the second Case of the Rule (Page 23.) there must be allowed Five Figures of the Dividend, viz. 59062 for the first Operation or Quotient; that so the first Figure 7 of the Divisor may be taken out of the two first Figures, viz. 59 of the Dividend, &c.

E

Then

Then I proceed (*per Case 2.*) and consider how often 7 may be taken from 59. and find it may be taken 8 times, for 8 times 7 is but 56. which I mentally subtract from 59. and there remains 3; to this 3 I mentally adjoyn the third Figure of the Dividend, *viz.* 0. which makes it 30. out of which I must take the second Figure of the Divisor, *viz.* 5. so often as I took the 7 from 59. which was 8 times. But that cannot be, for 8 times 5 is 40. which is more than 30. therefore 8 is too big a Figure to be placed in the Quotient; Yet, hence I conclude, that the next Less, *viz.* 7 may be taken without any further Tryal. I therefore place 7 in the Quotient, and with it multiply the Divisor, setting down their Product under the Dividend, and subtract it from thence, as in the other Example, and then the Work will stand

$$\begin{array}{r} \text{Thus } 7563) 590624922 \text{ (7} \\ \underline{52941} \\ 6121 \end{array}$$

In order to a second Operation, I make a Point under the next Figure of the Dividend, *viz.* under the 4. and bring it down to the Remainder 6121, which will then become 61214. with which I proceed in all respects as I did before with the 59062. and find the next Quotient Figure will be 8. with which I multiply the Divisor, &c. and subtract their Product from the said 61214. Then the Work will stand

$$\begin{array}{r} \text{Thus } 7563) 590624922 \text{ (78} \\ \underline{52941} \\ 61214 \\ \underline{60504} \\ 710 \end{array}$$

To this Remainder 710. I point and bring down the next Figure of the Dividend, *viz.* 9. which makes it 7109; now because the Divisor 7563 cannot be taken from 7109. I therefore place a Cypher in the Quotient.

And this must always be carefully observed, *viz.* That for every Figure or Cypher, which is brought down from the Dividend, in order to a new Operation, there must always be either a Figure or Cypher, set down in the Quotient. Then the Work will stand

Thus



Thus 7563) 590624922 (780  
52941...

61214  
60504

7109

To this 7109, I bring down another Figure of the Dividend, viz. 2, and then it will become 71092, then I consider how often 7 can be taken from 71, &c. (just as at the first Operation) and find it may be taken 9 times, therefore I set down 9 in the Quotient, and with it multiply the Divisor, setting down and subtracting their Product, as before; Then the Work will stand

Thus 7563) 590624922 (7809  
52941...

61214  
60504

71092  
68067

3025

To this Remainder 3025, I point and bring down the last Figure 2 of the Dividend, which makes it 30252, then proceeding in all Respects as before, I find the Quotient Figure to be 4, with it I multiply the Divisor, setting down and subtracting their Product as before, and then the Work will stand

Thus 7563) 590624922 (78094  
52941....

61214  
60504

71092  
68067

30252  
30252

(00000)

Here the Work is ended, and I find the Quotient to be 78094, being the true Multiplicand of the propos'd Example of Multiplication, Page 18.

That is, 7563 is contained in 590624922, just 78094 times, &c.

If the Work of this Example be considered and compared with the Rule (Page 22.) the whole Business of Division will be easie; for indeed the only Difficulty (as I said before) lies in making choice of a true Quotient Figure, which cannot well be done according to the common Method of Division, without Trials, yet those Trials need not be made with the whole Divisor (as appears by this last Example) for by the two first Figures of the Divisor all the rest are generally regulated; except the second Figure chance to be 2, 3, or 4, and at the same time the third Figure be 7, 8, or 9, then indeed Respect must be had to the third Figure according as the Rule directs.

However, if those Trials are thought too troublesom, they may be avoided, and the same Quotient Figures may both easily and certainly be found by Help of such a small Table made of the Divisor, as was of the Multiplicand in Page 20.

*Example 4.*

Let it be required to divide 70251807402 by 79863. See the Example of Multiplication, Page 20, and as there directed, make a Table of the Divisor 79863.

Thus,

	Divisor	Dividend
1	79863	70251807402
2	159726	638904.....
3	239589	-----
4	319452	636140
5	399315	559041
6	479178	-----
7	559041	770997
8	638904	718767
9	718767	-----
		522304
10	798630	479178

(879654. Quotient.

The Work of this Operation I presume may be easily understood. For those Figures in the Table are the Products of the Divisor into all the 9 Figures; consequently those Figures in the small Column do shew what Figure is to be placed in the Quotient; without any doubtful Trials of the Divisor with the Dividend, as before.

431260  
399315  
-----  
319452  
319452  
-----  
(000000)

This Method of tabulating the Divisor, may be of good Use to a Learner. Especially until he is well practised in Division; yea, and even then if the Divisor be large, and a Quotient of many Figures be required; as in resolving of high Æquations, and calculating of Astronomical Tables, or those of Interest, &c.

Hitherto



Hitherto I have made choice of Examples, wherein the Dividend is truly measured or divided off, by the Divisor, without leaving any Remainder, being those as were composed of the Divisor and Quotient. But it most usually falls out, that the Divisor will not exactly measure the Dividend; in that case the Remainder ( after Division is ended ) must be set over the Divisor with a small Line betwixt them adjoyning to the Quotient.

Example 5.

Suppose it were required to divide 379 by 5.

5) 379 ( $75\frac{4}{5}$  the Remainder,  
the Divisor.

35

29

25

Remains (4)

Example 6.

Again, Let it be required to divide 43789 by 67.

67) 43789 ( $653\frac{3}{7}$  The true Quotient required.

402

358

335

239

201

Remains (38)

How such Remainders thus placed over their Divisors ( which are indeed Vulgar Fractions ) may be otherwise managed, shall be shew'd farther on.

N. B. When the Divisor happens to be an Unit, viz. 1. with a Cypher, or Cyphers annexed to it, As 10, 100, 1000, &c. Division is truly performed by cutting off with a Point or Comma, so many Figures of the Dividend as there are Cyphers in the Divisor; then are those Figures so cut off to be accounted a Remainder, and the rest of the Figures in the Dividend will be the true Quotient required, because an Unit or 1 doth neither multiply nor divide.

Example 7.

Let it be required to divide 57842 by 100. The Work may stand thus, 100) 57842 the Quotient required, or thus 100) 57842 ( $578\frac{42}{100}$  the same as before.

Hence it follows, that if any Divisor have Cyphers to the Right-hand of it, you may cut off so many of the last Figures in

in the Dividend, and divide the other Figures of the Dividend, by those Figures of the Divisor that are left when the Cyphers are omitted. But when Division is ended, those Cyphers so omitted in the Divisor, and the Figures cut off in the Dividend are both to be restored to their own Places.

*Example 8.*

Suppose it were required to divide 675469 by 5400.

$$5400 \overline{) 675469} \quad (125$$

$$\underline{5400}$$

$$135$$

$$\underline{10800}$$

$$274$$

$$\underline{27000}$$

*Remains* (4) But the true Remainder is 469.

Consequently the true Quotient is  $125\frac{469}{5400}$ .

As to the Manner of proving the Truth of any Operation, either in Multiplication, or Division, I presume it may be easily understood, by what is deliver'd in *Page 21*, compared with the three first Examples of Division; For from thence it will be easie to conceive, that if the Divisor and Quotient be multiplied together, their Product (with what remains after Division being added to that Product) will be equal to the Dividend. As in the fifth Example, wherein the Dividend is 379, the Divisor is 5, the Quotient is 75, and the Remainder is 4.

I say,  $75 \times 5 = 375$ , to which add the Remainder 4, it will be 379.

Again, in the sixth Example, the Divisor is 67, the Quotient is 653, and the Remainder is 38.

Then  $653 \times 67 = 43751$ , and  $43751 + 38 = 43789$  the Dividend, &c.

There are several useful Contractions, both in Division and Multiplication, which I have purposely omitted until I come to treat of Decimal Arithmetick. Also I have omitted the Business of Evolution or Extracting of Roots, until further on; and so shall conclude this Chapter with a few Examples of Division, unwrought at large, leaving them for the Learner's Practice.

$$579 \overline{) 43800771} \quad (75649.$$

$$\text{Or } 75649 \overline{) 43800771} \quad (579$$

$$45007)$$



45007) 23884044718 (530674.  
 Or 530674) 23884044718 (45007.  
       356) 244572000 (687000.  
       59600) 57659066400 (967434.  
       10000) 679543820000 (67954382.  
       -79) 282016 (3569 $\frac{5}{9}$ .

## C H A P. III.

*Concerning Addition and Subtraction of Numbers of different Denominations, and how to reduce them from one Denomination to another.*

### S E C T. I.

#### 1. Of English Coin.

**T**H E least Piece of Money used in *England* is a Farthing, and from thence ariseth the rest, as in this Table.

<u>Farth.</u>
4 = <u>1 d. Pen.</u>
48 = 12 = <u>1 s. Shil.</u>
960 = 240 = 20 = <u>1 l. Pound Sterling.</u>

{

5 s. is a Crown.  
 10 s. is an Angel.  
 6 s. 8 d. a Noble.  
 13 s. 4 d. a Mark.

*Note,* When *l. s. d. q.* are placed over, (or to the Right-hand of) Numbers, they denote those Numbers to signify Pounds, Shillings, Pence, Farthings.

*l. s. d. q.*

As 35 10 6 2. Or 35 *l.* 10 *s.* 6 $\frac{1}{2}$  *d.* Either of these do signify 35 Pounds, 10 Shillings, 6 Pence, 2 Farthings.

The same must be understood of all the following Characters, belonging to their respective Tables, *viz.* Of Weights, Measures, &c.

#### 2. Troy Weight.

The Original of all Weights used in *England*, was a Corn of Wheat gathered out of the Middle of the Ear, and being well dried, 32 of them were to make one Penny Weight, 20 Penny Weight

Weight one Ounce, and 12 Ounces one Pound Troy. *Vide Statutes of 51. Hen. 3. 31. Edw. 1. 12. Hen. 7.*

But in later Times it was thought sufficient to divide the aforesaid Penny Weight into 24 equal Parts, called Grains, being the least Weight now in common Use; and from thence the rest are computed as in this Table.

<i>Gr. Gra.</i>	
24 =	1 <i>P. W. Penny Weight.</i>
480 = 20 =	1 <i>Oz. Ounce.</i>
7560 = 240 = 12 =	1 <i>lb Pound.</i>

Note, { By Troy Weight are weighed Jewels, Gold, Silver, Corn, Bread and all Liquors.

Besides the common Divisions of Troy Weight, I find in *Angliæ Notitia*, or, *The Present State of England*, Printed in the Year 1699, That the Moneyers, (as that Author calls them) do subdivide the Grain.

The { 24 Blanks = 1 Periot.  
20 Periot = 1 Droite.  
24 Droites = 1 Mite.  
20 Mites = 1 Grain, &c. as before.

### 3. Apothecaries Weights.

The Apothecaries divide a Pound Troy, as in this Table.

<i>Gr. Grain.</i>	
20 =	1 <i>℥ Scruple.</i>
60 = 3 =	1 <i>℥ Dram.</i>
480 = 24 = 8 =	1 <i>℥ Ounce.</i>
5760 = 288 = 96 = 12 =	1 <i>lb Troy, the same as before.</i>

By these Weights the Apothecaries compound their Medicines, but buy and sell their Drugs, by *Averdupois Weight*.

### 4. Averdupois Weight.

When *Averdupois Weight* became first in Use, or by what Law it was first settled. I cannot find out in the Statute Books; but on the contrary, I find that there should be but one Weight (and one Measure) used throughout this Realm, *viz.* that of Troy, (*Vide 14. Ed. 3. and 17. Ed. 3.*) So that it seems (*to me*) to be first introduced by Chance, and settled by Custom, *viz.* from giving good or large Weight to those Commodities as are usually weigh'd by it, which are such as are either very coarse and drossy, or very



# Chap. 3. Of Weights, Measures, &c. 33

very subject to waſt : As, all kind of Grocery Wares. And *Pitch, Tar, Roſin, Wax, Tallow, Soap, Flax, Hemp, &c. Copper, Tin, Steel, Iron Lead, &c.* Alſo, *Fleſh, Butter, Cheeſe, Salt, &c.* To theſe and the like (I preſume) it was thought convenient to allow a greater Weight than what the Laws had provided, which happened to be about a ſixth Part more : For I found by a very nice Experiment, that one Pound *Averdupois* is equal to 14 Ounces, 11 Penny-weight, and  $15\frac{1}{2}$  Grains Troy. And it is now computed as in the following Table.

<i>Drams.</i>		<i>lb</i>
16 = 1 oz. Ounces.		14 = a Stone.
256 = 16 = 1 lb Pounds.	And	28 = $\frac{1}{4}$ of C.
28672 = 1792 = 112 = 1 C. Hundred.		56 = $\frac{1}{2}$ of C.
573440 = 35840 = 2240 = 20 = 1 Tun.		84 = $\frac{3}{4}$ of C.

## 5. Long Measure.

As the leaſt Part of Weight came at firſt from a Wheat Corn, ſo (it's generally ſaid) the leaſt Part of *Long Measure* was at firſt a Barly Corn, taken out of the Middle of the Ear, and being well dried three of them in length were to make one Inch; and thence the reſt, as in this Table.

<i>Barly Corns</i>		And	$\left\{ \begin{array}{l} 4 \text{ Nails} = \frac{1}{4} \text{ of a Yard.} \\ 1\frac{1}{4} \text{ Yard} = 1 \text{ Ell.} \\ 2 \text{ Yards} = 1 \text{ Fathom.} \end{array} \right.$
3 = 1 In. Inches			
36 = 12 = 1 F. Feet			
108 = 36 = 3 = 1 T. Yards			
594 = 198 = $16\frac{1}{2}$ = $5\frac{1}{2}$ = 1 P. Poles			
23760 = 7920 = 660 = 220 = 40 = 1 Furlong			
190080 = 63360 = 5280 = 1760 = 320 = 8 = 1 Mile			

*Note,* That forty Poles (or Perches) in Length, and Four in Breadth do make a Statute Acre of Land.

That is, 220 Yards, multiplied into 22 Yards = 4840 Square Yards are a Statute Acre.

And according to the Transactions of the *French Academy*, Anno 1687, a *Paris Foot Royal* is = 12 8 Inches *English*; Six of thoſe Feet make a Toiſe; and 57060 Toiſe = 365184 *English* Feet, are the Measure of one Degree of a great Circle upon the Surface of the Earth. So that one Degree is 69 Miles and 288 Yards, which is very near to our Country-man Mr. *Normwood's* Experiment made betwixt *London* and *York*, Anno 1635; who found that 367196 Feet = 69 Miles, and 858 Yards do make a Degree. And not 60 Miles, according to the common received Opinion and Practice of the Navigators or Seamen.

F

Hence,

Hence, according to the *French Account*, the Circumference of the Earth (supposing it to be a true Spherical Figure) is 24899 *English Miles*.

### 6. Of *Liquid Measure*.

All Measures of Capacity, both Liquid and Dry, were at first made from *Troy Weight*. *Vide Statutes 9. H. 3. 51. H. 3. 12. H. 7. &c.* wherein it is Enacted, that eight Pound *Troy Weight* of Wheat, gathered out of the Middle of the Ear, and well dried, should make one Gallon of Wine Measure: And that there should be but one Measure for Wine, Ale and Corn, throughout this Realm (*Vide Stat. 14. Ed. 3. 15. Ric. 2.*) But Time and Custom hath alter'd Measures, as they have done Weights (and perhaps for one and the same Reason) for now we have three different Measures, *viz.* one for Wine, one for Ale or Beer, and one for Corn.

I have inserted Tables of each as they are now computed by Cubic Inches, and practised in the Art of Gaging, &c.

The common Wine Gallon sealed at *Guild-Hall* in *London*; by which all Wines, Brandies, Spirits, Strong-waters, Mead, Perry, Sider, Vinegar, Oyl and Honey, &c. are measured and sold; is supposed to contain 231 Cubic Inches, and from thence the rest are computed, as in this Table.

Cubic Inches		Gallons.	
231	= 1 <i>G. Gallons</i>	Note, {	18 = 1 Runlet, and 31½ makes a Wine or Vinegar Barrel. ( <i>Vide 1. R. 3.</i> )
9702	= 42 = 1 <i>Terce</i>		
14553	= 63 = 1½ = 1 <i>Hogshead</i>		
19404	= 84 = 2 = 1⅓ = 1 <i>Puncion</i>		
29106	= 126 = 3 = 2 = 1½ = 1 <i>Butt or Pipe</i>		
58212	= 252 = 6 = 4 = 3 = 2 = 1 <i>Tun</i>		

But Doctor *Wybard* in his *Tactometry*, Page 289, doth suppose the Wine Gallon to contain but 224, or 225 Cubic Inches at the most, and pursuant to this Account an Experiment was made by Mr. *Richard Walker* and Mr. *Philip Shales*, two General Officers in the Excise. They caused a Vessel to be very exactly made of Brass, in Form of a Parallelopipedon, each Side of its Base was 4 Inches; and its depth 14 Inches; so that its just Content was 224 Cubic Inches. This Vessel was produced at *Guild Hall* in *London* (May 25th, 1688) before the Lord-Mayor, the Commissioners of Excise, the Reverend Mr. *Flamsteed* Astr. Reg. Mr. *Halley*, and several other ingenious Gentlemen



# Chap. 3. Of *Weights, Measures, &c.* 35

Gentlemen, in whose Presence Mr. *Shales* did exactly fill the afore-said Brazen Vessel with clear Water, and very carefully emptied it into the old Standard Wine Gallon kept in *Guild-Hall*, which did so exactly fill it, that all then present were fully satisfied the Wine Gallon doth contain but 224 Cubic Inches. (*This notable Experiment I saw tried.*) However, for several Reasons, it was at that Time thought convenient to continue the Former supposed Content of 231 Cubic Inches to be the Wine Gallon, and that all Computations in Gaging should be made from thence, as above.

The Beer or Ale Gallon. (which are both one) is much larger than the Wine Gallon, it being (as I presume) made at first to correspond with *Averdupois Weight*, as the Wine Gallon did with *Troy Weight*: For (as I said before Page 33) one Pound *Averdupois* is equal to 14 Ounces 12 Penny Weight *Troy*, very near.

And, as one Pound *Troy* is in Proportion to the Cubic Inches in a Wine Gallon, so is one Pound *Averdupois* to the Cubic Inches in an Ale Gallon. That is,  $12 : 231 :: 14\frac{1}{2} : 281\frac{1}{2}$ , very near the Cubic Inches contained in an Ale Gallon, as appears from an Experiment made by one *Nicholas Gunton*, General Gager in the Excise, about 60 Years ago, who by such a Vessel mentioned before in the last Page, did find the Standard Ale-Quart (kept in the Exchequer, *Vid.* 12. *Car.* 2) to contain just  $70\frac{1}{2}$  Cubic Inches, consequently the Ale Gallon must contain 282 Cubic Inches, and from thence the following Tables are computed.

## *Ale-Measure.*

*Cubic Inches*

282	=	1	Gallon
2256	=	8	= 1 Ferkin
4512	=	16	= 2 = 1 Kilderkin
9024	=	32	= 4 = 2 = 1 Barrel
13536	=	48	= 6 = 3 = $1\frac{1}{2}$ = 1 Hogshhead.

Note, { A Ferkin of Soap and of  
Herrings are the same  
with that of Ale.

## *Beer-Measure.*

*Cub. Inches*

282	=	1	Gallon
2538	=	9	= 1 Ferkin
5076	=	18	= 9 = 1 Kilderkin
10152	=	36	= 4 = 2 = 1 Barrel
15228	=	54	= 6 = 3 = $1\frac{1}{2}$ = 1 Hogshhead.

N. B. This Distinction or Difference betwixt *Ale* and *Beer-Measure*, is now only used in *London*. But in all other Places of *England* the following Table of *Beer* or *Ale*, whether it be strong or small, is to be observed, according to a Statute of Excise, made in the Year 1689.

Cub. Inches	
282	= 1 Gallon
2397	= $8\frac{1}{2}$ = 1 Ferkin
4794	= 17 = 2 = 1 Kild.
9588	= 34 = 4 = 2 = 1 Barrel
14382	= 5 = 6 = 3 = $1\frac{1}{2}$ = 1 Hogshead.

### 7. Of Dry Measure.

*Dry Measure* is different both from the *Wine* and *Ale-Measure*, being as it were a Mean betwixt both, tho' not exactly so; which upon Examination I find to be in Proportion to the aforesaid old Standard *Wine Gallon*, as *Averdupois Weight* is to *Troy Weight*; That is, as one Pound *Troy* is to one Pound *Averdupois*, so are the Cubic Inches contained in the old *Wine Gallon*, to the Cubic Inches contained in the *Dry* or *Corn Gallon*.

*Viz.*  $12 : 14\frac{1}{2} :: 224 : 272\frac{1}{2}$  which is very near to  $272\frac{1}{4}$ , the common received Content of a *Corn Gallon*, altho' now it is otherwise settled by an Act of Parliament made in April 1697, the Words of that Act are these:

Every round Bushel with a plain and even Bottom, being made eighteen Inches and a half wide throughout, and eight Inches deep, should be esteem'd a legal Winchester Bushel, according to the Standard in his Majesty's Exchequer.

Now a Vessel being thus made will contain 2150.42 Cubic Inches, consequently the *Corn Gallon* doth contain but  $268\frac{2}{3}$  Cubical Inches.

Cub. Inches		Note, {	4 Bushels = a Comb. 10 Combs = a Wey, and 2 Weys = a Last of Corn.
268,8	= 1 Gallon		
537,6	= 2 = 1 Peck		
2150,4	= 8 = 4 = 1 Bushel		
17202,2	= 64 = 32 = 8 = 1 Quarter.		

I observ'd amongst the Lead-Mines in *Derbyshire*, (Anno 1692) that the Miners bought and sold their Lead-Ore, by a Measure which they call'd an Ore Dish; whose Dimensions I carefully took and found them

Thus { Length 21.3  
Breadth 6.  
Depth 8.4 } Inches

Conse-



Consequently it's Content is 1073,52 Cubic Inches, which is very near equal to 4 Corn Gallons, according to the abovementioned Settlement.

Nine of those Dishes they call a Load of Ore, which if it be pretty good, will produce about 3 hundred Weight of Lead.

## 8. Of Time.

It is not an easy Thing to give a true Definition of *Time*; for (according to the *Philosophic Poet*)

‘ Time of it self is nothing, but from Thought  
 ‘ Receives its Rise, by labouring Fancy wrought  
 ‘ From Things consider’d, whilst we think on some  
 ‘ As present, some as past, or yet to come.  
 ‘ No Thought can think on Time, that’s still confest,  
 ‘ But thinks on Things in Motion or at Rest.

And so on, *Vide Lucretius, Book I.*

That is, Time only shews the Duration or Mutation of Things, a Year being the Standard or Integer, by which such Continuance or Change is computed. And a Year is that Space of Time in which the Sun (apparently) completes its Revolution from any one Point in the Ecliptic (an imaginary Circle in the Heavens) to the same Point again, which according to Modern Observations is perform’d in 365 Days, 5 Hours, 48 Minutes, 57 Seconds, 21 Thirds, &c. But a Second being the least Part of Time that can be truly measured by the Motion of any Mechanical Engin, as a Clock, &c. (a Third being less than the Twinkling of an Eye) I begin the following Table with Seconds.

<i>Seconds. "</i>			
60	=	1	<i>Minutes</i>
3600	=	60	= 1 <i>Hours</i>
86400	=	1440	= 24 = 1 <i>Days</i>
31556937 = 525349 = 8765 = 365 + 5 + 48 + 57 = 1 Year, call'd a Solar			
(Year.			

But the common Year, usually call'd the *Julian* Year, doth consist of 365 Days and 6 Hours, and is divided into twelve unequal Months, called Calendar Months, whose Names and Number of Days are the Subject of every *Almanack*.

To

To these Tables it may not be amiss to give a brief Account of such Coins, Weights and Measures as are frequently mention'd in the Scriptures. As I have deduc'd them from those which seem to be the most correct, inserted in the *Index* to the large *Bible*, Printed *Anno* 1702, and compared with those used in *England*, by the Lord Bishop of *Peterborough*.

The Hebrew Weights, compared with  $\left\{ \begin{array}{l} \text{Troy Weight.} \\ \text{Oz. Pw. Grains.} \end{array} \right.$

A Gerah=	0	.	0	.	10 $\frac{1}{2}$ $\frac{2}{3}$
10 Gerahs=a Bekah=	0	.	4	.	13 $\frac{1}{2}$
2 Bekahs=a Shekel=	0	.	9	.	3
100 Shekels=a Menah=	45	.	12	.	12

Note, A *Shekel* is said to be their Original Weight.

Their Coin.  $\left\{ \begin{array}{l} \text{English Coin.} \\ \text{l. s. d.} \end{array} \right.$

A Silver Menah=	7	.	1	.	5 $\frac{1}{4}$	Weight 60 Shekels.
Talent of Silver=	357	.	11	.	10 $\frac{1}{2}$	Weight is 3000 Shekels.
Talent of Gold=	5075	.	15	.	7 $\frac{1}{2}$	The same Weight men-
The Gold Dram=	1	.	0	.	4	tioned in Ez. 2. 19.

The Roman Money mentioned in the *New Testament*.

A Denarius, or Silver Penny=	7 d.	3 Farthings.
Asses of Copper=	0 . 3	Farthings.
Assarium=	0 . 1	$\frac{1}{2}$ Farthing.
Quadrans=	0 .	$\frac{3}{4}$ of a Farthing.
A Mite=	0 .	$\frac{1}{3}$ of a Farthing.

Their Long Measures compared with  $\left\{ \begin{array}{l} \text{English Measure.} \\ \text{Yar. Feet. Inch Parts} \end{array} \right.$

A Finger's Breadth=	0.	0 .	0,912
4 Fingers=a Hand's Breadth=	0.	0 .	3,648
2 Hands=the least Span=	0.	0 .	7,296
3 Hands Breadth=the longest Span=	0.	0 .	10,944
2 Spans=the longest Cubit=	0.	1 .	9 888
4 Cubits=a Fathom=	2.	1 .	3,552
6 Cubits=Ezekiel's Reed=	3.	1 .	11 328
400 Cubits=a Stadium=	243.	0 .	7,2
10 Stadiums=a Mile=	2432.	0 .	0
3 Miles=a Parasang=	7296.	0 .	0
Which is 4 English Miles and	256.		

Their



# Chap. 2. Addition of Weights, &c. 39

Their Measures of Capacity, compared with  $\left\{ \begin{array}{l} \text{Gal.} \\ \text{Pints} \\ \text{Inch.} \end{array} \right.$  *English Wine.*

A Cotyla =	0	.	0 $\frac{1}{2}$	3.037
A Log =	0	.	0 $\frac{1}{2}$	9,83
4 Logs=a Cab =	0	.	3	10,458
10 Cotyla's=an Omer =	0	.	6	1,5
3 Cabs=a Hin =	1	.	2	2,5
2 Hins=a Seah =	2	.	4	5,
3 Seahs=an Epha =	7	.	4	15,
10 Epha's=a Chomer =	75	.	5	5,625

## S E C T. 2. Addition of Weights, &c.

The foregoing Tables being so well understood, as that you can readily tell (without pausing) how many Units of any one Denomination, do make one of the next superior Denomination (especially in those Tables as are most useful for your Business) it will then be as easy to add, or subtract them, as to add, or subtract whole Numbers, due care being taken in placing all Numbers that are of one Denomination exactly underneath each other. That is to say, in Money place Pounds under Pounds, Shillings under Shillings, Pence under Pence, &c. Understand the like in Weights and Measures. &c. According to their several Denominations: Then in Addition observe this Rule.

### R U L E.

Always begin with those Figures of the lowest or least Denomination, and add them all together into one Sum, then consider how many of the next superior Denomination are contained in that Sum, so many Units you must carry to the said next superior Denomination to be added together with those Figures that stand there; and if any thing remain over or above those Units so carried, that Overplus must be set down underneath its own Denomination: And so proceed on from one Denomination to another until all be finished.

### Example in Coin.

Let it be required to add 35*l.* 14*s.* 06*d.* and 27*l.* 02*s.* 10*d.* and 54*l.* 13*s.* 04*d.* and 10*l.* 17*s.* 09*d.* into one Sum.

These parricular Sums being placed, as before directed, will stand as in the Margin following.

Then according to the Rule, I begin with the Pence (being here the lowest or least Denomination) and adding them all together, I find their Sum to be 29*d.* that is 2*s.* and 5 Pence; (for

(for  $24 d. = 2 s.$  and  $29 - 24 = 5$ ) the  $5 d.$  I set down underneath its own Denomination, and carry the  $2 s.$  to the Place of Shillings, adding them and all the Shillings together, I find the Sum to be  $48 s.$  viz.  $2 l. 8 s.$  I set down the  $8 s.$  underneath its own Place of Shillings, and carry the  $2 l.$  to the Place of Pounds, adding them and all the Pounds together, I find their Sum is  $128 l.$  consequently the total Sum required is  $128 l. 08 s. 05 d.$

Now, for as much as it often happens in keeping Books of Ac-compts, (and in other Business) that it is required to add up large Sums of Money, consisting of 30, 40, or more several particular Sums, nay, perhaps filling up the whole length of a Sheet of Paper, I humbly conceive in those Cases the best and easiest Way will be to part them into Parcels, not exceeding 10 or 12 particular Sums in each Parcel, and Sum up (on a by-Paper) the Particulars in each Parcel; that done, add together all the Sums of those Parcels into one Sum, and that will be the total Sum required.

Also to avoid the making of Points, or other Marks amongst your Figures, it will be convenient to get the following Tables by heart.

The Pence-Table.

d.	s.	d.	s.
12	= 1.	72	= 6.
24	= 2.	84	= 7.
36	= 3.	96	= 8.
48	= 4.	108	= 9.
60	= 5.	120	= 10.

The Shillings Table.

s.	l.	s.	l.
20	= 1.	120	= 6.
40	= 2.	140	= 7.
60	= 3.	160	= 8.
80	= 4.	180	= 9.
100	= 5.	200	= 10.

The Use of these Tables is so obvious, that I presume 'tis needless to explain them.

## Examples in Addition of Weights.

## Troy Weight.

lb	Oz.	Pw.	Gr.
3	. 09	. 00	. 10
5	. 08	. 15	. 21
10	. 10	. 12	. 22
0	. 11	. 19	. 23

Sum 21 . 04 . 09 . 04

## Averdupois Weight.

Tun.	C.	Q	lb	Oz.
12	. 15	. 2	. 24	. 12
7	. 10	. 3	. 21	. 15
0	. 18	. 1	. 14	. 11
1	. 19	. 3	. 27	. 15

Sum 23 . 05 . 0 . 05 . 05

Example



# Chap. 3. Subtraction of Weights. 41

*Examples in Addition of long Measure.*

<i>Tards</i>	<i>Qrs.</i>	<i>Nails</i>	<i>Miles.</i>	<i>Fur.</i>	<i>Poles</i>	<i>Tards</i>	<i>Feet</i>	<i>Inch.</i>
35	2	3	2	6	32	4	2	9
17	3	1	0	7	27	3	1	10
129	1	2	1	3	39	1	2	11

---

182 . 3 . 2      Sum   5 . 2 . 19 . 5 . 0 . 0

I think it needless to set down more Examples of this kind, for if these 5, especially the last, be well understood, they will be sufficient to shew how any other may be performed.

---

## Sect. 3. Subtraction of Weights, &c.

Subtraction is but the Converse of the precedent Work, and may be performed by observing this *Rule*.

*R U L E.*

Begin with the lowest or least Denomination, as before in Addition, and take or subtract the Figure, or Figures, in that Place of the Subtrahend, from the Figure, or Figures, that stand over them of the same Denomination; setting down the Remainder (as in *Page 12.*) But if that cannot be done, then you must increase the upper Figure, or Figures, with one of the next superior Denomination, and from that Sum make Subtraction; and so proceed to the next superior Denomination, where you must pay the one borrowed, by adding Unity to the Subtrahend in that Place, &c. as in whole Numbers.

*Example in Coin.*

<i>l.</i>	<i>s.</i>	<i>d.</i>	<i>l.</i>	<i>s.</i>	<i>d.</i>
From 386	09	08	From 569	10	06
Take 173	04	06	Subt. 389	15	08
<hr/>			<hr/>		
Remains	213	05 . 02		179	14 . 10

The first of these Examples is self-evident. In the second Example, beginning at the Place of Pence, being here the least Denomination, I am to take 8 *d.* from 6 *d.* but because that cannot be done, I must, according to the Rule, borrow one of the next Denomination, *viz.* 1 *s.* and add it to the 6 *d.* which makes it 18 *d.* (for 1 *s.* = 12 *d.* and 12 + 6 *d.* = 18 *d.*) then I take 8 *d.* from that 18 *d.* and there remains 10 *d.* to be set down underneath the Place of Pence; that done, I proceed to the Place of Shillings, where I must now pay the 1 *s.* saying one borrowed, and 15 makes 16 from 10 cannot be, but 16 from 30 and there remains 14. That is, I

G

borrow

borrow one of the next Denomination, viz. 1 *l.* and add to it the 10 *s.* which makes it 30 *s.* (for 1 *l.* = 20 *s.* and 20 + 10 = 30) having set down the remaining 14 *s.* underneath its own Place of Shillings, I proceed to the Place of Pounds, where paying the 1 *l.* borrowed, it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there remains 9, and so on as in whole Numbers, until all be finished; and the Remainder will be 179 *l.* 14 *s.* 10 *d.*

This Example being a little considered will render all others in this Rule easie.

### Examples in Weight.

Troy Weight.			
lb	oz.	pwt.	gr.
From	9	10	16
Take	5	09	18
<hr/>			
	4	00	17
			20

Averdupois Weight.			
c.	qr.	lb	oz.
From	17	2	15
Take	14	3	18
<hr/>			
	2	2	24
			14

### Examples in long Measure.

	yards	qrs.	nails
From	78	3	2
Take	29	3	3
<hr/>			
Rests	48	3	3

	miles	fur.	pol.	yds.	feet	inches
From	22	3	26	3	1	11
Take	18	6	29	4	2	7
<hr/>						
	3	4	36	4	0	10

### Example in Time.

	days	'	"
From	27	18	35
Subtract	16	21	46
<hr/>			
Remains	10	20	48

The Proof of Addition and Subtraction in these Numbers of different Denominations, is the very same with that of whole Numbers in Page 13. I shall therefore refer you to that Place, and omit repeating it here.

## Sect 4. Of Reduction.

By Reduction, Numbers of different Denominations are brought into one Denomination.

That is, it alters or changes any superior Denomination proposed, into any inferior or lesser Denomination required; still keeping



keeping them equivalent in Value. And by that means they become fitly prepared for Multiplication and Division; which otherwise could not so conveniently be performed. Therefore the Business of Reduction is very useful in the Rule of Proportion, commonly called the *Golden Rule*, or *Rule of Three*, especially to those who do not understand either Vulgar or Decimal Fractions: And 'tis thus performed.

## R U L E.

Consider how many Units of the Denomination required, make one of that Denomination proposed to be reduced; which is easily known by its respective Table, and with that Number of Units, multiply the Denomination proposed, and their Product will be the Number required.

### Example in Coin.

Let it be required to reduce or change 357 *l.* into Shillings, and those Shillings into Pence, which shall still be equal in Value with the 357 *l.*

Multiply with  $\begin{array}{r} 357 \\ 20 \end{array}$  the Shillings in one Pound

Multiply with  $\begin{array}{r} 7140 \\ 12 \end{array}$  the Pence in one Shilling

1428

714

85680 = the Pence in 357 *l.* as was required.

Or 357 *l.* may be reduced into Pence, at one Operation: Thus,

Multiply with  $\begin{array}{r} 357 \text{ } l. \\ 240 \end{array}$  the Pence contained in one Pound.

1428

714

85680 = the Pence in 357 *l.* as before.

But when the Numbers proposed to be reduc'd are of several Denominations, and it is required to bring them all to the lowest; you must reduce the highest or greatest Denomination to the next less, Adding the Numbers that are of that less Denomination together, then reduce their Sum to the next lower Denomination, Adding together all the Numbers that are of that Denomination, and so proceed gradually on until all is done.

*Example.*

Let it be required to reduce 375 *l.* 17 *s.* 10 *d.* 3 *q.* into one Denomination, *viz.* into Farthings.

375 *l.* 17 *s.* 10 *d.* 3 *q.*

20

7500 = the Shillings in 375 *l.*  
+ 17 *s.*

7517 = the Shillings in 375 *l.* 17 *s.*

12

15034

7517

90204 = the Pence in 375 *l.* 17 *s.*  
+ 10 *d.*

90214 = the Pence in 375 *l.* 17 *s.* 10 *d.*

4

360856 = the Farthings in 375 *l.* 17 *s.* 10 *d.*  
+ 3 *q.*

360859 Farthings = 375 *l.* 17 *s.* 10 *d.* 3 *q.* as was required.

The Work of this Example, and all other Operations of this kind, may be somewhat shortned by observing the following Method.

375 *l.* 17 *s.* 10 *d.* 3 *q.*

20 Multiply and add in the 17 *s.*

7517

12 Multiply and add in the 10 *d.*

15034

7518

90214

4 Multiply and add in the 3 *qrs.*

360859 the Farthings as before.

*Example in Troy Weight.*

Suppose it be required to reduce 29 *lb* 8 *oz.* 18 *pwt.* 21 *gr.* into the least Denomination, *viz.* into Grains,

Thus,



Thus, 29 lb 8 oz. 18 pwt. 21 gr.  
Multiply with 12 the oz. in 1 lb and add in the 8 oz.

$$\begin{array}{r} 66 \\ 29 \\ \hline \end{array}$$

Multiply with 356—the Ounces in 29 lb 8 oz.  
20 the pwt. in 1 oz. and add in the 18 pwt.

Multiply with 7138—the pwt. in 29 lb 8 oz. 18 pwt.  
24 the Grains in 1 pwt. and add in the 21 gr.

$$\begin{array}{r} 28553 \\ 14278 \\ \hline \end{array}$$

171333 the Grains=29 lb 8 oz. 18 pwt. 21 gr.

These two Examples at large being well understood, may suffice to shew how all Operations of this kind are performed; either in Weights, Measures, or Time. I shall only insert a few Examples of each Sort for the Learner's Practice.

1. In 23 C. 3 qrs. 21 lb 9 oz. Averdupois Weight; How many Ounces? *Answer.* 42905 Ounces.

2. In 252 English Miles, How many Yards, Feet and Inches? *Answer* 443520 Yards=1330560 Feet=15966720 Inches.

3. In 1692 common Years, How many Days, Hours, and Minutes? *Answer* 618003 Days, 14832072 Hours, 889924320 Minutes.

*Note,* a common Year=365 Days, 6 Hours, see Page 37.

4. In 5786 Pounds, 17 Shillings, 9 Pence Sterling: How many Shillings, Pence and Farthings?

*Answer,* 115737 Shillings, 1388853 Pence, or 5555412 Farthings. That is, 5786 l. 17 s. 9 d.=115737 s. 9 d.=1388853 d. &c.

The next thing will be to shew how to bring Numbers from a lesser to a greater Denomination, which by most Authors is called, tho' very improperly.

### Reduction Ascending.

This Work is the Converse of the last, and is performed by Division. Thus,

#### RULE.

Consider how many of the Denomination proposed make one of the Denomination required, and make that Number your Divisor, by which divide the Denomination proposed; and the Quotient will be the Number required.

Ex-

*Example.*

Let it be required to find how many Shillings and Pounds are contained in 85680 Pence.

The Pence in 1 s. are 12) 85680 (7140 s. = 85680 d.

Again the Shillings in 1 l. are 20) 7140 (357 l. the Answer required.

*Another Example in Coin.*

How many Pence, Shillings and Pounds, are contained in 264859 Farthings.

$$\begin{array}{r}
 \begin{array}{r}
 12) \\
 4) 264859 \text{ (} 66214 \text{ d. (} 5517 \text{ s. (} 275 \text{ l.} \\
 \hline
 24 \qquad 62 \qquad 151 \\
 \hline
 08 \qquad 21 \qquad 117 \\
 \hline
 05 \qquad 94 \qquad (17) \text{ s.} \\
 \hline
 19 \qquad (10) \text{ d.}
 \end{array}
 \end{array}$$

Remains (3) q. } Note, The Remainder is always of the same Denomination with the Dividend.

The last Quotient 275 l. together with the several Remainders, gives the Answer required.

*Viz.* 275 l. 17 s. 10 d. 3 q. = 264859 Farthings.

*Example in Troy Weight.*

Suppose it were required to find how many pwt. ozs. and lbs. are contained in 171333 Grains.

$$\begin{array}{r}
 \begin{array}{r}
 20) \qquad 12) \\
 24) 171333 \text{ gr. (} 7138 \text{ pwt. (} 356 \text{ (} 29 \text{ lb} \\
 \hline
 168 \dots \qquad 113 \qquad 24 \\
 \hline
 33 \qquad 138 \qquad 116 \\
 24 \qquad \qquad 108 \\
 \hline
 93 \qquad (18) \text{ pwt.} \qquad (8) \text{ oz.} \\
 72 \qquad \qquad \qquad \\
 \hline
 213 \\
 192 \\
 \hline
 \end{array}
 \end{array}$$

Remains (21) gr.

*Answer,* 29 lb 8 oz. 18 pwt. 21 gr. This and the last Example are the Reverse or Proof of those in Page 43, 45.

1. In 42905 Ounces Averdupois Weight; How many Pounds, &c. Thus



$$\begin{array}{r}
 \text{Thus } 16) 42905 \quad \begin{array}{r} 28) \\ (2681 \text{ lb} \end{array} \quad \begin{array}{r} 4) \\ (95 \text{ qrs. } (23 \text{ C.} \\
 \hline
 109 \quad 252 \quad 15 \\
 \hline
 130 \quad 161 \quad (3) \\
 \hline
 25 \quad 140 \\
 \hline
 (9) \quad (21) \text{ Answer, } 23 \text{ C. } 3 \text{ qrs. } 21 \text{ lb } 9 \text{ oz.}
 \end{array}$$

2. In 15966720 Inches; How many *English* Miles, &c.

*Answer*, 252 Miles, &c. As Occasion requires.

There are many useful Questions may be answered by Help of Reduction only: As the Changing of one Sort of Coin for another; and comparing of one Sort of Measure with another, &c.

For Instance; Suppose one had 347 Rix-Dollars, at 4 s. 6 d. per Dollar: and desired to know how many Pounds *Sterling* they make.

$$\begin{array}{r}
 347 \\
 54 = \text{the Pence in one Dollar, viz. } 4 \text{ s. } 6 \text{ d.} = 54 \text{ d.} \\
 \hline
 1388 \\
 1735 \quad 20) \\
 12) 18738 \text{ d. } (1561 \text{ s. } (78 \text{ l.} \\
 \hline
 67 \quad 161 \\
 \hline
 73 \quad (1) \text{ s.} \\
 \hline
 18 \\
 (6) \text{ d.}
 \end{array}$$

*Answer*, 78 l. 1 s. 6 d. *Sterling*, are = 347 Rix-Dollars.

*Quest.* 2. In 645 *Flemish* Ells; How many Ells *English*?

*Note*, 3 Quarters of a Yard *English* make one Ell *Flemish*, and  $1\frac{1}{4}$  or 5 Quarters of a Yard is an *English* Ell.

Therefore, 645

$$\begin{array}{r}
 3 = \text{the qrs. of a Yard in } 1 \text{ Ell } \textit{Flemish}. \\
 \hline
 \text{qrs. in } 1 \text{ Ell} = 5) 1935 (387 \text{ English Ells for the Answer.}
 \end{array}$$

*Quest.* 3. Suppose a Bill of Exchange were accepted at *London*, for the Payment of 400 l. *Sterling*, for the Value deliver'd at *Amsterdam* in *Flemish* Money at 1 l. 13 s. 6 d. for 1 l. *Sterling*. How much *Flemish* Money was delivered at *Amsterdam*?

*First*, 1 l. 13 s. 6 d. = 402 d. the Value of one Pound *Sterling* at *Amsterdam*.

Then  $402 \text{ d.} \times 400 = 160800 \text{ d.} = 670 \text{ l. } \textit{Flemish}$ , and so much was deliver'd at *Amsterdam*.

## C H A P IV.

## Of Vulgar Fractions.

## Sect. i. Of Notation.

**A** Fraction, or broken Number, is that which represents a Part or Parts of any thing proposed (*vide Page 3.*) and is expressed by two Numbers placed one above the other with a Line drawn betwixt them.

Thus,  $\left\{ \begin{array}{l} 3 \text{ Numerator.} \\ 4 \text{ Denominator.} \end{array} \right.$

The Denominator or Number placed underneath the Line, denotes how many equal Parts the Thing is supposed to be divided into (being only the Divisor in Division.) And the Numerator or Number placed above the Line, shews how many of those Parts are contained in the Fraction, it being the Remainder after Division. (See *Page 29.*) And these admit of three Distinctions:

Viz.  $\left\{ \begin{array}{l} \text{Proper or Simple,} \\ \text{Improper,} \\ \text{Compound,} \end{array} \right\} \text{ Fractions.}$

A proper, pure, or simple Fraction, is that which is less than an Unit. That is, it represents the immediate Part or Parts of any thing less than the Whole, and therefore its Numerator is always less than the Denominator.

As  $\left\{ \begin{array}{l} \frac{1}{4} \text{ is one Fourth Part.} \\ \frac{1}{3} \text{ is one Third Part.} \end{array} \right.$  And  $\left\{ \begin{array}{l} \frac{1}{2} \text{ is one Half.} \\ \frac{2}{3} \text{ is Two Thirds, \&c.} \end{array} \right.$

An improper Fraction is that which is greater than an Unit; That is, it represents some Number of Parts greater than the whole thing; and its Numerator is always greater than the Denominator.

As  $\frac{5}{3}$  Or  $\frac{2}{1}$  Or  $\frac{4}{1}$  &c.

A Compound Fraction is a Part of a Part, consisting of several Numerators and Denominators connected together with the Word [of].

As  $\frac{1}{3}$  of  $\frac{2}{4}$  of  $\frac{2}{5}$ , &c. and are thus read, The one Third of the three Fourths of the two Fifths of an Unit.

That is, when a Unit (or whole Thing) is first divided into any Number of equal Parts, and each of those Parts are subdivided



subdivided into other Parts, and so on: Then those last Parts are called compound Fractions, or Fractions of Fractions.

As for instance, suppose a Pound *Sterling*, or 20 s. be the Unit or Whole; then is 8 s. the  $\frac{2}{5}$  of it, and 6 s. the  $\frac{3}{4}$  of those two Fifths, and 2 s. is the  $\frac{1}{3}$  of those three Fourths. *Viz.* 2 s.  $= \frac{1}{3}$  of  $\frac{3}{4}$  of  $\frac{2}{5}$  of one Pound *Sterling*.

All compound Fractions are reduced to single ones, Thus,

## R U L E.

Multiply *all the Numerators into one another for a Numerator, and all the Denominators into one another for the Denominator.*

Thus the  $\frac{1}{3}$  of  $\frac{3}{4}$  of  $\frac{2}{5}$  will become  $\frac{6}{60}$ . Or  $\frac{1}{10}$ . For  $1 \times 3 \times 2 = 6$  the Numerator, and  $3 \times 4 \times 5 = 60$  the Denominator, but  $\frac{6}{60}$  or  $\frac{1}{10}$  of a Pound *Sterling* is 2 s. As above,

## Sect. 2. To Alter or Change different Fractions into one Denomination retaining the same Value.

In order to gain a clear Understanding of this Section, it will be convenient to premise this Proposition, *viz.* "If a Number multiplying two Numbers produce other Numbers, the Numbers produced of them shall be in the same Proportion that the Numbers multiplied are, 17 *Euclid* 7.

That is to say, If both the Numerator and Denominator of any Fraction be equally multiplied into any Number, their Products will retain the same Value with that Fraction.

As in these,  $\frac{2 \times 2}{3 \times 2} = \frac{4}{6}$ . Or  $\frac{2 \times 3}{3 \times 3} = \frac{6}{9}$ . Or  $\frac{2 \times 5}{3 \times 5} = \frac{10}{15}$ , &c,

That is,  $\frac{2}{3}$  and  $\frac{4}{6}$ . Or  $\frac{2}{3}$  and  $\frac{6}{9}$ . Or  $\frac{2}{3}$  and  $\frac{10}{15}$  are of the same Value in respect to the Whole or Unit.

From hence it will be easy to conceive how two, or more Fractions that are of different Denominations, may be alter'd or chang'd into others that shall have one common Denominator, and still retain the same Value.

*Example.* Let it be required to change  $\frac{2}{3}$  and  $\frac{3}{7}$  into two other Fractions that shall have one common Denominator, and yet retain the same Value.

According to the foregoing Proposition, if  $\frac{2}{3}$  be equally multiplied with 7, it will become  $\frac{14}{21}$  *viz.*  $\frac{2 \times 7}{3 \times 7} = \frac{14}{21}$ . Again if  $\frac{3}{7}$  be equally multiplied with 3, it will become  $\frac{9}{21}$  *viz.*  $\frac{3 \times 3}{7 \times 3} = \frac{9}{21}$ . And by this means I have obtained two new Fractions  $\frac{14}{21}$  and  $\frac{9}{21}$  that are of one Denomination, and the same Value with the two first proposed, *viz.*  $\frac{14}{21} = \frac{2}{3}$  and  $\frac{9}{21} = \frac{3}{7}$ .

H

And

And from hence doth arise the General Rule for bringing all Fractions into one Denomination.

### R U L E.

*Multiply all the Denominators into each other for a new (and common) Denominator. And each Numerator into all the Denominators but its own, for New Numerators.*

*Example.* Let the proposed Fractions be,  $\frac{1}{3}$ .  $\frac{2}{5}$ .  $\frac{3}{4}$ . and  $\frac{6}{7}$ .

Then by the Rule.

A new Denominator will be thus found.

And the new Numerators will be thus found.

3	1.	2.	3.	6.
5	5	3	3	3
—	—	—	—	—
15	5	6	9	18
4	4	4	5	5
—	—	—	—	—
60	20	24	45	90
7	7	7	7	4
—	—	—	—	—
420	140.	168.	315.	360.

Hence 420 is the common Denominator. And 140. 168. 315. 360 are the new Numerators, which being placed Fraction-wise are  $\frac{1}{4} \frac{40}{20}$ .  $\frac{1}{4} \frac{68}{28}$ .  $\frac{3}{4} \frac{15}{20}$ .  $\frac{3}{4} \frac{60}{28}$ . the new Fractions required.

That is,  $\frac{1}{4} \frac{40}{20} = \frac{1}{5}$ .  $\frac{1}{4} \frac{68}{28} = \frac{2}{5}$ .  $\frac{3}{4} \frac{15}{20} = \frac{3}{4}$ . and  $\frac{3}{4} \frac{60}{28} = \frac{6}{7}$ .

### Sect. 3. To bring mix'd Numbers into Fractions, and the contrary.

Mix'd Numbers are brought into improper Fractions by the following

### R U L E.

*Multiply the Integers or whole Numbers, with the Denominator of the given Fraction, and to their Product add the Numerator, the Sum will be the Numerator of the Fraction required.*

*Example.*  $9\frac{4}{5}$  by the Rule will become  $4\frac{2}{5}$ . For  $9 \times 5 = 45$ .

And,  $4\frac{2}{5} + \frac{4}{5} = 4\frac{6}{5}$  the improper Fraction required.

Again,  $12\frac{11}{15}$  will become  $2\frac{0}{15}$ . For  $12 \times 15 = 180$ .

And  $1\frac{0}{15} + \frac{11}{15} = 2\frac{0}{15}$ . And so for any other as Occasion requires.

To find the true Value of any improper Fraction given is only the Converse of this Rule. For if  $4\frac{2}{5} = 9\frac{4}{5}$  as before is evident :



evident: Then it follows that if 49 be divided by 5, the Quotient will give  $9\frac{4}{5}$ . And if 206 be divided by 15 it will give  $13\frac{11}{15}$ , &c. consequently it follows, That

If the Numerator of any improper Fraction be divided by its Denominator, the Quotient will discover the true Value of that Fraction.

*Examples.*

$\frac{35}{7} = 5$ . And  $\frac{50}{9} = 5\frac{5}{9}$ . And  $\frac{121}{20} = 6\frac{1}{20}$ . Or  $\frac{15}{4} = 3\frac{3}{4}$ , &c.

When whole Numbers are to be express'd Fraction-wise, it is but giving them an Unit for a Denominator. Thus 45 is  $4\frac{5}{1}$ . 9 is  $\frac{9}{1}$  and 25 is  $2\frac{5}{1}$ , &c.

#### Sect. 4. To Abbreviate or Reduce Fractions into their lowest or least Denomination.

This is done, not out of any Necessity, but for the more convenient Managing of such Fractions as are either proposed in large Terms; or swell into such, either by Addition or otherwise: Besides, 'tis most like an Artist to express or set down all Fractions in the lowest Terms possible; And to perform that, it will be necessary to consider of these following Propositions.

*Numbers are either Prime or Composed.*

1. A PRIME Number is that which can only be measured by an Unit. *Euclid 7. Defi. 11.*

That is, 3. 5. 7. 11. 13. 17. &c. are said to be Prime Numbers, because it is not possible to divide them into equal Parts by any other Number but Unity or 1.

2. Numbers Prime the one to the other, are such as only an Unit doth measure, being their common Measure. *Euclid 7. Defi. 12.*

For instance, 7 and 13 are Prime Numbers to each other, because they cannot be divided by any Number but an Unit. And 9 and 14 are also Prime Numbers to each other, for altho' 3 will measure or divide 9 without leaving a Remainder, yet 3 will not measure 14 without leaving a Remainder: Again, altho' 2 will measure 14 without any Remainder, yet 2 will not measure 9 without leaving a Remainder, &c.

3. A COMPOSED Number is that which some certain Number measureth. *Euclid 7. Defi. 13.*

For instance, 15 is a composed Number of 3 and 5, for  $3 \times 5 = 15$ , consequently 3 or 5 will justly measure 15. Also 20

is composed of 5 and 4, viz.  $5 \times 4 = 20$ , therefore 5 and 4 will each justly measure 20.

4. Numbers composed the one to the other, are they which some Number being a common Measure to them both doth measure. *Euclid 7. Defi. 14.*

That is, If two or more Numbers can be divided by one and the same Divisor; then are those Numbers said to be composed one to another.

For instance, 14 and 21 are Numbers composed the one to the other, because they can both be measured or divided by 7. For  $7 \times 2 = 14$ , and  $7 \times 3 = 21$ , therefore 7 is a common Measure to 14 and 21. So that if  $\frac{14}{21}$  were proposed to be abbreviated, it will become  $\frac{2}{3}$ .

$$\text{Thus } \left\{ \begin{array}{l} 7 \overline{) 14 = 2} \\ 7 \overline{) 21 = 3} \end{array} \right.$$

And how those greatest common Measures may be found, comes from *Euclid 7. pro. 1. 2. 3.* and is thus:

### R U L E.

Divide the greater Number by the lesser, and that Divisor by the Remainder (if there be any) and so on continually until there be no Remainder left: Then will this last Divisor be the greatest Common Measure (and if it happen to be 1, then are those Numbers Prime Numbers, and are already in their lowest Terms, but if otherwise) Divide the Numbers by that last Divisor, and their Quotient will be their least Terms required.

*Example.*

Let it be required to find the greatest common Measure of 72 and 108, viz. Of  $\frac{72}{108}$ .

$$\begin{array}{r} 72 \overline{) 108} (1 \\ \underline{72} \end{array}$$

$$\begin{array}{r} 36 \overline{) 72} (2 \\ \underline{72} \end{array} \quad \left\{ \begin{array}{l} \text{Here because there's no Remainder;} \\ 36 \text{ is the greatest common Measure.} \end{array} \right.$$

$$\text{Therefore, } \begin{array}{r} (0) \\ 36 \overline{) 72} \\ \underline{72} \end{array} \quad \left\{ \begin{array}{l} \frac{72}{108} = \frac{2}{3} \\ \text{Hence } \frac{72}{108} \text{ is abbreviated} \\ \text{to } \frac{2}{3} \text{ the lowest Terms.} \end{array} \right.$$

Again, to find the greatest common Measure of 744 and 899.

Thus,



Thus, 744) 899 (1  
 $\underline{744}$

155) 744 (4  
 $\underline{620}$

124) 155 (1  
 $\underline{124}$

31) 124 (4  
 $\underline{124}$

(0)

Here 31 is found to be the greatest common Measure by which 744 and 899 may be abbreviated to 24 and 29 their lowest Terms.

Thus,  $\frac{31}{31}) \frac{744}{899} (= \frac{24}{29}, \text{ \&c.}$

*Note,* If the proposed Numbers be even, they may be brought lower by a continued Halving of them, so long as they can be halved, viz. divided by 2.

*Example.*

'Tis required to reduce  $\frac{56}{84}$  to its least Terms.

First,  $\frac{2}{2}) \frac{56}{84} (= \frac{28}{42}$ . Again  $\frac{2}{2}) \frac{28}{42} (= \frac{14}{21}$ .

This done, you may easily perceive that 7 will be the common Measure to 14 and 21, viz.  $\frac{7}{7}) \frac{14}{21} (= \frac{2}{3}, \text{ \&c.}$

If the Numbers proposed to be reduced have each a Cypher or Cyphers annexed to them, they will be abbreviated by cutting off a like Number of Cyphers from both.

Thus,  $\frac{150}{300}$  will be  $\frac{15}{30}$ . And  $\frac{200}{300}$  will be  $\frac{2}{3}$ , &c.

That is,  $\frac{150}{300} = \frac{15}{30} = \frac{1}{2}$ . and  $\frac{200}{300} = \frac{2}{3}$ . also  $\frac{360}{400} = \frac{36}{40} = \frac{9}{10}$ .

## Sect. 5. Addition of Fractions.

What hath been done by the Rules in this Chapter; is chiefly to prepare and fit Fractions of different Denominations for Addition or Subtraction, as Occasion requires, viz. If they are compound Fractions, they must be reduced to simple or pure Fractions, per Rule, Sect. 1.

If they are of different Denominations, they must be altered or chang.d, per Rule, Sect. 2.

That is, all Fractions must be brought into one Denomination before they can either be added, or subtracted, and that being done, Addition is thus performed.

**R U L E.**

Add together all the Numerators, and their Sum will be a New Numerator, under which subscribe the common Denomination.

*Ex-*

*Examples in Simple Fractions.*

Let it be proposed to add  $\frac{1}{3}$ ,  $\frac{2}{5}$  and  $\frac{3}{4}$  together. First  $\frac{1}{3} = \frac{20}{60}$ ,  $\frac{2}{5} = \frac{24}{60}$ , and  $\frac{3}{4} = \frac{45}{60}$ . per Sect. 2.

Then  $\frac{20}{60} + \frac{24}{60} + \frac{45}{60} = \frac{89}{60}$ . the Sum required, which according to Section 3. is  $1\frac{29}{60}$ . viz.  $\frac{89}{60} = 1\frac{29}{60}$ .

*Examples in Compound Fractions.*

Let it be required to add  $\frac{3}{4}$  and  $\frac{2}{3}$  of  $\frac{3}{4}$  into one Sum. First  $\frac{2}{3}$  of  $\frac{3}{4}$  becomes  $\frac{6}{12}$  or  $\frac{1}{2}$  per Sect. 1. And (per Sect. 2.)  $\frac{3}{4}$  and  $\frac{1}{2}$  is  $\frac{6}{8}$  and  $\frac{1}{4}$  viz.  $\frac{3}{4} = \frac{6}{8}$  and  $\frac{1}{2} = \frac{2}{4}$  but  $\frac{6}{8} + \frac{2}{4} = \frac{10}{8}$  the Sum required, viz.  $\frac{3}{4} + \frac{2}{3}$  of  $\frac{3}{4} = \frac{10}{8}$ .

*Examples in Mix'd Numbers.*

'Tis required to add  $5\frac{2}{3}$  to  $7\frac{3}{4}$ . these per Sect. 3. will be  $1\frac{7}{3}$  and  $3\frac{1}{4}$ . But  $1\frac{7}{3}$  and  $3\frac{1}{4}$  will become  $\frac{6}{12}$  and  $\frac{9}{12}$ . per Sect. 2.

Then  $\frac{6}{12} + \frac{9}{12} = \frac{15}{12}$ , and  $1\frac{15}{12} = 13\frac{5}{2}$ . the Sum required.

Or you may bring only the Fractions to one Denomination, Thus,  $5\frac{2}{3}$  and  $7\frac{3}{4}$  will become  $5\frac{8}{12}$  and  $7\frac{9}{12}$ .

Then  $5\frac{8}{12} + 7\frac{9}{12} = 12\frac{17}{12}$ . That is,  $13\frac{5}{12}$ . As before.

## Sect. 6. Subtraction of Fractions.

Rule { Subtract one Numerator from the other (according as the Question requires) and their Difference will be a new Numerator, under which subscribe the common Denominator as in Addition.

*Example 1.*

Let it be required to take  $\frac{2}{5}$  out of  $\frac{3}{4}$ . First  $\frac{2}{5}$  and  $\frac{3}{4}$ . per Sect. 2. will become  $\frac{1}{2}$  and  $\frac{2}{3}$ . then  $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ , that is,  $\frac{3}{4} - \frac{2}{5} = \frac{1}{6}$ . As was required.

*Example 2.*

'Tis required to subtract  $\frac{2}{3}$  of  $\frac{8}{9}$  from  $\frac{1}{4}$ . First  $\frac{2}{3}$  of  $\frac{8}{9} = \frac{16}{27}$ . per Sect. 1. Again  $\frac{1}{4}$  and  $\frac{16}{27}$  will become  $\frac{2}{9}$  and  $\frac{3}{4}$  per Sect. 2. Then  $\frac{3}{4} - \frac{2}{9} = \frac{17}{36}$ .

*Example 3.*

From  $6\frac{1}{8}$  subtract  $3\frac{1}{4}$ . First,  $6\frac{1}{8} = 4\frac{9}{8}$ . and  $3\frac{1}{4} = 1\frac{2}{4}$ . per Rule Sect. 3. Again,  $4\frac{9}{8} = 2\frac{3}{2}$ , and  $1\frac{2}{4} = \frac{1}{2}$ . per Rule Sect. 2. Then  $2\frac{3}{2} - \frac{1}{2} = 1\frac{2}{2} = 1$ . Or otherwise thus: First,  $6\frac{1}{8} = 5\frac{7}{8}$ , then



then bring  $\frac{2}{8}$  and  $\frac{1}{4}\frac{2}{8}$  into one Denomination, viz.  $5\frac{2}{8}=5\frac{4}{8}\frac{2}{8}$  and  $3\frac{1}{4}\frac{2}{8}=3\frac{1}{2}\frac{2}{8}$ .

Then  $5\frac{4}{8}\frac{2}{8}-3\frac{1}{2}\frac{2}{8}=2\frac{2}{8}\frac{0}{8}=2\frac{3}{8}$ . As before.

## Example 4.

Let it be required to subtract  $\frac{2}{7}$  of  $\frac{5}{9}$  of  $\frac{2}{3}$  from 7.

First,  $\frac{2}{7}$  of  $\frac{5}{9}$  of  $\frac{2}{3}=\frac{2}{1}\frac{5}{8}\frac{2}{9}$ . And  $7=6\frac{1}{1}\frac{8}{8}\frac{2}{9}$ .

Then  $6\frac{1}{1}\frac{8}{8}\frac{2}{9}-\frac{2}{1}\frac{5}{8}\frac{2}{9}=6\frac{1}{1}\frac{5}{8}\frac{2}{9}=6\frac{5}{8}\frac{2}{9}=7-\frac{2}{7}$  of  $\frac{5}{9}$  of  $\frac{2}{3}$ . As was required.

If these few Examples be well understood, the whole Business of adding, and subtracting Vulgar Fractions will be easy; which is really much more difficult to perform than either Multiplication or Division, as will appear in the next Section.

## Sect. 7. Of Multiplication of Fractions.

In order to perform either Multiplication or Division, you must prepare the Terms to be multiplied ( or divided ) thus ; Reduce compound Fractions to simple ones, per Sect. 1. Bring Mixt Numbers into Improper Fractions, and express Whole Numbers Fractionwise, per Sect. 3. Also it will be convenient to Abbreviate them to their smallest Terms when it can be done. Then Multiplication may be thus performed.

Rule.  $\left\{ \begin{array}{l} \text{Multiply the Numerators one into another} \\ \text{for a New Numerator; and the Denominators} \\ \text{one into another, for a New Denominator. As} \\ \text{in these} \end{array} \right.$

## Examples.

1. The Product of  $\frac{2}{5}$  into  $\frac{3}{7}=\frac{6}{35}$ . That is,  $\frac{2 \times 3}{5 \times 7}=\frac{6}{35}$ .
2. And the Product of  $\frac{2}{1}\frac{0}{8}$  into  $\frac{2}{2}\frac{0}{7}=\frac{1}{4}\frac{8}{3}\frac{0}{2}$ . Or  $\frac{5}{1}\frac{2}{2}$ .
3. Again, the Product of  $\frac{7}{11}$  into  $\frac{2}{5}$  of  $\frac{5}{7}=\frac{7}{3}\frac{0}{8}\frac{5}{5}$ . Or  $\frac{2}{1}\frac{2}{1}$ . For  $\frac{2}{5}$  of  $\frac{5}{7}=\frac{1}{3}\frac{0}{5}$ . Then  $\frac{7}{11} \times \frac{1}{3}\frac{0}{5}=\frac{7}{3}\frac{0}{8}\frac{5}{5}=\frac{2}{1}\frac{2}{1}$ .
4. Let it be required to multiply 6 with  $3\frac{2}{5}$ . These prepared for the Work will stand thus.  $\frac{6}{1} \times \frac{1}{1}\frac{7}{5}$ . viz.  $6=\frac{6}{1}$  and  $3\frac{2}{5}=\frac{1}{1}\frac{7}{5}$ . Then  $\frac{6}{1} \times \frac{1}{1}\frac{7}{5}=\frac{1}{5}\frac{0}{2}$ . or  $20\frac{2}{5}$ . Or otherwise thus,  $6 \times 3=18$ . And  $\frac{2}{5} \times 6=\frac{1}{5}\frac{2}{5}=2\frac{2}{5}$ . Then  $18+2\frac{2}{5}=20\frac{2}{5}$ . As before.
5. Let it be required to multiply  $7\frac{4}{5}$  with  $5\frac{2}{7}$ . First  $7\frac{4}{5}=\frac{6}{1}\frac{7}{5}$ . and  $5\frac{2}{7}=\frac{3}{1}\frac{8}{7}$ . Then  $\frac{6}{1}\frac{7}{5} \times \frac{3}{1}\frac{8}{7}=\frac{2}{5}\frac{4}{3}\frac{6}{6}=40\frac{2}{3}$ .

Now the Reason of this Rule for multiplying of Fractions, and consequently of these Operations, and all other performed by it; will be evident from this following.

Viz.

*Viz.* If  $\frac{4}{2}$  be multiplied with  $1\frac{2}{3}$  according to the Rule, their Product will be  $4\frac{8}{3}$ . But  $4\frac{8}{3}=8$ .

Now  $\frac{4}{2}=2$ . and  $1\frac{2}{3}=4$  per Sect. 3. But  $4 \times 2=8$ . Ergo, &c.

### Sect. 8. Division of Fractions.

The FRACTIONS being first prepared as before directed, Division may be thus performed :

Rule.  $\left\{ \begin{array}{l} \text{Multiply the Numerator of the Dividend in-} \\ \text{to the Denominator of the Dividing Fraction} \\ \text{for a New Numerator: And Multiply the other} \\ \text{Numerator and Denominator together for a} \\ \text{New Denominator.} \end{array} \right.$

#### Examples.

1. Let  $\frac{6}{35}$  be divided by  $\frac{3}{7}$  viz.  $\frac{3}{7}$   $\frac{6}{35}$  ( $\frac{42}{105}=\frac{2}{5}$  the Quotient.

That is, according to the Rule  $6 \times 7=42$  the new Numerator, and  $35 \times 3=105$ , the new Denominator, &c. as above.

2. Let it be requir'd to divide  $\frac{20}{27}$  by  $\frac{5}{12}$  viz.  $\frac{5}{12}$   $\frac{20}{27}$  ( $\frac{240}{135}=1\frac{7}{9}$ .

For  $12 \times 20=240$  the new Numerator, and  $27 \times 5=135$  the new Denominator, &c. as before.

3. Suppose it were required to divide  $\frac{2}{11}$  by  $\frac{2}{5}$  of  $\frac{5}{7}$ .

First,  $\frac{2}{5}$  of  $\frac{5}{7}=\frac{10}{35}$ . Then  $\frac{10}{35}$   $\frac{2}{11}$  ( $\frac{70}{110}=\frac{7}{11}$ .

4. Let  $20\frac{2}{5}$  be divided by  $3\frac{2}{5}$  viz.  $1\frac{0}{5}$  by  $1\frac{7}{5}$ .

For  $20\frac{2}{5}=1\frac{0}{5}$ , and  $3\frac{2}{5}=1\frac{7}{5}$ . Then  $1\frac{0}{5}$   $1\frac{0}{5}$  ( $=6$  the Quotient.

5. Let it be required to divide  $40\frac{2}{3}$  by  $5\frac{3}{7}$ .

First,  $40\frac{2}{3}=2\frac{54}{3}$ , and  $5\frac{3}{7}=3\frac{8}{7}$ . Then  $3\frac{8}{7}$   $2\frac{54}{3}$  ( $1\frac{7822}{2394}$ .

But  $1\frac{7822}{2394}=7\frac{4}{9}$  the true Quotient required.

6. Suppose it were required to divide 13 by  $\frac{5}{7}$ .

First,  $13=1\frac{3}{1}$ . Then  $\frac{5}{7}$   $1\frac{3}{1}$  ( $\frac{91}{5}=18\frac{1}{5}$ , the Quotient.

7. Again, let it be required to divide  $\frac{5}{7}$  by 6.

*Viz.*  $\frac{5}{7}$   $\frac{5}{7}$  ( $\frac{5}{42}$  for the Quotient required.

N.B. From hence you may observe, that when any Whole Number is divided by a Fraction less than Unity or 1, the Quotient will be greater than the Number propos'd to be divided: But if any Fraction be divided by a Whole Number, greater than 1. Then the Quotient will be less than the Dividend: As in the two last Examples.

As



As to the Reason (or Proof) of this Rule for dividing Fractions: 'Tis only the Converse to that of Multiplication, and will be very evident from this following.

Let  $4\frac{8}{8}$  be divided by  $\frac{4}{2}$ . Which according to the Rule is thus,  $\frac{4}{2}) 4\frac{8}{8} (\frac{2}{2} = 4$ . The true Quotient. Now  $4\frac{8}{8} = 8$ . And  $\frac{4}{2} = 2$ . per Sect. 3. Consequently  $4\frac{8}{8}$  divided by  $\frac{4}{2}$  is but the same with 8 divided by 2 viz. 2) 8 (4. The Quotient as before.

I could have inserted Geometrical Demonstrations, for both the Rules of Multiplication and Division of Fractions; but supposing the Learner purely unacquainted with those kind of Demonstrations, I thought these might be more intelligible to him, especially in this Place.

## C H A P. V.

### Of Decimal Fractions.

**W**HEN, or by whom this excellent Invention of Decimal Arithmetick, was first introduced is uncertain, but doubtless its Improvements, and the Perfection 'tis now in, is owing to later Years.

#### Sect. I. Of Notation.

In Decimal Fractions, the Integer or Whole Thing (whether it be Coin, Weight, Measure, or Time, &c.) is suppos'd to be divided into ten equal Parts; and every one of those ten Parts are supposed to be subdivided into other ten equal Parts, &c. *ad infinitum*.

The Integer being thus divided (by Imagination) into 10, 100, 1000, 10000, &c. equal Parts, becomes the Denominator to the Decimal Fraction.

Thus,  $\frac{2}{10}$ .  $\frac{3}{100}$ .  $\frac{7}{1000}$ .  $\frac{53}{10000}$ .  $\frac{745}{100000}$ . &c.

Now these Denominators are seldom or never set down but only the Numerators; and those are either distinguished, or separated from Whole Numbers by a Point or a Comma.

Thus, 5.4 is  $5\frac{4}{10}$ . and 0.7 is  $\frac{7}{10}$ . 35.05 is  $35\frac{5}{100}$ , &c.

But before we proceed further in Notation, it will be convenient for the Learner to consider of the following Table (taken out of the learned Mr. Oughtred's *Clavis Mathematica*) which shews the very Foundation of Decimal Fractions.

I

Whole





Cyphers annex'd to Decimal Parts alter not their Value. As, ,50, and ,500, or ,5000, &c. are each but 5 Tenths of an Unit. For  $\frac{50}{100} = \frac{5}{10}$ . And  $\frac{500}{1000} = \frac{5}{10}$ . Or  $\frac{5000}{10000} = \frac{5}{10}$ . Per Sect. 4. of the last Chapter.

But Cyphers prefix'd to Decimal Parts decrease their Value, by removing them further from the Comma.

Thus,  $\left\{ \begin{array}{l} ,5 = 5 \text{ Tenth Parts.} \\ ,05 = 5 \text{ Parts of a Hundred.} \\ ,005 = 5 \text{ Parts of a Thousand.} \\ ,0005 = 5 \text{ Parts of Ten Thousands, \&c.} \end{array} \right.$

Consequently the true Value of all Decimal Parts are known by their Distance from the Units Place; the which being once rightly understood, the rest will be easy.

## Sect. 2. Addition and Subtraction of Decimals.

In setting down the proposed Numbers to be added, or subtracted, great Care must be taken in placing every Figure directly underneath those of the same Value, whether they be mix'd Numbers, or pure Decimal Parts, and to perform that, you must have a due Regard to the Comma's, or separating Points, which ought always to stand in a direct Line one under another; And to the Right-hand of them carefully place the Decimal Parts, according to their respective Values or Distances from Unit. Then

Rule.  $\left\{ \begin{array}{l} \text{Add, or subtract them as if they were all} \\ \text{Whole Numbers; And from their Sum or Dif-} \\ \text{ference, cut off so many Decimal Parts as are the} \\ \text{most in any of the given Numbers.} \end{array} \right.$

### Examples in Addition.

Let it be required to find the Sum of these following Numbers, viz. 34,5 + 65,3 + 128,7 + 25 + 87,8 + 7,9, which being truly placed, will stand

Thus,  $\left\{ \begin{array}{r} 34,5 \\ 65,3 \\ 128,7 \\ 95,0 \\ 87,8 \\ 7,9 \\ \hline \end{array} \right.$

Their Sum required, 419,2

Example 2.

Let it be required to find the Sum of 25,854 + 34,578 + 9,076 + 13,907.

$$\begin{array}{r} 25,854 \\ 34,578 \\ 9,076 \\ 13,907 \\ \hline \end{array}$$

83,415 The Sum required.

When the Decimal Parts propos'd to be added (or subtracted) have not the same Number of Places, you may for Convenience of Operation supply or fill up the void Places, by annexing Cyphers. As in these Examples.

Example 3.

$$\begin{array}{r} 45,0700 \\ 50,7580 \\ 123,0057 \\ 74,7020 \\ 24,8000 \\ \hline \end{array}$$

318,3357

Example 4.

$$\begin{array}{r} 574,678953 \\ 95,796430 \\ 78,054600 \\ 54,789000 \\ 8,900000 \\ \hline \end{array}$$

812,218983

Example 5.

$$\begin{array}{r} 0,975642 \\ 5745257 \\ 5000598 \\ 800700 \\ 640530 \\ \hline \end{array}$$

3,162727

### Examples in Subtraction.

Let it be required to find the Difference between 45,375 and 74,284.

Example 1.

That is, From 74,284  
Take 45,375

Remains 28,909

Example 2.

From 437,5  
Take 89,657

Remains 347,843

Example 3.

From 75,0034  
Take 57,875

Remains 17,1284

Example 4.

Let it be required to find the Difference between 562 and 93,5784.

Example 4.

That is, From 562,  
Take 93,5784

the Difference. 468,4216

Example 5.

From 345,7578  
Take 157,

the Difference 188,7578

Note, The two last Examples are supposed to be supply'd with Cyphers, which if actually done would stand thus,

$$\begin{array}{r} 562,0000 \\ 93,5784 \\ \hline \end{array}$$

Remains

468,4216

As before, 188,7578

Example



*Example 6.*  
From 0,547893  
Take 0,439758

0,108135

*Example 7.*  
From 1,000000  
Take 0,997543

0,002457

The Proof of Addition, and Subtraction in Decimals is the same with that of Whole Numbers, *Page 13, &c.*

## Sect. 3. Multiplication of Decimals.

Whether the Factors or Numbers to be multiplied are pure Decimals, or mix'd. Multiply them as if they were all Whole Numbers, and for the true Value of their Product observe this Rule.

*Rule.* { *Cut off, (viz. separate with a Comma) so many Places of Decimal Parts in the Product, as there are in both the Factors accounted together. As in these.*

*Example 1.*

3,024  
2,23

9072  
6048  
6048

6,74352

*Example 2.*

32,12  
24,3

9636  
12848  
6424

780,516

The Reason why such a Number of Decimal Parts must be cut off in the Product, may be easily deduced from these Examples. Thus,

In *Example 1.* 'Tis evident, that 3 the whole Number in the Multiplicand, being multiplied with 2, the whole Number in the Multiplier; can produce but 6 (*viz.*  $3 \times 2 = 6$ ) So that of Necessity all the other Figures in the Product must be Decimal Parts; according as the *Rule* directs.

Or, the *Rule* is evident from the Multiplication of Whole Numbers only: Thus, suppose 3000 were to be multiplied with 200, their Product will be 600000; That is, there will be so many Cyphers in the Product, as are in both the Factors. (*Vide Page 18.*) Now if instead of those Cyphers in the Factors, we suppose the like Number of Decimal Parts; then it follows, that there ought to be the same Number of Decimal Parts in the Product, as there were Cyphers in the Factors.

Again, the *Rule* may be otherwise made evident from Vulgar Fractions, thus: Let 32,12 be multiplied with 24,3, and their Product

duct will be 780,516 as in *Example 2.* above. Now  $32,12 = 32\frac{12}{100}$ . and  $24,3 = 24\frac{3}{10}$ . which being brought into Improper Fractions (per *Sect. 3. Page 50.*) will become  $32\frac{12}{100} = 3\frac{212}{100}$ . and  $24\frac{3}{10} = 2\frac{43}{10}$ .

Then  $3\frac{212}{100} \times 2\frac{43}{10} = 7\frac{80516}{1000}$ . per *Sect. 7. Page 55.*

But  $7\frac{80516}{1000} = 780\frac{516}{1000}$ . viz. 780,516 as before.

Any of these three Ways do, I presume, sufficiently prove the Truth of the abovesaid Rule, &c.

*Example 3.*

$$\begin{array}{r} 78,546 \\ 436 \\ \hline 471276 \\ 235638 \\ 314184 \\ \hline 34246,056 \end{array}$$

*Example 4.*

$$\begin{array}{r} 5745 \\ ,0675 \\ \hline 28725 \\ 40215 \\ 34470 \\ \hline 387,7875 \end{array}$$

N. B. It sometimes falls out in multiplying Parts with Parts, that there will not be so many Figures in the Product, as there ought to be Places of Decimal Parts by the Rule: In that Case you must supply their Defect by prefixing Cyphers to the Product; as in these Examples.

*Example 5.*

$$\begin{array}{r} ,2365 \\ ,2435 \\ \hline 11825 \\ 7095 \\ 9460 \\ 4730 \\ \hline ,05758775 \end{array}$$

*Example 6.*

$$\begin{array}{r} ,0347 \\ ,0236 \\ \hline 2082 \\ 1041 \\ 694 \\ \hline ,00081892 \end{array}$$

When any proposed Number of Decimals is to be multiplied with 10. 100. 1000. 10000, &c. 'Tis only removing the separating Point in the Multiplicand, so many Places towards the Right-hand, as there are Cyphers in the Multiplier.

Thus,  $,578 \times 10 = 5,78$ . And,  $,578 \times 100 = 57,8$   
Again,  $,578 \times 1000 = 578$ . Or,  $,578 \times 10000 = 5780$

These



These Things being consider'd, it will be easy to multiply Decimals, and determine their true Products. As in these following Examples.

57,056 Multiplied into 0,578 will produce 32,978368

7,6543 into 5,4246 will produce 41,52151578

$$0,56879 \times 0,05674 = 0,0322731446$$

$$0,03246 \times 0,02364 = 0,0007673544$$

$$87649 \times 0,03687 = 3231,61863$$

$$94,35786 \times 6,57869 = 620,7511100034$$

$$3,141592 \times 52,7438 = 165,6995001296$$

Now it oftentimes happens, that it will be needless to express all the Figures of the Product at large (especially when the Factors have each of them many Places of Decimal Parts, as in the two last Examples) only so many of them as may suffice for the intended Design; and yet the Product may be as true to so many Figures as are retained, as if the Factors had been multiplied at large. And such compendious Contractions are not only of Curiosity, but may also be found of great Ease and Use to the ingenious Practitioner; especially in Resolving Adfected *Æquations*, or in calculating of *Trigonometrical Problems* by the *Natural Sines* and *Tangents*, &c. All which may be thus perform'd.

*Viz.* Set the Units Place of the Multiplier directly underneath that Figure of the Multiplicand, whose Place you intend to keep in the Product; And place all the other Figures of the Multiplier in a quite contrary Order to the usual Way. Then in Multiplying always begin at that Figure of the Multiplicand which stands over the Figure wherewith you are then a Multiplying, setting down the first Figure of each particular Product, directly underneath one another; yet herein you must have a due Regard to the Increase which would arise out of the two next Figures to the Right-hand of that Figure in the Multiplicand which you then begin with.

## Example.

Let it be required to multiply 3,141592 with 52,7438 and let there be only four Places of Decimal Parts retained in the Product.

If the proposed Numbers were to be multiplied at large, they must stand in a direct Order as usual.

Thus

Thus,  $\left\{ \begin{array}{l} 3,141592 \\ 52,7438 \end{array} \right\}$  And would produce ten Places of Parts, as in the last Example.

But being 'tis required to have only four Places of those Parts in the Product; set them down as above directed and they will stand

Thus  $\begin{array}{r} 3,141592 \\ 8347.25 \end{array}$

$\begin{array}{r} 1570796 \\ 62832 \end{array}$

$\begin{array}{r} 21991 \\ 1257 \end{array}$

$\begin{array}{r} 94 \\ 25 \end{array}$

$\begin{array}{r} 165,6995 \end{array}$

The Multiplicand placed as before.

The Multiplier in a reverse Order.

The Product with 5, Regard had to 5 times 2.

The Product with 2, increased with  $9 \times 2$ .

Product with 7, increased with  $5 \times 7 + 9 \times 7$ .

Product with 4, increased with  $1 \times 4 + 5 \times 4$ .

Product with 3, increased with  $4 \times 3$ .

Product with 8, increased with  $4 \times 8 + 1 \times 8$ .

The true Product as was required.

The Reason of this Contraction is very obvious from the whole Operation wrought at large.

Thus  $\begin{array}{r} 3,141592 \\ 52,7438 \end{array}$

$\begin{array}{r} 25 \mid 132736 \\ 94 \mid 24776 \end{array}$

$\begin{array}{r} 1256 \mid 6368 \\ 21991 \mid 144 \end{array}$

$\begin{array}{r} 62831 \mid 84 \\ 1570796 \mid 0 \end{array}$

$\begin{array}{r} 165,6995 \mid 001296 \end{array}$

From hence 'tis evident that all the Figures in the Square to the Right-hand, are wholly omitted in the former Contraction; And that the last single Product here, is the first there; consequently the Reason of placing the Multiplier in a reverse Order, must needs appear very plain.

#### Example 2.

Suppose it were required to multiply 257,356 with 76,48 and to have only the entire Product of Integers.

257,356

84,67

$\begin{array}{r} 18015 \\ 1544 \end{array}$

$\begin{array}{r} 103 \\ 20 \end{array}$

$\begin{array}{r} 19682 \end{array}$

The same at large.  $\left\{ \begin{array}{l} 257,356 \\ 76,48 \end{array} \right\}$

$\begin{array}{r} 20 \mid 58848 \\ 102 \mid 9424 \end{array}$

$\begin{array}{r} 1544 \mid 136 \\ 18014 \mid 92 \end{array}$

$\begin{array}{r} 19682,58688 \end{array}$

The chiefest Care and Difficulty that attends these Contractions, is the true setting down of the Units Place in the Multiplier underneath the proper Figure of the Multiplicand, according to the design'd Product.



# Chap. 5. Of Decimal Fractions. 65

*Viz.* In *Example 1.* It was required to have four Places of Decimal Parts in the Product; therefore the Units Place of the Multiplier was set under the fourth Place of Decimals in the Multiplicand: And in *Example 2.*, because it was requir'd to have an entire Product of Integers only; therefore the Units Place of the Multiplier was set under the Units Place of the Multiplicand. This, I say, being once rightly understood, will render the Method easy in Practice.

## *Thom's* Sect. 4. Division of Decimals.

*Division* is accounted the most difficult Part of Decimal Arithmetick; In order therefore to make it plain and easy, it will be convenient to resume what has been said in *Page 25.*

*Viz.*  $\left\{ \begin{array}{l} \text{The Quotient Figure is always of the same Value} \\ \text{or Degree with that Figure of the Dividend, under} \\ \text{which the Units Place of its Product stands.} \end{array} \right.$

As for Instance, Let 294 be divided by 4.

$$\begin{array}{r} 4 \overline{) 294} \quad (7 \\ \underline{28} \phantom{0} \end{array}$$
 $\left\{ \begin{array}{l} \text{This is not 7 but 70, because the Units} \\ \text{Place of } 4 \times 7 \text{ stands under the Tens Place} \\ \text{of the Dividend.} \end{array} \right.$

$$\begin{array}{r} 14 \quad (3 \text{ But this is only 3.} \\ \underline{12} \phantom{0} \end{array}$$

Remains (2) Hence  $73\frac{2}{4}$  is the Quotient.

Now if to the Remainder 2 there be annexed a Cypher, thus, 2,0 and then divided on, it must needs follow that the Units Place of the Product arising from the Divisor into the Quotient, will stand under the annexed Cypher; Consequently the Quotient Figure will be of the same Value or Degree with the Place of that Cypher: But that's the next below the Units Place, therefore the Quotient Figure is of the next Degree or Place below Unity; that is, in the first Place of Decimal Parts.

Thus  $4 \overline{) 294,0} \quad (,5$

So that  $4 \overline{) 294,0} \quad (73,5$  is the true Quotient required.

This being well understood; Division of Decimals may (in all the various Cases) be easily perform'd. However, that it may be render'd plain and easy, even to the meanest Capacity, if possible; let Division be again defin'd, as in *Page 21.*

K

*Viz.*

Viz. If that Number which divides another, be Multiplied with the Number which is produced, their Product will be the Number divided.

This Definition alone (if compar'd with the Rule, Page 61) will afford a general Rule for discovering the true Value of the Quotient Figure in Division of Decimals.

**Rule** *The Places of Decimal Parts in the Divisor and Quotient, being counted together, must always be equal in Number with those in the Dividend. And from this general Rule arise four Cases.*

**Case 1.** When the Places of Parts in the Divisor and Dividend are equal, the Quotient will be whole Numbers.

As in these Examples.

$$\begin{array}{r} 8,45 \overline{) 295,75} \quad (35. \\ \underline{253 \phantom{5}} \\ \phantom{2}42 \phantom{5} \end{array}$$

$$\begin{array}{r} \phantom{2}42 \phantom{5} \\ \underline{\phantom{2}42 \phantom{5}} \\ \phantom{00} \end{array}$$

(0)

$$\begin{array}{r} 0,0078 \overline{) .4368} \quad (56. \\ \underline{\phantom{0}390} \\ \phantom{0}468 \end{array}$$

$$\begin{array}{r} \phantom{0}468 \\ \underline{\phantom{0}468} \\ \phantom{00} \end{array}$$

(0)

**Case 2.** When the Places of Parts in the Dividend, exceed those in the Divisor; cut off the Excess for Decimal Parts in the Quotient. As in these Examples.

$$\begin{array}{r} 24,3 \overline{) 780,516} \quad (32,12 \\ \underline{729} \\ \phantom{0}515 \end{array}$$

$$\begin{array}{r} \phantom{0}515 \\ \underline{\phantom{0}486} \\ \phantom{00}291 \end{array}$$

$$\begin{array}{r} \phantom{00}291 \\ \underline{\phantom{00}243} \\ \phantom{000}486 \end{array}$$

$$\begin{array}{r} \phantom{000}486 \\ \underline{\phantom{000}486} \\ \phantom{0000} \end{array}$$

(0)

$$\begin{array}{r} 534 \overline{) 30438} \quad (57 \\ \underline{2670} \\ \phantom{0}3738 \end{array}$$

$$\begin{array}{r} \phantom{0}3738 \\ \underline{\phantom{0}3738} \\ \phantom{00} \end{array}$$

(0)

$$\begin{array}{r} 436 \overline{) 34246,056} \quad (78,546 \\ \underline{3052} \\ \phantom{0}3726 \end{array}$$

$$\begin{array}{r} \phantom{0}3726 \\ \underline{\phantom{0}3488} \\ \phantom{00}2380 \end{array}$$

$$\begin{array}{r} \phantom{00}2380 \\ \underline{\phantom{00}2180} \\ \phantom{000}2005 \end{array}$$

$$\begin{array}{r} \phantom{000}2005 \\ \underline{\phantom{000}1744} \\ \phantom{0000}2616 \end{array}$$

$$\begin{array}{r} \phantom{0000}2616 \\ \underline{\phantom{0000}2616} \\ \phantom{00000} \end{array}$$

(0)

**Case 3.** When there are not so many Places of Parts in the Dividend, as are in the Divisor; annex Cyphers to the Dividend to make them equal. Then will the Quotient be whole Numbers, as in Case 1.

Examples.



## Examples.

Let it be required to divide 192,1 by 7,684, And 441, by ,7875

7,684) 192,100 (25.

,7875) 441,0000 (560.

15368

39375

38420

47250

38420

47250

(0)

(00)

*Case 4.* If after Division is finished, there are not so many Figures in the Quotient, as there ought to be Places of Parts by the general Rule; supply their Defect by prefixing Cyphers to it.

## Examples.

Let it be required to divide 7,25406 by 957.

957) 7,25406 (,00758 the true Quotient required.

6699

5550

Again ,575) ,0007475 (,0013

4785

575

7656

1725

7656

1725

(0)

(0)

*Note,* When Decimal Numbers are to be Divided by 10. 100. 1000. 10000. &c. That is, when the Divisor is an Unit with Cyphers; Division is perform'd by removing or placing the separating Point in the Dividend, so many Places towards the Left-hand, as there are Cyphers in the Divisor.

## Examples.

10) 5784 (578,4

100) 5784 (57,84

1000) 5784 (5,784

10000) 5784 (,5784

*Note,* These Operations are the direct Converse to those in Pa. 62.

I presume it's needless to give more Examples at large, only insert a few Dividends, and Divisors, with their Quotients; wherein are contained all the Varieties that can happen in Division of Decimals.

574) 493066 (859.

5,74) 49,3066 (8,59

574) 493,066 (,859

5,74) 493066,00 (85900.

574) 49,3066 (,0859

,0574) 493,0660 (8590.

5,74) 4930,66 (859.

,0574) ,463066 (8,59

K 2

There

There is also a compendious Way of contracting Division; like unto that of Multiplication, *Page 64.* by which much Labour may be saved; especially when the Divisor hath many Places of Decimal Parts in it: And it's thus performed.

Having determined how many Places of whole Numbers there will be in the Quotient, if any at all; or if none, of what Value or Place the first Figure in the Quotient will be: Then omit. or prick off one Figure of the Divisor at each Operation; *viz.* for every Figure you place in the Quotient, prick off one in the Divisor; having a due Regard to the Increase which wou'd arise from the Figure so omitted.

*Example.*

Let it be required to divide 70,23 by 7,9863.

The Work contracted.

$$\begin{array}{r}
 7,9863 \overline{) 70,2300} \quad (8,7938 \\
 \underline{63 \ 8901} \\
 6 \ 3396 \\
 \underline{5 \ 5904} \\
 7492 \\
 \underline{7187} \\
 305 \\
 \underline{239} \\
 66 \\
 \underline{64} \\
 (2)
 \end{array}$$

The same at Large.

$$\begin{array}{r}
 7,9863 \overline{) 70,2300} \quad (8,7938 \\
 \underline{63 \ 8904} \\
 6 \ 33960 \\
 \underline{5 \ 59041} \\
 749190 \\
 \underline{718767} \\
 304230 \\
 \underline{239589} \\
 646410 \\
 \underline{638904} \\
 07506
 \end{array}$$

The Work contracted I presume is so obvious (if compared with the same at large) that it's needless to give any farther Explanation of it.

**Sect. 5. To Reduce Vulgar Fractions into Decimals, and the contrary.**

Any Vulgar Fraction being given it may be reduced, or rather changed into Decimal Parts equivalent to it. Thus,

**Rule.** { *Annex Cyphers to the Numerator, and then divide it by the Denominator, the Quotient will be the Decimal Parts equivalent to the given Fraction; or at least so near it as may be thought necessary to approach.*

*Example.*



## Example.

'Tis required to change or reduce  $\frac{3}{4}$  into Decimals.

4) 3,00 (,75 The Decimal Parts required.

That is,  $\frac{3}{4} = \frac{75}{100} = .75$ .

Again  $\frac{1}{2} = .5$  Thus 2) 1,0 (,5 And  $\frac{1}{4} = .25$  4) 1,00 (,25

Suppose it were required to change  $\frac{4}{7}$  into Decimals.

7) 4,0000000000 (,5714285714 &c. =  $\frac{4}{7}$

Note, When the last Figure of the Divisor (That is, the Denominator of the proposed Fraction) happens to be one of these Figures; viz. 1. 3. 7. or 9. (as in the last Example) then the Decimal Parts can never be precisely equal to the given Fraction; yet by continuing the Division on. you may bring them to be very near the Truth. As in this Example; Suppose it required to change  $\frac{1}{3}$  into Decimal Parts.

13) 1,0000 (,07692307692307 &c. *ad infinitum*.

91..

90

78

120

117

30

26

40

39

10

&c.

That is,  $0,07692307692307 = \frac{1}{13}$  fere.

And from hence it may be further observed; That in these imperfect Quotients, the Figures do return again and circulate in the same order as before: As you may easily perceive they begin to do in the seventh Place of both these last Examples.

As at first.

These being understood, it will be easy to find the Decimal Parts equivalent to any known Part, or Parts of Coin, Weights, Measures, or Time, &c. If you first reduce the given Parts of Coin, &c. into a Vulgar Fraction, whose Denominator is the Number of those known Parts contained in the Integer, and the given Parts its Numerator.

## Examples in Coin, &c.

1. Let it be required to find the Decimals of 16 s. 6 d. First  $16 \text{ s.} = \frac{16}{20}$  of one Pound, and  $6 \text{ d.} = \frac{6}{40}$  of 1 l.

But  $\frac{16}{20} + \frac{6}{40} = \frac{38}{40}$ . Then 40) 38,000 (,95 the Decimal Parts required: That is,  $.95 = 16 \text{ s. } 6 \text{ d.}$

Again, Suppose it were required to find the Decimals equal to 3 l. 13 s. 4 d.

Here

Here 3*l.* is 3 *Integers*, and 13*s.* =  $\frac{1}{20}$  of 1*l.* and 4*d.* =  $\frac{1}{240}$ .  
 But  $\frac{1}{20} + \frac{1}{240} = \frac{1}{24}$ . Then 240) 160,000 (0,666666, &c.  
 Hence 3*l.* 13*s.* 4*d.* = 3,666666, &c. As was required.

2. What are the Decimals equal to  $7\frac{3}{4}$  Inches, one Foot being made the Integer.

First, 7 Inches are  $\frac{7}{12}$  of 1 Foot, and  $\frac{3}{4}$  of 1 Inch are  $\frac{3}{48}$ .  
 But  $\frac{7}{12} + \frac{3}{48} = \frac{31}{48}$ . Then 48) 31,000 (,64583, &c. =  $7\frac{3}{4}$  Inches.

3. Let it be required to change 8  $\frac{3}{4}$  19 *Pwt.* 8 *Grains* into Decimals; one *lb Troy* being the Integer.

These being reduced into their least Terms, and added together will become  $\frac{43}{5760}$  of 1 *lb.*

Then 5760) 4304,000 (,74722, &c. The Decimals required:

And thus may any proposed Parts of Coin, Weights, Measures, &c. be reduced or changed into Decimal Parts; which perhaps may at first seem somewhat tedious in Practice, but being a little acquainted with them it will be found very easy; and the ingenious Practitioner will (with a little Consideration) soon find how to reduce them almost mentally; or with the help of a very few Figures; without the Use of such large Tables as are usually inserted in Books of Decimal Arithmetick, or at most they may be contracted into such as these following; which if duly applied to those Tables in Chap. 3. will be found very useful.

### Decimal Tables.

<i>In English Coin.</i>	<i>Averdupois Weight.</i>
0,05 ..... = 1 <i>s.</i>	0,0625 ..... = 1 <i>Ounce.</i>
0,00416667 = 1 <i>d.</i>	0,00390625 = 1 <i>Drachm.</i>
0,00104167 = 1 <i>Farthing.</i>	1, <i>lb</i> being the Integer.
<i>Troy Weight.</i>	<i>Averdupois Great Weight.</i>
0,05 ..... = <i>Pwt.</i>	0 25 ..... = $\frac{1}{4}$ <i>C.</i>
0,00208333 = 1 <i>Grain.</i>	0,00892857 = 1 <i>lb.</i>
1, $\frac{3}{4}$ being the Integer.	0,00055803 = 1 <i>Ounce.</i>
<i>Apothecaries Weights.</i>	1, <i>C.</i> being the Integer.
0,125 ..... = 1 <i>Drachm.</i>	<i>Time.</i>
0,04166667 = 1 <i>℥</i>	0,04166667 = 1 <i>Hour.</i>
0,00208333 = 1 <i>Grain.</i>	0,00069444 = 1 <i>Minute.</i>
1, $\frac{3}{4}$ being the Integer.	0,00001159 = 1 <i>Second.</i>
	1, Day or 24 Hours being made the Integer.

The Use of these Tables will be evident by the following.

*Example.*



## Example.

Let it be required to find the Decimal Parts equivalent to 17 s. 9 d. 2 Farthings.

First,  $0,05 = 1$  s. Therefore  $17 \times 0,05 = 85 \dots = 17$  s.

And  $,004166 = 1$  d. Therefore  $,004166 \times 9 = ,037494 = 9$  d.

And  $,00104167 = 1$  Farth. Also  $2 \times ,00104167 (= ,002083 = \frac{1}{2})$  d.

Consequently their Sum, viz.  $0,889577 = 17$  s.  $9\frac{1}{2}$  d.

Now to find the Value of Decimals, in known Parts of Coin, or Weights, &c. is only the Converse of the former Work. And is thus performed.

Multiply the given Decimals with the Denominator of the Vulgar Fraction required: That is, Multiply the Decimals with such a Number of Units as are contained in the next lower Denomination of that Kind or Species which your Decimal is of: And the Product will be the Number required.

## Example.

I. What is the Value of  $0,825$  Decimals of 1 Pound Sterling. That is, how many Shillings, Pence, &c.  $= 825$ . First, the next lower Denomination is 20, because 20 s. make one Pound.

Therefore  $0,825$

	20	
Shillings	16,500	And Parts of 1 Shilling.
	12	

Pence 6,000      Answer  $0,825 = 16$  s. 6 d.

Again, What are the known Parts of English Coin equal to  $3,666666$  Decimals.

Here the 3 Integers are 3 Pounds. Then  $,666666$

	20
Shillings	13,333320
	12

Answer  $3,666666 = 3$  l. 13 s. 4 d. because the Fraction is so large one Penny more is added.

666640
333320

Pence 3,999840

What is the Value of  $0,74722$  Parts of 1 lb. Troy.

First,  $,74722$

Then,  $,96664$

Again,  $,33280$

12
1 49444
7 4722

	20
Pwts.	19,33280

24
1 3312
6 656

3, 8,96664

3. Pwt. Gr.

Grains 7,98720

These collected are 8. 19. 8. very near.

And

And thus any proposed Number of Decimals may be turn'd or chang'd into the known Parts of what they represent. *viz.* Whether they be Parts of Coin, Weights, Measures, or Time, &c.

I have omitted inserting more Examples of this kind, because I take the Excellency, and indeed the chief Use of Decimal Fractions to consist more in Geometrical Computations, than in the Common or Practical Parts of Arithmetick (as will appear further on) although even in those they are very useful upon several Accounts; especially in the Computations of Interest and Annuities, &c. (But of that more in its proper Place.) I shall therefore conclude this Chapter with a Remark or two upon the Nature and Properties of Fractions in general.

If any given Number (whether it be Whole or Mix'd) be multiplied with a Fraction either Vulgar or Decimal, the Product will be less than the Multiplicand, in such a Proportion, as the Multiplying Fraction is less than an Unit or 1.

That is, *As the Denominator of the Fraction is to its Numerator, so will the given Number be to the Product.*

Therefore, whenever any Number is to be multiplied with a Fraction, whose Numerator is an Unit. Divide that Number by the Denominator of the Fraction, and the Quotient will be the Product required. Thus  $12 \times \frac{1}{4} = 3$ . And  $12 \div 4 = 3$ . Again,  $12 \times \frac{1}{2} = 6$ . And  $12 \div 2 = 6$ , &c.

From hence it follows, that if any Number be divided by a Fraction, the Quotient will be greater than the Dividend; by such a Proportion as Unity is greater than the Dividing Fraction.

Thus  $12 \div \frac{1}{4} = 48$ , *viz.*  $\frac{1}{4} : 1 :: 12 : 48$ , &c. But the Truth of these will be best understood after the next Chapter.

## C H A P. VI.

*Of Continued Proportions, and how to Change or Vary the Order of Things.*

*Sect. I. Concerning Arithmetical Progression, usually called Arithmetical Proportion Continued.*

**W**HEN any Rank or Series of Numbers do either Increase or Decrease by an Equal Interval or Common Difference; those Numbers are said to be in *Arithmetical Progression*.

As



As,  $\{ 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \&c. \}$   $\{ \}$  Here the Interval or  
 $\{ 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 : \}$   $\{ \}$  Common Difference is 1.

Or,  $\{ 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot \&c. \}$   $\{ \}$  Here the Common  
 $\{ 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot \&c. \}$   $\{ \}$  Difference is 2.

And so of any other Series, whose Common Difference is  
 $3 \cdot 4 \cdot 5 \cdot \&c.$

*Lemma 1.*

“ If any three Numbers be in Arithmetical Progression; the  
 “ Sum of the two Extremes (*viz.* the first and last) will be equal  
 “ to the Double of the Mean or middle Number.

As in these  $2 \cdot 4 \cdot 6$ . Or  $3 \cdot 6 \cdot 9$ . Or  $3 \cdot 7 \cdot 11$ .

*Viz.*  $2+6=4+4$ . Or  $3+9=6+6$ . And  $3+11=7+7$ . &c.

*Lemma 2.*

“ If any four Numbers are in Arithmetical Progression, the Sum  
 “ of the two Extremes will be equal to the Sum of the two Means.

As in these.  $2 \cdot 4 \cdot 6 \cdot 8$ . Or  $3 \cdot 6 \cdot 9 \cdot 12$ .

*Viz.*  $2+8=4+6$ . And  $3+12=6+9$ . &c.

*Corollary 1.*

“ From these two *Lemma's* 'tis easy to conceive; that if never  
 “ so many Numbers be in Arithmetical Progression, the Sum of the  
 “ two Extremes will be equal to the Sum of any two Means, that  
 “ are equally distant from those Extremes.

As in these,  $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16$ .

Then  $2+16=4+14=6+12=8+10$ .

Or if the Number of Terms be odd as these.

$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \cdot 18 \cdot \&c.$

Then  $2+18=4+16=6+14=8+12=10+10$ .

*Lemma 3.*

“ Every Series of Numbers in Arithmetical Progression is  
 “ composed of the Interval or Common Difference, so often re-  
 “ peated as there are Terms in the Progression except the First.

As in these,  $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot \&c.$

Here the Interval or common Difference being two, it will be  
 $1+2=3$ .  $3+2=5$ .  $5+2=7$ .  $7+2=9$ .  $9+2=11$ .  $11+2=13$ .  
 $13+2=15$ .  $15+2=17$ , &c.

*Corollary 2.*

“ Hence 'tis evident, that the Difference betwixt the two Ex-  
 “ tremes (*viz.* 1 and 17) is composed of the common Difference;  
 “ multiplied into the Number of all the Terms excepting the first.

As in the aforesaid Progression,  $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17$ .

L

The

The Number of Terms without the First is 8 }  
 The Common Difference is 2 } Multiply

The Difference betwixt the two Extremes 16

*Proposition 1.*

In any *Series* of Numbers in *Arithmetical Progression*, the two *Extremes*, and the *Number of Terms* being given; thence to find the *Sum* of all the *Series*.

Theorem. { Multiply the Sum of the two Extremes into the Number of all the Terms; and divide the Product by 2. The Quotient will be the Sum of all that Series. Per Corol. 1.

*Example 1.*

'Tis required to find the Number of all the Strokes a Clock strikes in one whole Revolution of the Index, viz. in twelve Hours.

Here  $1+12=13$  the Sum of the two Extremes.

And  $\frac{12}{2}$  the Number of all the Terms.

26

13

Then  $2) 156$  (78. The Number of Strokes required.

*Example 2.*

Suppose one Hundred Eggs were placed in a right Line a Yard distant from one another; and the first Egg were a Yard from a Basket; whether may a Man gather up those 100 Eggs singly one after another, still returning with every Egg to the Basket and put it in, before another Man can run four Miles. That is, which will run the greater Number of Yards.

In this Question  $200+2=202$  Is the Sum of the two Extremes.

And  $\frac{100}{2}$  Is the Number of all the Terms.

Then  $2) 20200$  (10100 { The Number of Yards he runs that takes up the Eggs.

Now 4 Miles = 7040 Yards { The Yards he runs that takes up  
 But  $10100-7040=3060$  { the Eggs more than the other.

*Proposition 2.*

In any *Series* of Numbers in *Arithmetical Progression*, the two *Extremes* and *Number of Terms* being given; thence to find the *Common Difference* of all the *Terms* in that *Series*.

Theorem. { The Difference betwixt the two Extremes, being divided by the Number of Terms less an Unit or 1. The Quotient will be the common Difference of the Series. Per Corol. 2.

*Example.*



## Example 1.

One had twelve Children that differed alike in all their Ages; the youngest was nine Years old, the eldest was thirty six and a half; what was the Difference of their Ages, and the Age of each?

Here  $36,5 - 9 = 27,5$  The Difference of the two Extremes.

And  $12 - 1 = 11$ . The Number of Terms less an Unit.

Then  $11 \div 27,5 = 2,5$ . The common Difference required.

Consequently  $9 + 2,5 = 11,5$  The Age of the youngest but one.

And  $11,5 + 2,5 = 14$  The Age of the youngest but two. And so on for the rest. *Per Corol. 2.*

## Example 2.

A Debt is to be discharged at Eleven several Payments to be made in Arithmetical Progression. The first Payment to be twelve Pounds ten Shillings, and the last to be sixty three Pounds. What's the whole Debt, and what must each Payment be?

*Per Theorem 1.* Find the whole Debt thus:

$12,5 + 63 = 75,5$  The Sum of the Extremes.

$11$  The Number of Terms.

---

755

755

---

2)  $830,5$  ( $415,25 = 415$  l. 5 s. The whole Debt.

Then *per Theorem 2.* Find the common Difference of each Payment.

Thus  $63 - 12,5 = 50,5$  The Difference of the Extremes.

And  $11 - 1 = 10$  The Number of Terms less 1.

Then  $10 \div 50,5 = 5,05 = 5$  l. 1 s. The common Difference.

l. s. l. s. l. s.

Consequently  $12$  l.  $10$  s.  $+ 5$  l.  $1$  s.  $= 17$  l.  $11$  s. The second Payment.

l. s. l. s. l. s.

And  $17$  l.  $11$  s.  $+ 5$  l.  $1$  s.  $= 22$  l.  $12$  s. The third Payment, &c.

## Example 3.

A Man is to travel from London to a certain Place in ten Days; and to go but two Miles the first Day, encreasing every Day's Journey by an equal Excess; so that the last Day's Journey may be twenty nine Miles; what will each Day's Journey be, and how many Miles is the Place he goes to distant from London?

L 2

First

First  $29 - 2 = 27$  The Difference of the Extremes.

And  $10 - 1 = 9$  The Number of Terms less 1.

Then  $9 \times 3 = 27$  The common Difference.

Consequently  $2 + 3 = 5$ . The second Day's Journey.

And  $5 + 3 = 8$ . The third Day's Journey, &c.

Again  $29 + 2 = 31$  The Sum of the Extremes.

$10$  The Number of Terms.

$2 \times 310 = 620$  The Distance required.

There are eighteen *Theorems* more relating to Questions in Arithmetical Progression; but because they would require a great many Words to shew the Reason of them: I therefore refer the Reader to the second Part, *viz.* That of *Algebra*, where he may find their *Analytical Investigation*.

## Sect. 2. Concerning Geometrical Proportion Continued; sometimes called Geometrical Progression.

When a Rank or Series of Numbers do either increase by one common Multiplicator, or decrease by one common Divisor; those Numbers are said to be in Geometrical Proportion continu'd.

As  $\{ 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot \&c.$  here 2 is the common Multiplier.

$\{ 64 \cdot 32 \cdot 16 \cdot 8 \cdot 4 \cdot \&c.$  here 2 is the common Divisor.

Or  $\{ 2 \cdot 6 \cdot 18 \cdot 54 \cdot 162 \cdot \&c.$  here 3 is the common Multiplier.

$\{ 162 \cdot 54 \cdot 18 \cdot 6 \cdot 2 \cdot$  here 3 is the common Divisor.

Note, the common Multiplier (or Divisor) is called the *Ratio*; and it shews the Habitude or Relation the Numbers have to one another, *viz.* whether they are double, triple, quadruple, &c. Which *Euclid* thus defines.

Ratio (or Rate) is the mutual Habitude or Respect of two Magnitudes (consequently two Numbers) of the same kind each to other, according to Quantity. Eu. 5. Def. 3.

Proportion (rather Proportionality or Analogy) is a Similitude of Ratio's. Eu. 5. Def. 4.

So that there cannot be less than three Terms to form a Proportionality or Similitude of Ratio's; and if but three Terms, the second must supply the Place of two. As in these  $2 \cdot 4 \cdot 8$ . That is.  $2 : 4 :: 4 : 8$ . (of  $::$  see Page 5.)

Here 4 the middle Term supplies the Place of two Terms, *viz.* of the second and third; 8 bearing the same Reason, Likeness



Likeness or Proportion to 4. As 4 doth to 2. *Viz.* As 2 : is to 4 :: So is 4 : to 8.

*Lemma 1.*

*If three Numbers are proportional, the Rectangle or Product of the two Extremes; viz. of the first and last Terms, will be equal to the Square of the Mean or Middle Term. ( 20 Eucl. 7. )*

As in these  $2 : 4 :: 4 : 8$  Here  $8 \times 2 = 16$  the Product of the Extremes.

And  $4 \times 4 = 16$  the Square of the Mean: *Ergo*  $8 \times 2 = 4 \times 4$ .

*Corol. 1.*

‘ Hence it follows, that if the Product of any two Numbers be equal to the Square of a third Number; those three Numbers will be in Proportion.

*Lemma 2.*

*If four Numbers are proportional, the Product of the two Extremes, will be equal to the Product of the two Means. ( 19 Euclid 7. )*

As in these,  $2 : 4 :: 8 : 16$ . Here  $16 \times 2 = 32$ .

And  $8 \times 4 = 32$ . Consequently  $16 \times 2 = 8 \times 4$ .

*Corol. 2.*

‘ From hence it follows, that if the Product of any two Numbers, be equal to the Product of any other two Numbers, those four Numbers are Proportionals.

And from these two *Lemma*’s, it will be easy to conceive, that if never so many Numbers are in continued Proportion; the Product of the two Extremes, will be equal to the Product of any two Means, that are equally distant from the Extremes.

As in these  $2 . 4 . 8 . 16 . 32 . 64 . \&c.$

Here  $64 \times 2 = 32 \times 4 = 16 \times 8$ . &c. And if the Number of Terms be odd,

As in these  $2 . 4 . 8 . 16 . 32 . 64 . 128 . \&c.$

Then  $128 \times 2 = 64 \times 4 = 32 \times 8 = 16 \times 16$ .

*Note,* The Character made use of to signify continued Proportionals is  $\div$

In

In every Series of  $\div$  (viz. of continued Proportionals) that Number which is compared to another, is called the Antecedent of the *Ratio*; and that Number to which it is compared, is called its Consequent.

As in these,  $2 : 4 :: 4 : 8$ . Here 2 is the Antecedent, and 4 is the Consequent; and 4 the middle Term is an Antecedent to 8 its Consequent: Whence it follows, that in every Series of  $\div$  all the middle Terms between the first and last are both Antecedents and Consequents.

As in these,  $2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64 \cdot \&c.$  Here  $4 \cdot 8 \cdot 16 \cdot 32$  are both Consequents and Antecedents.

For  $2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 32 :: 32 : 64 \&c.$

So that all the Terms except the last are Antecedents. And all the Terms except the first are Consequents.

### Lemma 3.

*If never so many Numbers are proportional, it will be: As any one of the Antecedents is to its Consequent: So will the Sum of the Antecedents be; To the Sum of all the Consequents. (12 Euclid 5.)*

That is. in the foregoing Series.

$$2 : 4 :: 2 + 4 + 8 + 16 + 32 : 4 + 8 + 16 + 32 + 64.$$

For 'tis evident, that  $4 + 8 + 16 + 32 + 64$  the Sum of all the Consequents, is double to  $2 + 4 + 8 + 16 + 32$  the Sum of all the Antecedents; As 4 is to 2, according to the *Ratio*, and would have been triple, or quadruple, &c. had the *Ratio* been 3 or 4, &c.

Note, In every Series of  $\div$  increasing the *Ratio* is found by dividing any of the Consequents by its Antecedent.

As in these,  $2 : 6 :: 6 : 18 :: 18 : 54 :: 54 : 162$   
Here 2) 6 (3 the *Ratio*. Or 6) 18 (3 &c.

From the second and third Lemma's may be raised two general Theorems or Rules. for finding the Sum of any Series in  $\div$  without a continued Addition of all the Terms.

Let the Series  $2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64 \cdot 128 \cdot$  be given to find its Sum.

Suppose  $z$  = the Sum of all the Terms.

Then  $z - 128$  = the Sum of all the Antecedents.

And  $z - 2$  = the Sum of all the Consequents.

But  $2 : 4 :: z - 128 : z - 2$ . per Lemma 3.

Ergo  $4z - 512 = 2z - 4$ . per Lemma 2.

Conse.



Consequently  $4z - 2z = 512 - 4$ .

Theorem.  $\left\{ \begin{array}{l} z = \frac{512 - 4}{4 - 2} \text{ In Words at Length thus.} \end{array} \right.$

Theorem 1.  $\left\{ \begin{array}{l} \text{From the Product of the second and last Terms, subtract} \\ \text{the Square of the first Term; that Remainder being divided} \\ \text{by the second Term less the first; will give the Sum of all the} \\ \text{Series.} \end{array} \right.$

Or if the first Term, the common Ratio, and the last Term be only given. Then

Theorem 2.  $\left\{ \begin{array}{l} \text{Multiply the last Term into the Ratio, and from their} \\ \text{Product subtract the first Term; divide that Remainder} \\ \text{by the Ratio less Unity or 1, and it will give the Sum of} \\ \text{all the Series.} \end{array} \right.$

For  $4z - 2z = 512 - 4$ . As above.  
Consequently  $2z - z = 256 - 2$  viz. the last divided by 2.

Then  $z = \frac{256 - 2}{2 - 1}$  Theorem 2.

Example. Let 2 . 6 . 18 . 54 . 162 . 486 . be the given Series!  
Here 2 is the first Term, 3 is the Ratio, and 486 the last Term.

But  $486 \times 3 = 1458$ . And  $1458 - 2 = 1456$ .  
Then  $3 - 1 = 2$ )  $1456$  (728 the Sum required.  
That is  $728 = 2 + 6 + 18 + 54 + 162 + 486$ .

In either of these Theorems it is required to have the last Term known (the which in a long Series of  $\div$  will be very tedious to come at by a continued Multiplication, &c. It will therefore be convenient to shew how to obtain either the last Term or any other Term, whose Place is assigned; without producing all the Terms.

In order to that, it will be necessary to premise the Coherence or Similitude that is betwixt Numbers in Arithmetical Progression, and those in Geometrical Proportion.

If in any Series of Numbers in  $\div$  when the first Term is not an Unit or 1, there be assigned a Series of Numbers in Arithmetical Progression, beginning with an Unit or 1, and whose common Difference is 1. Called Indices or Exponents.

Thus,  $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \quad \text{Indices.} \\ 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64 \cdot 128 \text{ \&c. } \div \end{array} \right.$

Then

Then will the Addition or Subtraction of any two of those Indices ( or Numbers in Arithmetical Progression ) directly correspond with the Product, or Quotient of their respective Terms in the Series of  $\div$ .

That is,  $\begin{cases} \text{As } 3 + 4 = 7 \\ \text{So } 8 \times 16 = 128 \end{cases}$  the seventh Term in  $\div$ .

Again,  $\begin{cases} \text{As } 6 + 4 = 10. \\ \text{So } 64 \times 16 = 1024. \end{cases}$  The tenth Term in  $\div$ .

Or,  $\begin{cases} \text{As } 7 - 3 = 4. \\ \text{So } 128 \div 8 = 16. \end{cases}$  Or,  $\begin{cases} \text{As } 6 - 2 = 4. \\ \text{So } 64 \div 4 = 16 \end{cases}$  &c.

But if the Series of  $\div$  begin with an Unit, the Indices must begin with a Cypher.

As in these,  $\begin{cases} 0 . 1 . 2 . 3 . 4 . 5 . 6. & \text{c.} \\ 1 . 2 . 4 . 8 . 16 . 32 . 64. \end{cases}$

Now by the Help of these Indices, and a few of the first Terms in any Series of  $\div$ . It is plain that any Term whose Place or Distance from the first Term is assigned, may be speedily obtained without producing the whole Series.

### Example 1.

A Man bought a Horse, and was to give a Farthing for the first Nail, two for the second, four for the third, &c. In  $\div$  The Number of Nails was to be 7 in every Shoe, viz. 28 Nails in all. What must he have paid for the Horse?

First  $\begin{cases} 0 . 1 . 2 . 3 . 4 . 5. & \text{Indices.} \\ 1 . 2 . 4 . 8 . 16 . 32. & \text{Farthings in } \div \end{cases}$

Then,  $\begin{cases} 5 + 5 = 10 \\ 32 \times 32 = 1024 \end{cases}$  And  $\begin{cases} 10 + 10 = 20 \\ 1024 \times 1024 = 1048576 \end{cases}$

Again,  $\begin{cases} 4 + 3 = 7 \\ 16 \times 8 = 128 \end{cases}$  Lastly,  $\begin{cases} 20 + 7 = 27 \\ 1048576 \times 128 = 134217728 \end{cases}$

Which is here to be accounted the 28. and last Term. Because the first Term in the Series is 1. Which doth neither multiply nor divide.

Now this 134217728 being the Number of Farthings to be paid for the last Nail, by it the common Ratio which is 2, and the first Term which is 1. may be found the Sum of all the Series. Per Theorem 2.



134217728

2

268435456 From this Product subtract 1.

*Viz.* 268435456—1=268435455. Then 2—1=1 the Divisor.

Consequently 268435455 is the Sum of all the Series or Price of the Horse in Farthings; which being brought into Pounds, &c. (See Page 46.) will be 279620 l. 5 s. 3 d. 3 qrs.

## Example 2.

A cunning Servant agreed with a Master (unskill'd in Numbers) to serve him eleven Years without any other Reward for his Service but the Produce of one *Wheat Corn* for the first Year; And that Product to be sow'd the second Year; and so on from Year to Year until the End of the Time. Allowing the Increase to be but in a ten-fold Proportion.

'Tis required to find the Sum of the whole Produce.

First  $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \end{array} \right.$  Indices or Years.  
 $\left\{ \begin{array}{l} 10 \cdot 100 \cdot 1000 \cdot 10000 \cdot 100000 \end{array} \right.$  Wheat Corns in  $\div$

Then  $\left\{ \begin{array}{l} \text{As } 4 + 2 = 6 \\ \text{So } 10000 \times 100 = 1000000. \end{array} \right.$  the 6th Year's Produce.

And  $\left\{ \begin{array}{l} 6 + 5 = 11 \\ 1000000 \times 100000 = 100000000000. \end{array} \right.$  The eleventh or last Year's Produce.

Then, (either by *Theorem 1* or 2) the Sum of all the Series will be, 11111111110 Corns. Now it may be computed from Page 31 and 34, that 7680 Wheat Corns round and dry out of the Middle of the Ear, will fill a Statute Pint. If so;

Then 7680) 11111111110 (14467592 Pints, but 64 Pints are contained in a Bushel.

Therefore 64) 14467592 (2260561  $\frac{1}{8}$  Bushels. Suppose it to be sold for 3 Shillings the Bushel;

Then  $\left\{ \begin{array}{l} 2260561 \frac{1}{8} \\ 3 \end{array} \right.$

Shillings 678168  $\frac{3}{8}$  = 33908 l. 8 s. 4  $\frac{1}{2}$  d. A very good Recommendation for 11 Years Service.

There are several pretty Questions resolved by Numbers in Arithmetical Progression; and by those in  $\div$  which the ingenious Learner will easily perceive hereafter; *viz.* When we come to the Solution of Questions, relating to Interest and Annuities, &c.

M

There

There is also a third Kind of Proportion, called *Musical*; which being but of little or no common Use, I shall therefore give but a short Account of it.

Musical Proportion or *Habitude* is, when of three Numbers the first hath the same Proportion to the third; As the Difference between the first and second hath to the Difference between the second and third.

As in these, 6 . 8 . 12. viz.  $6 : 12 :: 8 - 6 : 12 - 8$

If there are four Numbers in Musical Proportion, the first will have the same Proportion to the fourth; As the Difference between the first and second hath to the Difference between the third and fourth.

As in these 8 . 14 . 21 . 84.

Here  $8 : 84 :: 14 - 8 = 6 : 84 - 21 = 63$ .

That is,  $8 : 84 :: 6 : 63$ .

The Method of finding out Numbers in Musical Proportion is best expressed by Letters; as shall be shewed in the Algebraic Part.

### Sect. 3. *How to Change or Vary the Order of Things, &c.*

This being a Thing not treated of in any common Books of Arithmetick (that I have had the Opportunity of perusing) made me think it would be acceptable to the young Learner to know how oft 'tis possible to vary or change the Order or Position of any proposed Number of Things.

As how many several Changes may be rung upon any proposed Number of Bells; or how many several Variations may be made of any determined Number of Letters; or any other things exposed to be varied.

The Method of finding out the Number of Changes, is by a continual Multiplication of all the Terms in a Series of Arithmetical Progressionals; whose first Term, and common Difference is Unity or 1. And last Term the Number of Things proposed to be varied, viz.  $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$ , &c. As will appear from what follows.

1. If the Things proposed to be varied are only two, they admit of a double Position, as to Order of Place; And no more.

$$\text{Thus, } \left\{ \begin{array}{c} 1 \\ 2 \end{array} : \begin{array}{c} 2 \\ 1 \end{array} \right\} = 2 = 1 \times 2.$$

2. And if three things are proposed to be varied, they may be



be changed six several Ways; as to their Order of Places; and no more.

For beginning with 1 there will be { 1 . 2 . 3  
1 . 3 . 2  
Next beginning with 2 there will be { 2 . 1 . 3  
2 . 3 . 1  
Again, beginning with 3 it will be { 3 . 1 . 2  
3 . 2 . 1  
Which in all make 6 or 3 times 2. viz. 1 X 2 X 3 = 6

3. Suppose four things are proposed to be varied;  
Then they will admit of 24 several Changes, as to their Order of different Places.

For beginning the Order with 1 it will be { 1 . 2 . 3 . 4  
1 . 2 . 4 . 3  
1 . 3 . 2 . 4  
1 . 3 . 4 . 2  
1 . 4 . 2 . 3  
1 . 4 . 3 . 2  
Here is six different Changes.

And for the same Reason there will be 6 different Changes when 2 begins the Order, and as many when 3 and 4. begins the Order; which in all is 24=1X2X3X4. And by this Method of Proceeding, it may be made evident, that 5 Things admit of 120 several Variations or Changes; and 6 Things of 720, &c. As in this following Table.

The Number of Things proposed to be varied.	The Manner how their several Variations are produced.	The different Changes or Variations every one of the proposed Numbers can admit of.
1	1X1	=1
2	1X2	=2
3	2X3	=6
4	6X4	=24
5	24X5	=120
6	120X6	=720
7	720X7	=5040
8	5040X8	=40320
9	40320X9	=362880
10	362880X10	=3628800
11	3628800X11	=39916800
12	39916800X12	=479001600

These may be thus continued on to any assigned Number. Suppose to 24 the Number of Letters in the Alphabet, which will admit of 620448401733239439360000 several Variations.

From these Computations may be started several pretty, and indeed very strange Questions.

### Examples.

Six Gentlemen that were travelling met together by chance at a certain Inn upon the Road, where they were so pleased with their Host, and each others Company, that in a Frolick they made a Contract to stay at that Place, so long as they, together with their Host, could sit every Day in a different Order or Position at Dinner; which by the foregoing Computations will be found near 14 Years. For they being made 7 with their Host, will admit of 5040 different Positions; but 5040 being divided by  $365\frac{1}{4}$  the Number of Days in one Year, will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told, (That before the great Fire of London, which happen'd Anno 1666) there was 12 Bells in St. Mary le Bowes Church in Cheapside, London. Suppose it were required to tell how many several Changes might have been rung upon those 12 Bells: and at a moderate Computation how long all those Changes would have been ringing but once over.

First,  $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 479001600$  the Number of Changes.

Then supposing there might be rung 10 Changes in one Minute: viz.  $12 \times 10 = 120$  Strokes in a Minute, which is 2 Strokes in a Second of Time; now according to that rate there must be allowed 47900160 Minutes to ring them once over in all their different Changes; viz.  $10 \times 479001600$  (479001600).

In one Year there is 365 Days, 5 Hours and 49 Minutes; which being reduced into Minutes, is 525949.

Then  $525949 \div 47900160$  (91 Years, and 26 Days.

So long would those 12 Bells have been continually ringing without any Intermission, before all their different Changes could have been truly rung but once over. 'Tis strange, and seems almost incredible, that a few Things should produce such Varieties.

But



But that which seems yet more strange and surprising, yea, even impossible to those who are not a little vers'd in the Power of Numbers, is, that if two Bells more had been added to the aforesaid 12, they would have advanc'd the Number of Changes (and consequently the Time) beyond common Belief. For 14 Bells would require (at the same rate of Ringing as before) about 16575 Years to ring all their different Changes but once over.

And if it were possible to ring 24 Bells in Changes, and at the same Rate of 10 Changes in a Minute, which is 4 Strokes in one Second; they would require more than 11700000000000000 Years to ring them but once over in all their different Changes; as may easily be computed from the precedent Table.

## C H A P VII.

*Of Proportion Disjunct; commonly called the Golden Rule.*

**P**ROPORTION Disjunct, or the Golden Rule, is either Direct or Reciprocal, called Inverse. And those are both Simple and Compound.

### S E C T. I.

Direct Proportion is, when of four Numbers, the first beareth the same Ratio or Proportion to the Second; as the third doth to the fourth.

As in these  $2 : 8 :: 6 : 24$ .

Consequently, the greater the second Term is, in respect to the first; the greater will the fourth Term be, in respect to the third.

That is, as 8 the second Term, is 4 times greater than 2 the first Term: So is 24 the fourth Term, 4 times greater than 6 the third Term.

Whence it follows, that if four Numbers are in direct Proportion, the Product of the two Extremes, will always be equal to the Product of the two Means, as well in Disjunct as in continued Proportion; according to Lemma 2. Page 77.

For As  $2 : 2 \times 4 :: 6 : 6 \times 4$ . Or As  $3 : 3 \times 5 :: 6 : 6 \times 5$ .

But  $2 \times 6 \times 4 = 2 \times 4 \times 6$ . Or  $3 \times 6 \times 5 = 3 \times 5 \times 6$ .

That is, the Product of the Extremes is equal to that of the Means.

Again;

Again, the less the second Term is, in respect to the first; the less will the fourth Term be, in respect to the third.

As in these  $18 : 6 :: 12 : 4$ .

That is,  $18 : 18 \div 3 :: 12 : 12 \div 3$ .

But  $18 \times 12 \div 3 = 18 \div 3 \times 12$ . *Viz.*  $18 \times 4 = 6 \times 12$ .

Consequently  $2 . 8 . 6 . 24$ . And  $18 . 6 . 12 . 4$ . are true Proportionals, *per Corol. 2. Page 77*.

From these Considerations. comes the Invention of finding a fourth Number in Proportion to any three given Numbers. Whence 'tis called, *the Rule of Three*.

For if the second Number multiplied into the third, be equal to the first multiplied into the fourth, it is easy to conceive, that if the Product of the second and third be divided by the first, the Quotient must needs be the fourth Number. For if that Number which divides another, be multiplied into the Quotient produced by that Division; their Product will be equal to the Number divided. See *Page 21*.

As in these  $2 : 8 :: 6 : 24$ . Here  $8 \times 6 = 48 = 24 \times 2$ .

But if  $24 \times 2 = 48$ . then will  $48 \div 2 = 24$ . Or  $48 \div 24 = 2$ .

*Note*, Any four Numbers in direct Proportion may be varied several Ways. As in these.

*Viz.* If  $2 : 8 :: 6 : 24$ . Then  $2 : 6 :: 8 : 24$ .

And  $6 : 24 :: 2 : 8$ . Or  $24 : 6 :: 8 : 2$ . &c.

These Variations being well understood, will be of no small Use in the true Stating of any Question in this *Rule of Three*.

When three Numbers are given; and it is required to find a fourth Proportional; the greatest Difficulty (if there be any) will be in the right stating the Question. or abstracting the Numbers out of the Words in the Question, and placing them down in their proper Order.

Now this will be very easy if it be truly considered, that always two of the three given Terms, are only supposed, and assign or limit the Ratio or Proportion. The third moves the Question; and the fourth gives the Answer.

As for Instance; If 3 Yards of Cloth cost 9 Shillings: What will 6 Yards cost at the same Rate or Proportion?

Here 3 Yards, and 9 Shillings, are two supposed Numbers that imply the Rate; as appears by the Word [if], *viz.* if 3 Yards cost 9 Shillings (then comes the Question) What will 6 Yards cost?

*N. B.* The Term which moves the Question hath generally some of these Words before it; *viz.* What will? How many? How far? How long? or, How much?

Then



Then (carefully observe this; viz.) The first Term in the Supposition must always be of the same Kind and Denomination with that Term which moves the Question. And the Term sought will always be of the same Kind and Denomination with the second Term in the Supposition.

Thus,  $\begin{matrix} \text{Yds.} & \text{Shil.} & \text{Yds.} & \text{Shil.} \\ 3 & : 9 & :: 6 & : \text{---} \end{matrix}$

Then

All Questions in direct Proportion may be answered by three several Theorems.

Theorem 1.  $\left\{ \begin{array}{l} \text{Multiply the second and third Terms together, and divide their Product by the first Term; the Quotient will be the Answer required.} \end{array} \right.$

Thus  $\begin{matrix} \text{Yds.} & \text{Shil.} & \text{Yds.} & \text{Shil.} \\ 3 & : 9 & :: 6 & : 18. \end{matrix}$  The Answer.

$\frac{18}{3} = 6$   
3) 54 (18 Shillings.  $\left\{ \begin{array}{l} \text{Because the second} \\ \text{Term was Shillings.} \end{array} \right.$

Theorem 2.  $\left\{ \begin{array}{l} \text{Divide the second Term by the first, then} \\ \text{Multiply the Quotient into the third Term;} \\ \text{and the Product will be the Answer requir'd.} \end{array} \right.$

Thus  $\begin{matrix} \text{Yds.} & \text{Shil.} & \text{Yds.} & \text{Shil.} \\ 3 & : 9 & :: 6 & : 18. \end{matrix}$   
3) 9 (=3. Then  $3 \times 6 = 18$ , as before.

Theorem 3.  $\left\{ \begin{array}{l} \text{Divide the third Term by the first, then} \\ \text{multiply that Quotient into the second Term,} \\ \text{and their Product will be the Answer.} \end{array} \right.$

Thus  $\begin{matrix} \text{Yds.} & \text{Shil.} & \text{Yds.} & \text{Shil.} \\ 3 & : 9 & :: 6 & : 18. \end{matrix}$   
3) 6 (=2. And  $9 \times 2 = 18$ . as before.

Here you see that all the three Theorems are equally true; but the first is more general, and usually practised. Yet the two last may be readily performed when either the second or third Term can be divided by the first; and will be found of singular Use in the Rules of Fellowship, &c. as will appear further on.

Quest.

*Quest. 2.* If 8 lb of Tobacco cost 14 Shillings; What will half a hundred Weight (*viz.* 56 lb) cost at the same Rate?

Thus 8 lb : 14 s. :: 56 lb : 4 l. 18 s. The Answer.

$$\begin{array}{r} 14 \\ \hline 224 \\ 56 \end{array}$$

$$8) 784 (=98 \text{ s.} = 4 \text{ l. } 18 \text{ s.})$$

Or thus 8) 56 (=7. Then  $14 \times 7 = 98 \text{ s.}$  as before.

*Quest. 3.* If 14 Shillings will buy 8 lb of Tobacco; How much will 4 l. 18 s. buy after the same Rate?

Stated thus, 14 s. 8 lb :: 4 l. 18 s. = 98 s. : —

Then  $98 \times 8 = 784$ . And 14) 784 (56 lb. The Answer.

*Quest. 4.* If half a hundred Weight of Tobacco be worth 4 l. 18 s. How much may I buy for 14 Shillings at that Rate?

Stated thus, 4 l. 18 s. = 98 s. : 56 lb :: 14 s. : —

Then  $56 \times 14 = 784$ . And 98) 784 (8 lb. The Answer.

*Quest. 5.* Suppose 4 l. 18 s. will buy 56 lb of Tobacco; What will 8 lb of the same Tobacco cost?

This Question is thus stated, 56 lb : 4 l. 18 s. = 98 s. :: 8 lb. : —

Then  $98 \times 8 = 784$ . And 56) 784 (14 s. The Answer.

*Note,* The three last Questions are only the second varied, being proposed purely to give an Instance how any Question in this Rule of Three may be varied, according to Page 86.

*Quest. 6.* What will  $\frac{3}{4}$  of a Yard of Velvet cost, when the Price of 21 Yards and a half is worth 22 l. 10 s. 6 d. This Question truly stated will stand

Thus,  $21 \frac{1}{2} \text{ Yds.} : 22 \text{ l. } 10 \text{ s. } 6 \text{ d.} :: \frac{3}{4}$ . To the Answer.

Which may be found three several Ways; *viz.* by Reduction; by Vulgar Fractions; and by Decimals.

1. By Reduction. Bring the first and third Terms into one Denomination; *viz.* into Quarters, and reduce the second Term into its least Denomination, *per Sect. 4. Page 42.*

Thus  $21 \frac{1}{2} = 86$  Quarters. And  $22 \text{ l. } 10 \text{ s. } 6 \text{ d.} = 5406$  Pence.

Then  $86 : 5406 :: 3 : 15 \text{ s. } 8 \frac{1}{2} \text{ d.}$  For  $5406 \times 3 = 16218$ .

And



And 86) 16218 ( $=188\frac{5}{8}$  d. Then  $188\frac{5}{8}$  Pence  $=15$  s. 8 d.  $2\frac{1}{4}$  Farthings; the Answer required.

2. The same Question stated in Vulgar Fractions will stand

Thus  $21\frac{1}{2} = \frac{43}{2} : 22\frac{2}{3} = \frac{68}{3} :: \frac{3}{4} : (\text{See Sect. 3. Page 50.})$

Then  $\frac{68}{3} \times \frac{3}{4} = \frac{2703}{160}$ . And  $\frac{43}{2} ) \frac{2703}{160} (= \frac{5406}{160}$  Page 55, 56:

These  $\frac{5406}{160}$  Parts of a Pound are brought into Shillings by multiplying the Numerator with 20, and dividing the Product by its Denominator, &c.

Thus  $5406 \times 20 = 108120$ . And 6880) 108120 (15 s.

And there remains 4920. Again  $4920 \times 12 = 59040$ .

Then 6880) 59040 (8 d. and  $\frac{5}{8}$  c, &c. as before.

3. The same wrought by Decimal Fractions will be thus;

$21\frac{1}{2} = 21,5$   $22$  l. 10 s. 6 d.  $= 22,525$  and  $\frac{3}{4} = 0,75$

Therefore  $21,5 : 22,525 :: 0,75 : \text{to the Answer.}$

Then  $22,525 \times 0,75 = 16,89375$

And  $21,5 ) 16,89375$  ( $7857$  l.  $= 15$  s. 8 d. 2 far.  $\frac{27}{160}$  c.

Quest. 7. If 2 C. 3 qrs. 21 lb of Sugar cost 6 l. 1 s. 8 d. what will 12 C. 2 qrs. cost at the same Rate?

That is, 2 C. 3 qrs. 21 lb : 6 l. 1 s. 8 d. :: 12 C. 2 qr. To what?

4	20	4
11 qrs.	121 s.	50 qrs.
28	12	28
88	250	1400 lb
22	121	

Viz.  $308 + 21 = 329$  lb : 1460 d. :: 1400 lb: —

Then  $1460 \times 1400 = 2044000$ . And 329) 2044000 ( $6212\frac{3}{4}$  d.  $= 25$  l. 17 s.  $8\frac{3}{4}$  d. the Answer required.

The same Question stated in Decimals will stand

Thus  $2,9375 : 6,0833 :: 12,5 : \text{To the Answer.}$

Then  $6,0833 \times 12,5 = 76,04125$  which being divided by 2,9375 will give 25 88663 &c. the Answer in Decimals, which brought into Coin, will be 25 l 17,83 d. as before.

Note, When the first Term is an Unit or 1, the Question is answered by Multiplication only.

Example. Suppose I give 5 Shillings 4 Pence for one Ounce of Silver, what must I pay for  $32\frac{1}{2}$  Ounces at the same Rate.

That is  $1 \text{ } \frac{3}{4} : 5 \text{ s. } 4 \text{ d.} :: 32 \text{ } \frac{1}{2} \text{ } \frac{3}{4} : \text{To } \&c$

Which is best stated thus  $1 : 64 \text{ d.} :: 32 \text{ s.} :$

N

Then

Then  $32,5 \times 64 = 2080$  d.  $= 8$  l. 13 s. 4 d. the Answer required. For 1 neither multiplies nor divides.

When the second, or third Term is an Unit or 1 then the Question is answered by Division only. As in this *Example*.

If a Silver Tankard, weighing 21 Ounces, cost 5 l. 19 s. What's that an Ounce?

Thus 21 oz. : 5 l. 19 s.  $= 119$  s. : : 1 : 5 s. 8 d. To the Answer.

That is 21) 119 ( $= 5$  s.  $\frac{1}{2}$   $\frac{4}{1} = 5$  s. 8 d.

The Proof of all Questions in the *Rule of Three Direct*, may be easily conceiv'd from what hath been already said; viz. That the Product of the first and fourth Terms, must always be equal to the Product of the second and third Terms.

Or otherwise, by varying the Question, as in the second, third, fourth and fifth Questions.

I shall conclude this *Section* with inserting a few Questions and their Answers; leaving their Work for the Learner's Practice.

*Quest. 1.* What will the Carriage of 17 C. 3 qrs. 11 lb come to, at the Rate of 7 s. the Hundred.

*Answer,* 6 l. 4 s. 11  $\frac{1}{4}$  d.

*Quest. 2.* If 6 l. 4 s. 11  $\frac{1}{4}$  d. be paid for the Carriage of 17 C. 3 qrs. 11 lb; What was paid for the Carriage of 1 lb?

*Answer,* 3 Farthings.

*Quest. 3.* A Grocer bought 3 C. 1 qr. 14 lb Weight of Cloves, at the Rate of 2 s. 4 d. per lb, and sold them for 52 l. 14 s. Whether did he gain or lose by the Bargain, and how much?

*Answer,* he gained 8 l. 12 s.

*Quest. 4.* A Draper bought of a Merchant Eight Packs of Cloth; every Pack had four Parcels in it; and each Parcel contained ten Pieces; every Piece was twenty six Yards; he gave after the Rate of four Pounds sixteen Shillings for 6 Yards, What came the eight Packs to; and what was it worth per Yard?

*Ans.* They came to 6656 l. And is worth 16 s. per Yard.

*Quest. 5.* A Merchant bought 436 Yards of Broad Cloth for 8 s. 6 d. per Yard. And sold it again for 10 s. 4 d. per Yard. What did he gain by 436 Yards?

*Answer,* he gain'd 39 l. 19 s. 4 d.

*Quest.*



*Quest. 6.* A Goldsmith bought a Wedge of Gold, which weighed 1 lb 3 oz. 8 pw. for 514 l. 4 s. What did he pay per Ounce?

*Ans.* 3 l. per Ounce.

*Quest. 7.* What will 48 oz. 17 pwt. 20 Grains of Silver Plate come to, at the Rate of 5 s. 6 d. per Ounce?

*Ans.* 13 l. 8 s. 10  $\frac{3}{4}$  d.

*Quest. 8.* If in four Weeks one spend 13 s. 4 d. How long will 53 l. 6 s. last at that Rate?

*Ans.* 6 Years, 47 Days, 2 Hours, 24.

*Quest. 9.* What will the eighth Part of a Ship be worth; when the half is valued at 1015 l. 10 s?

*Ans.* 253 l. 17 s. 6 d.

*Quest. 10.* The Sun is said to perform one entire Revolution, (or three hundred and sixty Degrees) in the Space of three hundred sixty five Days, five Hours, forty eight Minutes, and fifty seven Seconds of Time, called a Tropical or Solar Year. How much doth it move in one Day?

*Ans.* 59 . 8 . 19 &c.

*Quest. 11.* If  $\frac{5}{8}$  of a Yard of Velvet cost  $\frac{2}{3}$  of a Pound Sterling; what will  $\frac{1}{8}$  of a Yard cost of the same Velvet, at that Rate.

*Ans.*  $\frac{1}{4} \frac{6}{8} = 1$  s. 4 d.

*Quest. 12.* Suppose 2 l. and  $\frac{3}{8}$  of  $\frac{1}{3}$  of a Pound Sterling will buy 3 Yards and  $\frac{2}{3}$  of  $\frac{3}{5}$  of a Yard of Cloth; How much will  $\frac{3}{4}$  of a Yard cost at that Rate?

*Ans.*  $\frac{7}{8} \frac{2}{3} \frac{5}{5}$  of a Pound = 9 s. 4  $\frac{1}{2}$  d.

## Sect. 2. Of Reciprocal Proportion; usually called the Rule of Three Inverse.

*Reciprocal Proportion* is, when of four Numbers the Third (*viz.* that which moves the Question) beareth the same *Ratio* to the First; as the Second does to the Fourth.

Therefore, the less the third Term is, in respect to the First; The greater will the fourth Term be, in respect to the Second.

### Example 1.

If sixteen Men can do a Piece of Work in six Days; How many Days must eight Men require to do the same Work, at the same Rate of Working.

Here 'tis plain that eight Men must needs have more Time than 16 Men to do the same Work. Consequently the greater

greater the third Term is, in respect to the First, the lesser will the fourth Term be, in respect to the Second.

*Example 2.* If 8 Men can do a Piece of Work in 12 Days; how many Days will 16 Men require to do the same Work. Here it is plain the fourth Term must be less than the Second, because 16 Men undoubtedly can do the same Work in less Time than 8 Men can.

From these Considerations, compared with those in Page 85. 'twill be easy to perceive whether the Terms of any proposed Question are in Direct or Reciprocal Proportion.

For when according to the true Meaning or Design of any Question in Proportion. MORE requires MORE, or LESS requires LESS, the Terms are in Direct Proportion, as in the last Section.

But if MORE requires LESS, or LESS requires MORE (as above) then the Terms will be in Reciprocal Proportion.

The Manner of placing down the proposed Terms is the same in both Rules, viz. The first Term in the Supposition must be of the same Kind and Denomination with the third Term which moves the Question; and the Term sought must be of the same Kind and Denomination with the second Term in the Supposition. As in the two last Examples.

	Men   Days   Men   Days
Thus, in { <i>Example 1.</i>	16 : 6 :: 8 : —
{ <i>Example 2.</i>	8 : 12 :: 16 : —

The Question being truly stated, observe this Theorem.

Theorem. { *Multiply the first and second Terms together,*  
*and divide their Product by the third Term,*  
*the Quotient will be the Answer required.*

Thus in the second Example  $12 \times 8 = 96$ .

Then 16) 96 (= 6 Days the Answer required.

That is, 16 Men may do the same Work in 6 Days as 8 Men can do in 12 Days.

Now the Reason of this Operation, (and consequently of the Theorem) is grounded upon this Consideration; viz. If 8 Men require 12 Days to do the Work, 'tis plain that one Man would require 8 times 12 Days = 96 Days to do the same Work. but if one Man can do it in 96 Days, most certain 16 Men can do it in one 16th Part of that Time. Therefore 96 divided by 16 will give the Answer required, viz. 16) 96 (6 As before, &c.

*Quest. 3.* Suppose 800 Soldiers were besieged in a Town, and their Victuals were computed to serve them two Months (or 56 Days) How many of these Soldiers must depart the Garrison, that the same Victuals may serve the remaining Soldiers 5 Months.

The



The Question truly stated, will stand

Thus,  $\begin{array}{l} \text{Month Soldier} \quad \text{Month Soldier.} \\ 2 : 800 :: 5 : \text{---} \\ \quad \quad \quad 2 \end{array}$

5) 1600 (320      So many Soldiers may stay in the Garrison.

Consequently,  $800 - 320 = 480$  Soldiers that must go out of the Garrison, which is the Answer required.

Question 4. A. Borrowed of his Friend B. 250 l. for six Months promising to do him the like Kindness upon Demand: Some time after B. desires A. to lend him 400 l. the Question is how long B. may keep the 400 l. to be fully satisfied for his former Kindness to A?  
Answer, 3 Months 21 Days.

Thus,  $\begin{array}{l} 250 \text{ l.} : 6 \text{ Months} :: 400 \text{ l.} : \text{---} \\ \quad \quad \quad 6 \end{array}$

400) 1500 (3 Months

$\begin{array}{r} 12 \\ \text{---} \end{array}$

$\begin{array}{r} 3 \\ 28 \end{array}$  Days in one Month.

4) 84 (21 Days.

Question 5. If a Penny White Loaf ought to weigh eight Ounces Troy Weight, when Wheat is sold for six Shillings six Pence the Bushel, what must it weigh when Wheat is sold for four Shillings the Bushel.

Thus  $6 \text{ s. } 6 \text{ d.} = 78 \text{ d.} : 8 \text{ Oz.} :: 4 \text{ s.} = 48 \text{ d.}$  To the Answer, 13 oz.

48) 624 (13  $\frac{3}{4}$  the Answer required.

$\begin{array}{r} 144 \\ \text{---} \end{array}$

$\begin{array}{r} 144 \\ \text{---} \end{array}$

(0)

The Proof of this Inverse Rule is easily deduced from its Operations; viz. The Product of the first and second Terms, must be equal to the Product of the third and fourth Terms.

Note,

*Note,* Any Question that falls under this Inverse Rule or Reciprocal Proportion, may be so stated as to have its Terms in Direct Proportion; by only changing the Places of the first and third Terms in the Question. Thus,

*Question 6.* If a Field will feed eighteen Horses for seven Weeks, how long will it feed Forty two Horses at the same Rate of feeding.

First, 18 Horses : 7 Weeks :: 42 Horses : 3 Weeks.

Here the Terms are stated Inversely, as before.

Otherwise thus, 42 Horses : 7 Weeks :: 18 Horses : 3 Weeks.

Then  $18 \times 7 = 126$ . And  $126 \div 42 = 3$  Weeks. The Answer required.

### Sect. 3. Of Compound Proportion; commonly called the Double Rule of Three.

*Compound Proportion* (as 'tis here meant) is, when there are five Numbers given to find out a sixth Proportional; and this is generally performed by a double Position; that is, by stating and working the Question at two Operations, either in Direct, or Reciprocal Proportion, according as the Question requires.

And therefore it's called the Double Golden Rule; or Double Rule of Three.

The *Double Rule Direct* is, when the sixth Term, or Number sought, is found by two Operations, both of them in Direct Proportion.

*Example 1.* If a hundred Pounds gain six Pounds Interest in twelve Months; how much will three hundred Pounds gain in nine Months; at the same rate?

First 100*l.* : 6*l.* :: 300*l.* : 18*l.*

6		
100)	1800	(18 <i>l.</i>
Months		Months
Then,	12 : 18 <i>l.</i> :: 9 : 13 <i>l.</i> 10 <i>s.</i>	
	9	

} The Interest of 300*l.*  
 } for twelve Months.

12) 162 (13*l.* 10*s.* The Answer required.

I suppose the Learner will easily conceive the Reason of these two Operations. For, First it's plain by direct Proportion, that if 100*l.* gain 6*l.* in twelve Months, 300*l.* will gain 18*l.* in the same Time, and at the same Rate,

And



And by the same Rule 'tis plain, that if 12 Months will produce or give 18 *l.* Interest for 300 *l.* then 9 Months must needs give 13½ for the same Sum, *viz.* 300 *l.*

The *Double Rule of Three Inverse* is, when the sixth Term or Number sought is found at two Operations, (as before). But one of them requires an Answer in Reciprocal Proportion.

*Question 2.* If 6 Bushels of Oats will serve 4 Horses 8 Days, how many Days will 21 Bushels serve 16 Horses at the same Rate of Feeding?

This Question being parted into two Positions, the first will be thus:

If 6 Bushels of Oats will serve 4 Horses 8 Days, how many Days will 21 Bushels serve them?

Here 'tis plain that 21 Bushels will serve them longer than 6 Bushels; therefore the first Position falls in Direct Proportion.

$$\text{Thus, } \begin{array}{cccc} \text{Bush.} & \text{Days.} & \text{Bush.} & \text{Days.} \\ \{ & 6 : 8 :: 21 : 28 \end{array}$$

$$\begin{array}{r} 6 \overline{) 168} \quad (28 \text{ Days.} \end{array}$$

That is, if 6 Bushels will serve 4 Horses 8 Days, 21 Bushels will serve them 28 Days.

The next Position must be to find how long the said 21 Bushels will serve 16 Horses at the same Rate of Feeding: 'Tis plain, that 21 Bushels cannot serve 16 Horses so many Days as they will serve 4 Horses; therefore this second Position falls in Reciprocal Proportion.

$$\text{Thus, } \begin{array}{cccc} \text{Horses.} & \text{Days.} & \text{Horses.} & \text{Days.} \\ \{ & 4 : 28 :: 16 : 7 \end{array} \quad \text{the Answer required.}$$

After the like Manner any Question in the *Double Rule of Three* may be answered by two single Positions, if Care be taken in stating them right, *viz.* Whether their Operations must be performed by the *Single Rule Direct*, or *Inverse*.

But all Questions in this *Double Rule*, where five Numbers are proposed to find a sixth, may more easily and readily be answered by one *General Theorem*; which comprised both the *Direct* and *Inverse Rules*: without considering either of them, being deduced from the single Operations before-going.

But first you must carefully note, that in all Questions of this Nature, three of the five proposed Terms are always conditional and

and supposed; and that the other two move the Question. As for Instance in *Example 1.*

*Viz.* If 100 *l.* will gain 6 *l.* in 12 Months: These three Terms are only supposed or conditional. Then comes the Question; What will 300 *l.* gain in 9 Months? Now, in order to raise the General Theorem, let us suppose, instead of Numbers, these Letters.

*Viz.* Let  $\left\{ \begin{array}{l} P=100. \text{ The Principal.} \\ T=12. \text{ The Time.} \\ G=6. \text{ The Gain.} \end{array} \right.$   $\left. \begin{array}{l} \} \} \text{ In the Supposition} \\ \} \} \text{ of any proposed} \\ \} \} \text{ Question.} \end{array} \right.$

And  $\left\{ \begin{array}{l} p=300. \text{ The Principal.} \\ t=9. \text{ The Time.} \\ g=13,5 \text{ The Gain.} \end{array} \right.$   $\left. \begin{array}{l} \} \} \text{ The Three Terms} \\ \} \} \text{ wherein the Que-} \\ \} \} \text{ stion lies.} \end{array} \right.$

Then  $P : G :: p : \frac{Gp}{P} = \left\{ \begin{array}{l} \text{The Product of the two Means} \\ \text{divided by the first Extreme.} \end{array} \right.$

That is,  $100 : 6 :: 300 \times 6$   
 $\qquad \qquad \qquad \frac{\qquad \qquad \qquad}{100} = 18.$   $\left. \begin{array}{l} \} \} \text{ Which is the} \\ \} \} \text{ first Part of the} \\ \} \} \text{ Question.} \end{array} \right.$

Then  $T : \frac{Gp}{P} :: t : g$   $\left. \begin{array}{l} \} \} \text{ Which is the} \\ \} \} \text{ second Part of} \\ \} \} \text{ the Question.} \end{array} \right.$

*Viz.*  $12 : 18 :: 9 : 13,5$

Ergo  $Tg = \frac{Gpt}{P}$   $\left. \begin{array}{l} \} \} \text{ That is, the Product of the Extremes} \\ \} \} \text{ is equal to that of the Means.} \end{array} \right.$

Consequently,  $TgP = Gpt.$  Is the *Theorem.*

This *Theorem* affords two *Rules* by which all Questions in this *Double Rule of Three*, or rather of five Numbers, may be resolved; due Regard being had to the true placing down of the proposed Terms, which must be thus:

Always place the three Conditional Terms in this Order; Let that Number which is the principal Cause of Gain, Loss, or Action, &c. (*viz.* *P.*) be put in the first Place; That Number which denotes the Space of Time or Distance of Place, &c. (*viz.* *T.*) be put in the second Place. And that Number which is the Gain, Loss, or Action, &c. (*viz.* *G.*) be put in the third Place. Now according to these Directions the Conditional Terms of the last Questions will stand thus; *P. T. G.*

That done, place the other two Terms which move the Question, underneath those of the same Name.

Thus,  $\left\{ \begin{array}{l} P \cdot T \cdot G. \\ p \cdot t \cdot \end{array} \right.$

Then



Then if the Blank or Term sought, fall under the third Place, as in this Question,

It will be  $\left\{ \frac{G p t}{P T} = g \right.$  Which gives this Rule.

Rule 1.  $\left\{ \begin{array}{l} \text{Multiply the three last Terms together for a} \\ \text{Dividend; and the two first together for a} \\ \text{Divisor, the Quotient arising from them will} \\ \text{be the sixth Term.} \end{array} \right.$

That is, in our proposed Example 1.

Thus  $6 \times 300 \times 9 = 16200$  The Dividend.

And  $100 \times 12 = 1200$  The Divisor.

Then  $1200 \overline{) 16200} (13\frac{1}{2}$  The Answer. As before.

But if the Blank or Term sought fall under the first Place, then

It will be  $\left\{ \frac{T g P}{t G} = p \right.$

Or if the Blank fall under the second Place,

It will be  $\left\{ \frac{T g P}{G p} = t \right.$  Either of these give this Rule.

Rule 2.  $\left\{ \begin{array}{l} \text{Multiply the first, second and last Terms toge-} \\ \text{ther for a Dividend: And the other two} \\ \text{together for a Divisor; the Quotient arising} \\ \text{from them will be the sixth Term.} \end{array} \right.$

And because our Example 2. falls under the Consideration both of Direct and Reciprocal Proportion. let it be here propos'd again.

Viz. If 6 Bushels of Oats will serve 4 Horses 8 Days; how many Days will 21 Bushels serve 16 Horses, &c.

If the Terms of this Question be placed down as before directed they will stand

Thus	{	Horses	Days.	Bushels.		Terms in the Supposition.
4		8	6			
16			21			

Here the Blank falls under the second Place, therefore it must be found by the second Rule.

Thus  $4 \times 8 \times 21 = 672$  the Dividend.

And  $16 \times 6 = 96$  the Divisor.

Then  $96 \overline{) 672} (7$  the Answer as before.

O

Quest.

*Quest. 3.* What Principal or Stock will gain 20 *l.* in 8 Months at 6 per Cent. per Annum.

Prin. Time. Gain

100 . 12 . 6  
8 . 20

Terms in the Supposition.

In this Question the Blank falls under the first Place, therefore it must be found by the second Rule.

Thus  $100 \times 12 \times 20 = 24000$  the Dividend.

And  $8 \times 6 = 48$  the Divisor.

Then  $48 \overline{) 24000}$  (500 *l.* the Answer required.

The Proof of all Questions in this Double Rule of Five Numbers, is best perform'd by varying the *Question*; viz. by stating it in another Order, as in the last *Example*: Thus,

If 100 *l.* gain 6 *l.* in 12 Months, what will 500 *l.* gain in 8 Months?

The Answer to this Question must be 20 *l.* if the Work of the last *Example* be true.

Prin. Time. Gain.

Stated thus  $\left\{ \begin{array}{l} 100 . 12 . 6 \\ 500 . 8 . \end{array} \right\}$  then per Rule 1.

$500 \times 8 \times 6 = 24000$ . And  $100 \times 12 = 1200$

Then  $1200 \overline{) 24000}$  (20. the Answer, &c.

*Quest. 4.* If Two Men can do 12 Rods of Ditching in 6 Days, How many Rods may be done by 8 Men in 24 Days, at the same Rate of Working.

*Ans.* 192 Rods.

*Quest. 5.* If the Carriage of 5 C. 3 qr. Weight, 150 Miles; cost 3 *l.* 7 s. 4 d. What must be paid for the Carriage of 7 C. 2 qr. 25 lb Weight, 64 Miles; at the same Rate.

*Ans.* 1 *l.* 18 s. 7  $\frac{1}{2}$  d.

*Quest. 6.* If 8 Men deserve 2 *l.* Wages for 5 Days Work, How much will 32 Men deserve for 24 Days; at the same Rate.

*Ans.* 38 *l.* 8 s.

*Quest. 7.* Suppose a Hundred Pounds would defray the Expences of five Men for twenty two Weeks and six Days. How long would twelve Men be in spending of one hundred and fifty Pounds, and at the same Rate.

*Ans.* 14 Weeks and 2 Days.

C H A P.



C H A P. VIII.

*Of Trading in Company; usually called the Rule of Fellowship; Also Bartering, and Exchanging of Coins, &c.*

**T**HE Rule of Fellowship, is that by which the Accompts of several Partners trading in a Company, are so adjusted or made up, that every Partner may have his just Part of the Gain, or sustain his just Part of the Loss; according to his Proportion or Share of Money he hath in the Joint-Stock: Now this falls under two Considerations, called the Single and Double Rules of Fellowship.

Sect. I. *The Single Rule of Fellowship; viz. That without Time.*

By the *Single Rule of Fellowship*, are adjusted the Accompts of those Partners that put all their several and perhaps different Sums of Money, into a common Stock at one and the same Time; and therefore 'tis usually call'd the *Rule of Fellowship without Time*: Now all Questions of this Nature are answered by so many several Operations in the *Rule of Three Direct*, as there are Partners in the Stock.

'For; as the total Sum of Money in the Stock is in Proportion to the whole Gain or Loss; so is every Man's particular Part of that Stock; to his particular Share of that Gain, or Loss.

*Quest. 1.* There are three Partners, suppose *A*, *B*, and *C*; make a Joint-Stock of 96 *l.* in this Manner.

*A*, puts in 24 *l.* *B*, puts in 32 *l.* and *C*, puts in 40 *l.* with this 96 *l.* they trade and gain 12 *l.* 'Tis required to find each Man's true Part of that Gain.

The first Operation for *A*'s Part of the Gain will stand

Thus  $96\text{ }l. : 12\text{ }l. :: 24\text{ }l. : 3\text{ }l. = A\text{'s Gain.}$

Secondly,  $96\text{ }l. : 12\text{ }l. :: 32\text{ }l. : 4\text{ }l. = B\text{'s Part of the Gain.}$

Again,  $96\text{ }l. : 12\text{ }l. :: 40\text{ }l. : 5\text{ }l. = C\text{'s Part of the Gain.}$

Proof  $3\text{ }l. + 4\text{ }l. + 5\text{ }l. = 12\text{ }l.$  the whole Gain.

That is, if the Sum of each Man's particular Gain, amount to the whole Gain. the Work is true; if not, some Error is committed which must be found out.

*Note,* These Operations will be very much abbreviated, if you work them by *Theorem 2. Page 87.* For here 96 is a common Antecedent, and 12 is the common Consequent in all the three Proportions,

Therefore  $96 : 12 :: 1 : 0,125$  is a common Multiplier.

Then  $24 \times 0,125 = 3 \text{ l. for } A\text{'s Part}$

And  $32 \times 0,125 = 4 \text{ l. for } B\text{'s Part}$

Again  $40 \times 0,125 = 5 \text{ l. for } C\text{'s Part}$

} As before.

Now this Method is more readily perform'd than the other, especially when the Partners are many; because one single Division serves for all the Work.

*Quest. 2.* Three Merchants, *A*, *B*, and *C*, freight a Ship with 248 Tun of Wine: thus, *A* loaded 98 Tun, *B* 86 Tun, and *C* 64 Tun. By Extremity of Weather the Seamen were forc'd to cast or throw 93 Tun of it over-board. How much of this Loss must each Merchant sustain?

First  $248 : 93 :: 1 : 0,375$  the common Multiplier.

Then  $98 \times 0,375 = 36,75$  for *A*'s Loss.

And  $86 \times 0,375 = 32,25$  for *B*'s Loss.

Again  $64 \times 0,375 = 24$  for *C*'s Loss.

Proof  $93,00 =$  the whole Loss.

Now if the Question were to find how much of the remaining Wine that was saved, belongs to *A*, to *B*, and to *C*.

Then  $98 - 36,75 = 61,25$  belongs to *A*.

And  $86 - 32,25 = 53,75$  belongs to *B*.

And  $64 - 24 = 40$  belongs to *C*.

That is: *A* ought to have 61 Tun and 63 Gallons. *B* ought to have 53 Tun and 189 Gallons. And *C* ought to have 40 Tun of what was left.

*Quest. 3.* Suppose six Men, viz. *A*, *B*, *C*, *D*, *E*, and *F*, make a Joint-Stock of 2558 *l*.

	<i>l</i> .	<i>s</i> .	Decimals.
Thus { <i>A</i> puts in	654	10	= 654,5
{ <i>B</i> ———	543	15	= 543,75
{ <i>C</i> ———	480	00	= 480,
{ <i>D</i> ———	254	10	= 254,5
{ <i>E</i> ———	365	05	= 365,25
{ <i>F</i> ———	260	00	= 260,

The whole Stock  $2558 \cdot 00 = 2558,00$ , according to the Question.

With



## Chap. 8. Rule of Fellowship, &c. 101

With this Stock of 2558 *l.* they trade eighteen Months, and gain 831 *l.* 7 *s.* 'Tis required to find every Man's Part or Share of that Gain.

*Note*, Altho' the Time of Trading, *viz.* Eighteen Months be mentioned in the Question; yet 'tis no Way concerned in answering of it; as you may observe by the following Work.

First, 2558 *l.* : 831,35 *l.* :: 1 *l.* : 0,325 Decimal Parts.

Consequently, 1 *l.* : 0,325 :: 654,5 : 212,7125. That is,

A's Stock 654,5  $\times$  0,325 = 212,7125 for A.

B's Stock 543,75  $\times$  0,325 = 176,71875 for B.

C's Stock 480,  $\times$  0,325 = 156,000 for C.

D's Stock 254,5  $\times$  0,325 = 82,7125 for D.

E's Stock 365,25  $\times$  0,325 = 118,70625 for E.

F's Stock 260,  $\times$  0,325 = 84,5 for F.

That is, 

	<i>l.</i>	<i>parts</i>	<i>l.</i>	<i>s.</i>	<i>d.</i>
A Gains	212,7125	=	212	. 14	. 03
B ———	176,71875	=	176	. 14	. 04 $\frac{1}{2}$
C ———	156,0000	=	156	. 00	. 00
D ———	82,7125	=	82	. 14	. 03
E ———	118,70625	=	118	. 14	. 01 $\frac{1}{2}$
F ———	84,5	=	84	. 10	. 00

Proof. Sum 831,35 = 831 . 07 . 00 the Gain.

I have omitted resolving this Question according to the usual Method (as before directed) or finding every Man's particular Part of the Gain by the *Golden Rule*, as in the first Work of Example 1. leaving that for the Learner's Practice.

### Sect. 2. The Double Rule of Fellowship; or that with Time.

This is usually called the *Double Rule of Fellowship*, because every particular Man's Money is to be considered with relation to the Time of its Continuance in the Joint-Stock.

*Question 1.* A and B join in Partnership upon these Terms, *viz.* A agrees to lay down 100 *l.* and to employ it in Trade 3 Months: Then B is to lay down his 100 *l.* and with the whole Stock of 200 *l.* they are to trade 3 Months more. Now at the End of that Time, they find their whole Gain to be 21 *l.* 'Tis required to know what each Man's Part of the Gain ought to be, according to his Stock, and the Time of employing it.

Here

Here it is but reasonable to conclude, that *A.* ought to gain more than *B.* notwithstanding their Stocks of Money are equal; because *A.* imploy'd his Money a longer Time than *B.*

Now for solving of this Question, let us suppose *A.*'s 100 *l.* employed the first 3 Months to gain  $Z =$  a Sum as yet unknown; then it must gain  $2Z$ , in 6 Months; and to find what *B.* must gain it will be,

$$\begin{array}{lcl} l. & . & Months \\ 100. & 6 & . \quad 2Z = A's \text{ Gain} \\ 100. & 3 & . \quad \text{To } B's \text{ Gain} \end{array} \left. \vphantom{\begin{array}{l} 100. \\ 100. \end{array}} \right\} \text{ per Rule 1. Page 97.}$$

$$\text{Ergo } \frac{100 \times 3 \times 2Z}{100 \times 6} = B's \text{ Gain.}$$

But *A.*'s Gain added to *B.*'s Gain must  $= 21$  *l.* the whole Gain the Question,

$$\text{Therefore } 2Z + \frac{100 \times 3 \times 2Z}{100 \times 6} = 21 \text{ } l.$$

$$\text{That is, } 100 \times 6 \times 2Z + 100 \times 3 \times 2Z = 21 \times 100 \times 6,$$

$$\text{Which contracted is, } 900 \times 2Z = 21 \times 600.$$

$$\text{Consequently } 2Z = \frac{21 \times 600}{900} \text{ which gives the following Analogy,}$$

$$\text{Viz. } 900 : 21 :: 600 : 2Z = 14 \text{ } l. \quad \text{for } A's \text{ Gain.}$$

$$\text{And } 900 : 21 :: 100 \times 3 = 300 : 7 \text{ } l. \quad \text{for } B's \text{ Gain.}$$

Now this Way of Arguing hath not only resolved the present Question; but it also affords (and demonstrates) a general Rule for resolving all Questions of this Nature, be the Partners never so many.

Rule.  $\left\{ \begin{array}{l} \text{Multiply every particular Man's Stock, with} \\ \text{the Time it is employed; then it will be,} \\ \text{As the Sum of all those Products; Is to the} \\ \text{whole Gain (or Loss). So is every one of those} \\ \text{Products; to its proportional Part of that} \\ \text{whole Gain (or Loss)} \end{array} \right.$

Question 2. Three Merchants *A.* *B.* and *C.* enter into Partnership thus; *A.* puts into the Stock 65 *l.* for 8 Months; *B.* puts in 78 *l.* for 12 Months; and *C.* puts in 84 *l.* for 6 Months. With these they traffick, and gain 166 *l.* 12 *s.* 'Tis required to find each Man's Share of the Gain, proportionable to his Stock and Time of employing it.

1. *A.*'s



# Chap. 8. Rule of Fellowship, &c. 103

1. A's Stock 65 l. x 8 Months, the Time it was employed = 520
2. B's Stock 78 l. x 12 Months, the Time it was employed = 936
3. C's Stock 84 l. x 6 Months, the Time it was employed = 504

The Sum of those Product is, 1960

Then, according to the Rule, the several Proportions will stand thus,

$$1960 : 166,6 :: 520 : 44,2 = 44 \text{ l. } 4 \text{ s. } 0 \text{ d. for A.}$$

$$1960 : 166,6 :: 936 : 79,56 = 79 \text{ l. } 11 \text{ s. } 2 \frac{1}{2} \text{ d. for B.}$$

$$1960 : 166,6 :: 504 : 42,84 = 42 \text{ l. } 16 \text{ s. } 9 \frac{1}{2} \text{ d. for C.}$$

The whole Gain = 166 l. 12 s. 0 d.

Or you may work as in some of the former Examples, viz. by finding the proportional Part of the Gain due to one Pound, &c.

Thus, 1960 : 166,6 :: 1 : 0,085 the common Multiplier.

$$\text{Then } 520 \times 0,085 = 44,2 \text{ for A.}$$

$$\text{And } 936 \times 0,085 = 79,56 \text{ for B.}$$

$$\text{And } 504 \times 0,085 = 42,84 \text{ for C.}$$

&c. As before.

Question. 3. Six Merchants, viz. A. B. C. D. E. and F. enter into Partnership, and compose a Joint-Stock in this manner;

	l.	s.	
viz. {	A puts in	64	10
	B	78	15
	C	100	00
	D	80	10
	E	74	12
	F	125	15

for {  $\begin{matrix} 4\frac{1}{2} \\ 6 \\ 8\frac{1}{2} \\ 12 \\ 9\frac{1}{2} \\ 7 \end{matrix}$  } Months

They traffick, and gain 258 l. 18 s. 4  $\frac{1}{2}$  d. 'Tis required to find every Man's Share of the Gain, according to his Stock and Time it was employed.

The several Stocks of Money, and their respective Times being first brought into Decimals, and then multiplied together will produce these following Products.

l. Montbs.

$$\text{A's Stock } 64,5 \times 4,5 \text{ the Time it was employed} = 290,25$$

$$\text{B's Stock } 78,75 \times 6, \text{ the Time it was employed} = 472,5$$

$$\text{C's Stock } 100, \times 8,25 \text{ the Time it was employed} = 825,0$$

$$\text{D's Stock } 80,5 \times 12, \text{ the Time it was employed} = 966,0$$

$$\text{E's Stock } 74,6 \times 9,5 \text{ the Time it was employed} = 708,7$$

$$\text{F's Stock } 125,75 \times 7, \text{ the Time it was employed} = 880,25$$

The Sum of those Products = 4142,7

Then

Then if you work by the common way; it will be  
 $4142,7 : 258,91875 :: 290,25 : 18,140625 = 18 \text{ l. } 2 \text{ s. } 9\frac{3}{4} \text{ d.}$   
 for *A*'s Part of the Gain; and so on for the rest.

But if you work by the easiest Way, *viz.* by finding the proportional Part of the Gain due to one Pound.

Thus  $4142,7 : 258,91875 :: 1 : 0,0625$

	l.	s.	d.	
Then $290,25 \times 0,0625 = 18,140625 = 18$	2	9	$\frac{3}{4}$	for <i>A</i>
And $472,5 \times 0,0625 = 29,53125 = 29$	10	7	$\frac{1}{2}$	for <i>B</i>
$825, \times 0,0625 = 51,5625 = 51$	11	3		for <i>C</i>
$966, \times 0,0625 = 60,375 = 60$	7	6		for <i>D</i>
$708,7 \times 0,0625 = 44,29375 = 44$	5	10	$\frac{1}{2}$	for <i>E</i>
$880,25 \times 0,0625 = 55,015625 = 55$	0	3	$\frac{3}{4}$	for <i>F</i>

The whole Gain = 258 . 18 . 4 $\frac{1}{2}$

These few Examples being well understood, are, I presume, sufficient to shew the whole Business of Fellowship, &c.

### Sect. 3. Of Bartering.

When Merchants, or Tradesmen, exchange one Commodity for another, 'tis called *Bartering*; and the only Difficulty in this Way of Dealing, lies in the due proportioning the Commodities to be exchanged; so as that neither Party sustain Loss.

*Question 1.* Two Merchants, *A.* and *B.* Barter; *A.* would exchange 5 C. 3 qrs. 14 lb of Pepper, which is worth 3 l. 10 s. per C. with *B.* for Cotton, worth 10 d. per lb Weight. How much Cotton must *B.* give to *A.* for his Pepper?

*Note,* In order to the resolving of this Question, (and all other Questions of this Nature) you must first find, by the Rule of Three, (or otherwise) the true Value of that Commodity whose Quantity is given; (which in this Question is *Pepper*). And then find how much of the other Commodity will amount to that Sum, at the Rate proposed.

First 5 C. 3 qrs. 14 lb = 5,875 } in Decimals.  
 And 3 l. 10 s. 0 d. = 3,5

Then  $1 : 3,5 :: 5,875 : 20,5625 = 20 \text{ l. } 11 \text{ s. } 3 \text{ d.}$  the true Value of the *Pepper*.

Next, 'tis easy to conceive, that *A.* ought to have as much Cotton at 10 d. per lb. as will amount to 20 l. 11 s. 3 d. which may be thus found;

$10 \text{ d.} : 1 \text{ lb.} :: 20 \text{ l. } 11 \text{ s. } 3 \text{ d.} = 4235 \text{ d.} : 493,5 \text{ lb.}$

That



## Chap. 8. Rule of Fellowship, &c. 105

That is, 4 C. 1 qr. 17  $\frac{1}{2}$  lb of Cotton. And so much *B* must give to *A* in Exchange for his 5 C. 3 qrs. 14 lb of Pepper.

*Question 2.* Two Merchants *A.* and *B.* Barter thus, *A.* hath 86 Yards of broad Cloth worth 9 s. 2 d. per Yard, ready Money; but in Barter he will have 11 s. per Yard. *B.* hath Shalloon worth 2 s. 1 d. per Yard ready Money; 'Tis required to find how many Yards of the Shalloon *B.* must give to *A.* for his Cloth, to make his Gain in the Barter equal to that of *A.*'s.

The Method of resolving this, and the like Questions, differs a little from the last Case; for in this you must first find what Advance *B.* ought to make per Yard upon his Shalloon, in Proportion to what *A.* hath done upon a Yard of his Cloth.

Thus  $\begin{matrix} \text{s.} & \text{d.} & & \text{s.} & & \text{s.} & \text{d.} & & \text{s.} & \text{d.} & \\ \{ & 9. & 2=110 : & 11=132 : & : & 2. & 1=25 : & 2. & 6=30 \end{matrix}$

the advanced Price for a Yard of *B.*'s Shalloon. Then proceed as before in the last Example.

Thus 1 Yard : 11 s. :: 86 Yards : 946 s. = 47 l. 6 s. the advanced Value of all the Cloth.

Next, If 2 s. 6 d. will buy one Yard of Shalloon, at its advanced Price, how many Yards will 47 l. 6 s. buy?

Thus 2,5 : 1 : : 946 : 378,4 Yards.

That is, *B.* must give 378  $\frac{2}{5}$  Yards of his Shalloon to *A.* for his 86 Yards of broad Cloth.

These two Examples are sufficient to shew the Learner, that the Method of Bartering, or Exchanging Commodities for Commodities, wholly depends upon a clear Understanding of the Golden Rule; which indeed is so called because of its Universal Use.

---

### Sect. 4. Of Exchanging Coins.

Exchanging the Coins of one Country for those of another is like the Business of Bartering Commodities. That is, it consists in finding what Sum of one Country Coin will be equal in Value to any proposed Sum of another Country Coin. And in order to perform that, it will be very necessary to have a true Account at all Times of the just Values of those Foreign Coins which are to be exchanged, as they are compared in Value with our *English* Coin.

I say, at all Times, because the *Par* of Exchange (as the Merchants call it) differs almost every Day from *London* to other Countries. That is, it rises and falls, according as Money is plenty or scarce; or according to the Time allowed for Payment of the Money in Exchange, &c.

P

Those

Those that desire to be fully satisfied in the common Values of Foreign Coins, may find them in a Book called the *Merchants Map of Commerce*, which for Brevity's sake I have omitted Transcribing, and only collected these few of Coin.

Foreign Coin.		English Coin.	
		l.	s. d.
French Coins.	A Denier =	0.0.0	$\frac{3}{4}$
	12 Deniers = 1 Soulz =	0.0.0	$\frac{6}{16}$
	20 Soulz = 1 Livre =	0.1.6	
	3 Livres = 1 Crown =	0.4.6	
Low-Country Coin.	A Stiver =	0.0.1	$\frac{1}{5}$
	6 Stivers = 1 Flemish Shilling =	0.0.7	$\frac{1}{3}$
	20 Stivers = 1 Gilder =	0.2.0	
	10 Gilders = $33\frac{1}{3}$ Shillings } or a Flemish Pound }	1.0.0	
	A {	Embden Doller =	0.2.3 $\frac{3}{5}$
		Campen Doller =	0.2.7 $\frac{1}{5}$
		Zealand Doller =	0.3.0
		Lyons Doller =	0.4.0
		Specie Doller =	0.5.0
Germany.	{	Ducatoon =	0.6.3 $\frac{2}{5}$
		A Rix Doller of the Empire =	0.4.5 $\frac{1}{4}$
	{	A Gilder of Nuremberg =	0.7.1
		The Livre at Leghorn =	0.0.9
		Florence Crown Courrant =	0.5.3
		Venice Ducat de Banco =	0.4.4
		The Currant Ducat =	0.3.4
In Italy and Spain.	{	The Naple Ducat =	0.5.0
		The Cadiz Ducat =	0.5.6 $\frac{1}{4}$
		The Barcelona Ducat =	0.6.0
		The Valentia Ducat =	0.5.3
		The Bergonia Ducat =	0.4.4
		The Portugal Testoon =	0.1.3
		The Piece of Eight =	0.4.6

Note, The *English* generally reckon their Exchange with other Countries by Pence, viz. other Countries value their Crowns, Dollers, or Ducats, &c. by *English* Pence, except with some Parts of the *Low-Countries*, with whom the Exchange is in Pounds Sterling.

Quest. 1. How many Dollers at 4 s. 6 d. per Doller, may one have for 162 l. 18 s.

Answer, 724 Dollers.

Thus



Thus 162 l. 18 s. = 3258 s. and 4 s. 6 d. = 45 s.

Then 45 : 1 :: 3258 : 724 the Answer.

Quest. How many *Saragosa* Ducats of 5 s. 6 d. the Ducat may be had for 275 *Bergonia* Ducats, at 4 s. 4 d. the Piece.

Answer, 216 and 3 s. 8 d. over.

Thus 5 s. 6 d. = 66 d. and 4 s. 4 d. = 52 d.

Then  $275 \times 52 = 14300$  d. = 275 Ducats.

Consequently 66) 14300 (216 $\frac{2}{3}$  the Answer requir'd.

Quest. 3. A Traveller would change 233 l. 16 s. 8 d. *Sterling* Money; for *Venice* Ducats at 4 s. 9  $\frac{1}{2}$  d. per Ducat. How many Ducats must he have?

Answer, 976 Ducats.

Thus 4 s. 9  $\frac{1}{2}$  d. = 57,5 d. and 233 l. 16 s. 8 d. = 56120 d.

Then 57,5 d.) 56120 d. (976 the Answer required.

Quest. 4. A Cashier hath received 759 Ducats, at 7 s. 6 d. per Ducat; And 579 Dollers at 4 s. 8 d. per Doller: Which he would exchange for *Flemish* Marks at 14 s. 3 d. per Piece. How many ought he to have?

Answer, 589 Marks, and 15 d. over.

For 7 s. 6 d. = 90 d. and 4 s. 8 d. = 56 d.

Then  $\begin{cases} 759 \times 90 = 68310 \text{ d. the Value of the Ducats: } \\ 579 \times 56 = 32424 \text{ d. the Value of the Dollers.} \end{cases}$

their Sum = 100734 d.

And 14 s. 3 d. = 171 d. the *Flemish* Mark in Pence.

Consequently 171) 100734 (589 &c. the Answer required.

Quest. 5. A Bill of Exchange was accepted at *London* for the Payment of 400 l. *Sterling*, for the like Value delivered in *Amsterdam*, at 1 l. 13 s. 6 d. for 1 l. *Sterling*. How much Money was delivered at *Amsterdam*?

Answer, 670 l. *Flemish*.

For 1 l. = 240 d. and 1 l. 13 s. 6 d. = 402 d.

Then 240 : 402 :: 400 : 670 The Answer required.

Quest. 6. When the Exchange from *Antwerp* to *London* is at 1 l. 4 s. 7 d. *Flemish*, for 1 l. *Sterling*: How many Pounds *Sterling* must be paid at *London*; to ballance 236 l. *Flemish* at *Antwerp*?

Answer, 192 l. *Sterling*.

Thus 1 l. 4 s. 7 d. = 295 d. and 1 l. = 240 d.

Then 295 : 240 :: 236 : 192. The Answer.

*Quest. 7.* A Merchant delivered at London 120 l. Sterling, to receive 147 l. Flemish in Amsterdam. How much was 1 l. Sterling valued at, in Flemish Money?

*Answer,* 1 l. 4 s. 6 d.

Thus  $120 : 147 :: 240 d. : 294 d. = 1 l. 4 s. 6 d. \&c.$

*Quest. 8.* A Factor hath sold Goods at Cadiz for 1468 Pieces of Eight; valued at 4 s. 6½ d. Sterling per Piece. How much Sterling Money do those Pieces of Eight amount to.

*Answer,* 333 l. 7 s. 2 d.

Thus, if  $1 = 54.5 d.$  then  $1468 \times 54.5 = 80006 d. \&c.$

*Quest. 9.* A Traveller would have an equal Number of Crowns at 5 s. 6 d. per Crown; and Dollers at 4 s. 5 d. per Piece; How many of each Sort may he have for 309 l. 8 s.

*Ans.* 624 of each.

Thus  $309 l. 8 s. = 74256 d.$

And  $5 s. 6 d. + 4 s. 5 d. = 119 d.$

Then  $119 \overline{) 74256}$  (624. the *Answer* required.

*Quest. 10.* Suppose I would exchange 527 l. 17 s. 6 d. for Dollers at 4 s. 6 d. a Piece, Ducats at 5 s. 8 d. a Piece, and Crowns at 6 s. 1 d. a Piece. And would have 2 Dollers for 1 Ducat, and 3 Dollers for 2 Crowns. How many of each sort must I have?

*Ans.* 927 Dollers, 463½ Ducats, and 618 Crowns.

For  $\left\{ \begin{array}{l} 54 d. = 1 \text{ Doller.} \\ 68 d. = 1 \text{ Ducat.} \\ 73 d. = 1 \text{ Crown.} \end{array} \right\} \text{ per Question.}$

And  $126690 d. = 527 l. 17 s. 6 d.$

Now if the Crowns, Dollers, and Ducats were to be equal in Number; then  $73 + 54 + 68$  must have been the Divisor, by which 126690 must have been divided, and the Quotient would have been the Answer to the Question. As in the last Example.

But here instead of their Sum, such Parts of them must be taken as are assigned or limited by the Question; that so the Number of some one of them may be found.

And because there must be  $\left\{ \begin{array}{l} 2 \text{ Dollers for 1 Ducat, and} \\ 3 \text{ Dollers for 2 Crowns.} \end{array} \right.$

Therefore it will be but ½ of a Ducat for one Doller, and ⅔ of a Crown for one Doller.

Con-



Consequently,  $54 + \frac{68}{2} : + \frac{2}{3}$  of 73 =  $136\frac{2}{3}$  Or  $4\frac{1}{3}^0$  will be the Divisor to find the Number of Dollers.

Thus  $4\frac{1}{3}^0$  126690 (927 the Number of Dollers.

Then  $\frac{1}{2}$  of 927 =  $463\frac{1}{2}$  is the Number of Ducats.

And  $\frac{2}{3}$  of 927 = 618 is the Number of Crowns.

Or if you please you may form Divisors to find either the Ducats or Crowns first: For if it be 2 Dollers for 1 Ducat, and 3 Dollers for 2 Crowns, as before;

Then will 6 Dollers be for 3 Ducats, and 6 Dollers for 4 Crowns.

Therefore,  $\left\{ \begin{array}{l} \frac{3}{2} \text{ of a Doller} \\ \frac{3}{4} \text{ of a Ducat} \end{array} \right\}$  will be for 1 Crown.

Consequently,  $\frac{3}{2}$  of 54 :  $+ \frac{3}{4}$  of 68 :  $+ 73 = 205$  will be the Divisor to find the Crowns first, &c.

*Quest. 11.* A Cashier is to receive 500*l*. He is offered Crowns at 6*s*.  $1\frac{1}{2}$  *d*. per Crown, which are worth but 6*s*. Or he may have Dollers at 4*s*. 5 *d*. the Piece, which are worth but 4*s*. 4 *d*. Which of these shall he receive to have the least Loss? And how much will he lose in the Payment?

1  $\left\{ \begin{array}{l} 1 \text{ Crown} = 72 \text{ d.} \\ 1 \text{ Doller} = 52 \text{ d.} \end{array} \right\}$  according to the true Values.

2  $\left\{ \begin{array}{l} 1 \text{ Crown} = 73,5 \text{ d.} \\ 1 \text{ Doller} = 53 \text{ d.} \end{array} \right\}$  the advanced Values.

Now to find which will be the least Loss; find what the advanced Value of a Doller ought to be in Proportion to that of 1 Crown.

Thus  $72 : 73,5 :: 52 : 53,22$ , &c. But he may have Dollers at 53*d*. per Piece, therefore the Payment in Dollers will be the least Loss; viz. 53 is less than 53,22. &c.

Next, to find what the whole Loss will be, divide 120000 *d*. = 500*l*. by 52. and by 53. The Difference of their Quotients will be the Loss.

Thus 52) 120000 (2307  $\frac{6}{13}$ . And 53) 120000 (2264  $\frac{8}{13}$ .

Then  $2307\frac{6}{13} - 2264\frac{8}{13} = 43\frac{2}{13}$  Dollers at 4*s*. 4 *d*. is the Loss; viz. 9*l*. 8*s*.  $10\frac{2}{3}\frac{5}{9}$  *d*.

There are other Ways of answering the last Question, but this I take to be the easiest.

*Quest. 12.* Suppose I exchange 4*l*. 10*s*. 10 *d*. for 11 Crowns and 7 Dollers; And at another time I have 4 Crowns and 3 Dollers

Dollers for 1 l. 15 s. each being of the same Value with the first. What is the Value of a Crown, and of a Doller?

First 11 Crowns + 7 Dollers = 1090 d. }  
 Second 4 Crowns + 3 Dollers = 420 d. } by the Question.

Then in order to find the Value of 1 Crown, you must cast off the Dollers by making them of the same Number; Thus,

33 Crowns + 21 Dollers = 3270 d. the first Multipl. with 3.

28 Crowns + 21 Dollers = 2940 d. the second Multipl. with 7.

Then 5 Crowns = 330 d. being their Difference.

Consequently 5) 330 (66 = 5 s. 6 d. is the Value of 1 Crown.

And 4 Crowns = 264 d.

Then will 3 Dollers = 420 d. — 264 d. = 156 d.

Consequently 3) 156 (52 d. = 4 s. 4 d. the Value of 1 Doller.

## CHAP. IX.

### Of Alligation.

**W**HEN it is required to mix several Sorts of Ingredients together; as different Sorts of Corns, Wines, Wooll, Spices, or Metals; or to compose Medicines, &c. the Method of proportioning such Mixtures, is called the *Rule of Alligation*. And is divided into two Parts or Branches; called **MEDIAL** and **ALTERNATE**.

#### Sect. I. Of Alligation Medial.

Alligation Medial, is that by which the mean Rate or Price of any Mixture is found, when the particular Quantities of the Mixtures and their Rates are given: And is thus performed.

First find the Sum of all the Quantities proposed to be mix'd; And also the Sum of all their particular Rates.

Then the Proportion will be.

Rule.  $\left\{ \begin{array}{l} \text{As the Sum of all the Quantities : Is to the} \\ \text{Sum of all their Rates :: So is any Part of} \\ \text{the Mixture : To the Mean Rate or Price of} \\ \text{that Part.} \end{array} \right.$

*Quest. I.* Suppose 15 Bushels of Wheat at 5 s. the Bushel, and 12 Bushels of Rye at 3 s. 6 d. the Bushel, were mix'd together; What



What is the Mean Rate or Price, it may be sold for a Bushel, without Loss or Gain;

This Question prepared as directed above will stand

Thus  $\left\{ \begin{array}{l} 15 \text{ Bushels of Wheat at } 5 \text{ s. per Bushel, comes to } 900 \text{ d.} \\ 12 \text{ Bushels of Rye at } 3 \text{ s. } 6 \text{ d. each, comes to } 504 \text{ d.} \end{array} \right.$

27 = their Sum.

And their Total Value = 1404 d.

Then 27 Bushels : 1404 d. :: 1 Bushel : 52 d. = 4 s. 4 d. the Answer required.

*Quest. 2.* A Grocer mixeth 36 lb of Tobacco, worth 1 s. 6 d. a Pound, with 12 lb of another Sort at 2 s. a lb. And 12 lb of a third Sort at 1 s. 10 d. the Pound. How may he sell the Mixture per Pound?

	lb	s.	d.				
First	{	36 . at 1 .	6	}	per lb Amounts to		
		12 . at 2 .	0			{	648 d.
		12 . at 1 .	10				288 d.
					264 d.		

60 = the Number of lbs. their Value = 1200 d.

Then 60 lb : 1200 d. :: 1 lb : 20 d. = 1 s. 8 d. the Answer required:

*Quest. 3.* A Vintner mixeth  $31\frac{1}{2}$  Gallons of Malaga Sack, worth 7 s. 6 d. the Gallon; with 18 Gallons of Canary at 6 s. 9 d. the Gallon;  $13\frac{1}{2}$  Gallons of Sherry at 5 s. the Gallon; And 27 Gallons of White Wine at 4 s. 3 d. the Gallon. 'Tis required to find what one Gallon of this Mixture is worth.

	Gal.	s.	d.		Pence		
First,	{	$31\frac{1}{2}$ at 7 .	6	}	per Gallon comes to		
		18 . at 6 .	9			{	2835
		$13\frac{1}{2}$ at 5 .	0				1458
		27 . at 4 .	3				810
					1377		

90 = the Number of Gallons. Their Value = 6480

Then 90 : 6480 :: 1 : 72 d. = 6 s. The Rate or Price of one Gallon, as was required.

The Proof of all Operations in these Sort of Mixtures, is done by comparing the Value of all the Mixture, being sold at the Mean Rate; with the total Value of all the particular Quantities, supposing they had been sold at their respective Rates unmix'd; if those Sums are equal the Work is true.

Sect.

## Sect. 2. Of Alligation Alternate.

*Alligation Alternate*, is that by which the particular Quantities of every Ingredient concern'd in any Mixture are found; when the particular Rates of every one of those Ingredients, and the Mean Rate are given; being (as it were) the Converse to *Alligation Medial*; as will appear by the following Operations, which admit of three Cases.

*Case I.* The particular Rates of any Ingredients proposed to be mixed, and the Mean Rate of the whole Mixture being given. To find how much of each Ingredient is requisite to compose the Mixture; when the whole Quantity, or any Part thereof is not limited.

*Quest. 1.* How much Wheat at 5 s. the Bushel, and Rye at 3 s. 6 d. the Bushel, will compose a Mixture that may be sold for 4 s. 4 d. the Bushel?

*Note,* In all Questions of this Nature, it will be convenient to place the Mean Rate so, as that it may be easily compared with the particular Rates, in order to find every one of their Differences from the Mean Rate, by Inspection only.

Thus, the Mean Rate = 52 d.  $\left\{ \begin{array}{l} \text{Wheat } 60 \text{ d.} \\ \text{Rye } 42 \text{ d.} \end{array} \right.$

Then take the several Differences between the Mean Rate, and the particular Rates; setting down those Differences alternately, and they will be the Quantities required.

Thus 52  $\left\{ \begin{array}{l} 60 \\ 42 \end{array} \right\} \left\{ \begin{array}{l} 10 = 52 - 42 \\ 8 = 60 - 52 \end{array} \right.$

That is 52 - 42 = 10 for the Quantity of Wheat.

And 60 - 52 = 8 for the Quantity of Rye, that will compose the Mixture required.

The Proof by *Alligation Medial*.

Add  $\left\{ \begin{array}{l} 10 \text{ Bushels of Wheat at } 60 \text{ d. per Bushel} = 600 \text{ d.} \\ 8 \text{ Bushels of Rye at } 42 \text{ d. per Bushel} = 336 \text{ d.} \end{array} \right.$

18 = The Number of Bushels = 936 d.

Then 18 : 936 :: 1 : 52 d. = 4 s. 4 d. the Mean Rate.

*Note,* Altho' 10 and 8 do answer the Question, as plainly appears by the Proof; yet they are not the only two Numbers; for this Question, and all others of this Kind, will admit of various Answers, and all in whole Numbers; for any two Numbers that are in the same Proportion to one another, as 10 is to 8, will as truly answer the Question.

*Viz.*



$$\text{viz. } 10 : 8 :: \left\{ \begin{array}{l} 5 : 4 \\ 15 : 12 \\ 20 : 16 \\ 25 : 20 \end{array} \right\} \text{ \&c. ad infinitum.}$$

*Quest. 2.* A Grocer would mix three Sorts of Tobacco together, viz. One Sort of 18 d. per lb. another Sort of 22 d. per lb. and a third Sort of 2 s. the lb. How much of each Sort must he take, that the whole Mixture may be sold for 20 d. the Pound.

Having set down the given Rates, as before: Then find each of their Differences from the proposed Mean Rate; and place those Differences Alternately. Thus,

$$\text{Mean Rate } 20 \left\{ \begin{array}{l} 18 \\ 22 \\ 24 \end{array} \right\} \left\{ \begin{array}{l} 4+2=24-20 \text{ and } 22-20 \\ 2=20-18 \\ 2=20-18 \end{array} \right\}$$

These Differences, viz. 6 . 2 . 2 are the Quantities required.

$$\text{Proof } \left\{ \begin{array}{l} 6 \text{ lb. of Tobacco at } 18 \text{ d. per lb. comes to } 108 \\ 2 \text{ lb. at } 22 \text{ d. the Pound comes to } 44 \\ 2 \text{ lb. at } 24 \text{ d. the Pound comes to } 48 \end{array} \right\} \text{ d.}$$

10 = the Number of lbs. Their Value = 200 d.

Then 10) 200 (20 the Mean Rate.

Or indeed any three Numbers that have the same Ratio to one another, as 6 . & 2 have, will answer the Question.

$$\text{That is } 6 : 2 :: \left\{ \begin{array}{l} 9 : 3 \\ 12 : 4 \\ 15 : 5 \end{array} \right\} \text{ \&c.}$$

But if only one of the three given Rates had been greater than the Mean Rate; as suppose 14 d. per lb. 18 d. per lb. and 24 d. per lb. And the Mean Rate 20 d. as before. Then their Differences must have been placed,

$$\text{Thus } 20 \left\{ \begin{array}{l} 14 \\ 18 \\ 24 \end{array} \right\} \left\{ \begin{array}{l} 4 \\ 4 \\ 6+2 \end{array} \right\} \text{ \&c. As before.}$$

*Quest. 3.* A Vintner would make a Mixture of Malaga, worth 7 s. 6 d. per Gallon, with Canary at 6 s. 9 d. per Gallon. Sherry at 5 s. per Gallon, and White Wine. at 4 s. 3 d. per Gallon; What Quantity of each Sort must he take, that the Mixture may be sold for 6 s. a Gallon.

In all Questions of this kind. wherein it is required to mix four Things together, two of them having their Prices greater, and

and two lesser than the Mean Rate; you must always Alligate or Compare a greater and lesser Price with the Mean Price, setting down their Differences Alternately, as in the first *Example* of this *Section*.

Thus, Mean Rate = 72 d.  $\left\{ \begin{array}{l} \text{Malaga } 90 \text{ d.} \\ \text{White } 51 \text{ d.} \\ \text{Sherry } 60 \text{ d.} \\ \text{Canary } 81 \text{ d.} \end{array} \right\} \left\{ \begin{array}{l} 21 = 72 - 51 \\ 18 = 90 - 72 \\ 9 = 81 - 72 \\ 12 = 72 - 60 \end{array} \right.$

Hence 21 Gallons of *Malaga*, 12 of *Canary*, 9 of *Sherry*, and 18 of *White*, will compose the Mixture required.

Or thus, 72  $\left\{ \begin{array}{l} \text{Malaga } 90 \text{ d.} \\ \text{Sherry } 60 \text{ d.} \\ \text{Canary } 81 \text{ d.} \\ \text{White } 51 \text{ d.} \end{array} \right\} \left\{ \begin{array}{l} 12 \text{ Malaga} \\ 18 \text{ Sherry} \\ 21 \text{ Canary} \\ 9 \text{ White.} \end{array} \right\} \text{ will, \&c.}$

Either of these Mixtures equally answer the Question, which may be easily try'd as before in the last, &c.

*Case II.* The particular Rates of all the Ingredients proposed to be mix'd, the Mean Rate of the whole Mixture, and any one of the Quantities to be mix'd being given. Thence to find how much of every one of the other Ingredients is requisite to compose the Mixture.

*Note,* This is usually called *Alligation Partial*.

*Quest. 4.* How much Wheat at 5 s. the Bushel, must be mix'd with 12 Bushels of Rye at 3 s. 6 d. a Bushel; that the whole Mixture may be sold for 4 s. 4 d. the Bushel.

In this *Case* you must set down all the particular Rates with the Mean Rate, and find their Differences just as before; without any Regard had to the Quantity given.

Thus, Mean Rate 52 d.  $\left\{ \begin{array}{l} \text{Wheat } 60 \text{ d.} \\ \text{Rye } 42 \text{ d.} \end{array} \right\} \left\{ \begin{array}{l} 10 \\ 8 \end{array} \right.$

Then  $\left\{ \begin{array}{l} \text{As the Quantity found by the Differences of the} \\ \text{same Name with the Quantity given: Is to} \\ \text{the Quantity given: : So is any of the other} \\ \text{Quantities found by the Differences: To the} \\ \text{Quantity of its Name.} \end{array} \right.$

Thus 8 : 12 :: 10 : 15. the Quantity or Number of Bushels of Wheat required.

*Quest. 5.* How much *Malaga* at 7 s. 6 d. the Gallon, *Sherry* at 5 s. the Gallon, and *White Wine* at 4 s. 3 d. the Gallon, must be mix'd with 18 Gallons of *Canary* at 6 s. 9 d. the Gallon: That the whole Mixture may be sold for 6 s. the Gallon.

The



The Terms being set down, &c. as before, will stand

Thus, Mean Rate 72 d.  $\left\{ \begin{array}{l} \text{Malaga } 90 \text{ d.} \\ \text{White } 51 \text{ d.} \\ \text{Sherry } 60 \text{ d.} \\ \text{Canary } 81 \text{ d.} \end{array} \right\} \left\{ \begin{array}{l} 21 \\ 18 \\ 9 \\ 12 \end{array} \right.$

Then, As 12 : 18 ::  $\left\{ \begin{array}{l} 21 : 31\frac{1}{2} \text{ Gallons of Malaga.} \\ 18 : 27 \text{ Gallons of White.} \\ 9 : 13\frac{1}{2} \text{ Gallons of Sherry.} \end{array} \right.$

That is,  $31\frac{1}{2}$  Gallons of Malaga, 27 of White Wine, and  $13\frac{1}{2}$  of Sherry, being mix'd with 18 Gallons of Canary, will make the Mixture required.

Or Thus, 72  $\left\{ \begin{array}{l} \text{Malaga } 90 \\ \text{Sherry } 60 \\ \text{Canary } 81 \\ \text{White } 51 \end{array} \right\} \left\{ \begin{array}{l} 12 \\ 18 \\ 21 \\ 9 \end{array} \right.$

Then, As 21 : 18 ::  $\left\{ \begin{array}{l} 12 : 10\frac{6}{21} \text{ The Malaga} \\ 18 : 15\frac{9}{21} \text{ The Sherry} \\ 9 : 7\frac{5}{21} \text{ The White Wine} \end{array} \right\} \&c.$

Proof  $\left\{ \begin{array}{l} 10\frac{6}{21} \text{ at } 90 \text{ d. each} = 925\frac{15}{21} \\ 15\frac{9}{21} \text{ at } 60 \text{ d. each} = 925\frac{15}{21} \\ 7\frac{5}{21} \text{ at } 51 \text{ d. each} = 393\frac{25}{21} \\ 18 \text{ at } 81 \text{ d. each} = 1458 \end{array} \right.$

$51\frac{9}{21}$  Value =  $3702\frac{18}{21}$   
Then  $51\frac{9}{21}$   $3702\frac{18}{21}$  (72 d. = 6 s. the Mean Rate.)

Therefore the Quantities are as truly assigned here, as in the last Work.

Case III. The particular Rates of all the Ingredients proposed to be mix'd; and the Sum of all their Quantities, with the Mean Rate of that Sum being given; To find the particular Quantities of the Mixture.

This is called *Alligation Total*, and is thus performed. Set down all the particular Rates with the Mean Rate, and find their Differences, as before: Add together all the Differences into one Sum;

Then  $\left\{ \begin{array}{l} \text{As the Sum of all the Differences: Is to the Sum} \\ \text{of all the Quantities given :: So is every particu-} \\ \text{lar Difference: To its particular Quantity.} \end{array} \right.$

Quest. 6. Let it be required to mix Wheat at 5 s. the Bushel, with Rye at 3 s. 6 d. the Bushel; So as that the whole Quantity may be 27 Bushels, to be sold for 4 s. 4 d. a Bushel. What Quantity of each must be taken to make up the Mixture?

Q 2

Mean

$$\text{Mean Rate } 52 \left\{ \begin{array}{l} \text{Wheat } 60 \text{ d.} \\ \text{Rye } 42 \text{ d.} \end{array} \right\} \left\{ \begin{array}{l} 10 \\ 8 \end{array} \right\}$$

18 = their Sum.

$$\text{Then } 18 : 27 :: \left\{ \begin{array}{l} 10 : 15 \\ 8 : 12 \end{array} \right\} \left\{ \begin{array}{l} \\ \end{array} \right\} \text{The Quantities required.}$$

*Question 7.* Suppose it were required to mix *Malaga* at 7 s. 6 d. the Gallon, with *Canary* at 6 s. 9 d. the Gallon; *Sherry* at 5 s. the Gallon; and *White Wine* at 4 s. 3 d. the Gallon: So as that the whole Mixture may be 90 Gallons; to be sold for 6 s. the Gallon. How much of each Sort will compose that Mixture?

$$\text{Mean Rate} = 72 \text{ d.} \left\{ \begin{array}{l} \text{Malaga } 90 \\ \text{White } 51 \\ \text{Canary } 81 \\ \text{Sherry } 60 \end{array} \right\} \left\{ \begin{array}{l} 21 \\ 18 \\ 9 \\ 12 \end{array} \right\}$$

60 = their Sum

$$\text{Then } \left\{ \begin{array}{l} 60 : 90 :: 21 : 31\frac{1}{2} \\ 60 : 90 :: 18 : 27 \\ 60 : 90 :: 9 : 13\frac{1}{2} \\ 60 : 90 :: 12 : 18 \end{array} \right. \begin{array}{l} \text{The Gallons of Malaga.} \\ \text{The Gallons of White Wine.} \\ \text{The Gallons of Canary.} \\ \text{The Gallons of Sherry.} \end{array}$$

$$\text{Or thus } 72 \left\{ \begin{array}{l} \text{Malaga } 90 \\ \text{Sherry } 60 \\ \text{Canary } 81 \\ \text{White } 51 \end{array} \right\} \left\{ \begin{array}{l} 12 \\ 18 \\ 21 \\ 9 \end{array} \right\}$$

60 their Sum

$$\text{Then } \left\{ \begin{array}{l} 60 : 90 : 12 : 18 \\ 60 : 90 : 18 : 27 \\ 60 : 90 : 21 : 31\frac{1}{2} \\ 60 : 90 : 9 : 13\frac{1}{2} \end{array} \right. \begin{array}{l} \text{Gallons of Malaga} \\ \text{Gallons of Sherry} \\ \text{Gallons of Canary} \\ \text{Gallons of White Wine.} \end{array}$$

Either of these Ways do equally answer the Question, as may be easily tried by *Alligation Medial*. As before, &c.

*Note,* The Work of these Proportions may be much shortned (especially when there are many Ingredients to be mix'd) if you observe the same Method as was proposed in the *Rule of Fellowship*, Page 99, &c.

I have made use of the very same Examples both in *Alligation Medial*, and *Alternate*, throughout the three Cases; being, as I presume, much better than if they had been different ones; because the Learner may (if he consider a little on them) easily perceive, not only the Difference between the two Rules, but also where-  
in



in the chief Difference of each Case in the *Alternate Rule* depends, &c. Not but that I could have inserted many various *Examples*, as also the Manner of composing Medicines, &c. which, for Brevity's sake, I have omitted, and refer those that desire to see into that Business to Sir *Jonas More's* Arithmetick, wherein he will find it largely handled. And so I shall conclude with *Alligation Alternate*, which altho' it gives true Answers to Questions of that kind, with some little Variety, according as the Ingredients are more or less in Number; as appears by the foregoing Examples. Yet it will not give all the Answers as such Questions are capable of, nor perhaps those which suit best with the present Occasion: Nor can this Imperfection be remedied by common Arithmetick; but by an Algebraic Way of Arguing it may; whereby all the possible Answers to any Question may be clearly and easily discovered; As shall be shew'd further on in the second Part.

## CHAP X.

### Of Metals, and their Specific Gravities, &c.

#### Sect. 1. Of Gold and Silver.

**P**URE GOLD, free from Mixture of other Metals, usually called *Fine Gold*, is of such a Nature and Purity that it will endure the Fire without Wasting; although it were kept continually melted: And therefore some of the *Ancient Philosophers* have supposed the *Sun* to be a *Globe* of *Liquid* or *Melted Gold*.

*Silver* having not the Purity of *Gold*, will not endure the Fire like it; yet *Fine Silver* will wast but a very little by being in the Fire any reasonable Time; whereas *Copper*, *Tin* *Lead*, &c. will not only wast, but may be calcin'd or burnt to a Powder.

Both *Gold* and *Silver* in their Purity, are so very flexible or soft (like new *Lead*, &c.) that they are not so useful either in Coin, or otherwise (except to beat into *Leaf Gold* or *Silver*) as when they are allay'd, or mix'd and harden'd with *Copper* or *Brass*. And altho' most Places differ more or less in the Quantity of such Al-  
lay, yet in *England* it is generally agreed on, that,

Standard



## Standard for Gold.

22 Carraets of Fine Gold, and 2 Carraets of Copper, being melted together, shall be esteemed the true Standard for Gold Coin, &c. The *French* and *Spanish* Gold being very near of the same Standard.

That is, if any Quantity or Weight of *Fine Gold*, be divided into twenty four equal Parts, and 22 of those Parts be mix'd with 2 of the like Parts of Copper; that Mixture is called *Standard Gold*.

Whence you may observe, that a Carraet is not any certain Quantity or Weight, but  $\frac{1}{24}$  Part of any Quantity or Weight; and the *Minters* and *Goldsmiths* divide it into 4 equal Parts, which they call Grains of a Carraet; also they subdivide one of those Grains into Halves, Quarters, &c.

## Standard for Silver.

Eleven Ounces and Two-penny Weight of Fine Silver, and Eighteen Penny-weight of Copper being melted together, is esteem'd the true Standard for Silver Coin; called *Sterling Silver*. And so in Proportion for a greater or lesser Quantity; which is a less Proportion of Allay for Silver, than the other is for Gold.

*Note*, When either Silver or Gold, is finer than Standard, 'tis called *Better*; if courser, 'tis called *Worse*; and that Betterness or Worseness, is reckoned by Carraets and Grains of a Carraet in Gold; and by Penny-weights in Silver, and is thus discovered: The Goldsmiths, or Refiners, &c. do take a small Quantity of such Gold as they intend to try (which they call making an *Affay*) and weigh it very exactly, then they put it into a *Crucible*, and melt it in a strong Fire, so long that if there be any Copper, or other Allay mixt with it, that Allay may be consum'd or burnt away: When 'tis cold they weigh it very exactly again, and if it have lost nothing of its first Weight, they conclude it is Fine Gold, but if the Loss be  $\frac{1}{24}$  Part, they call it 23 Carraets Fine, or one Carraet better than Standard: If it have lost  $\frac{2}{24}$  Parts, 'tis 22 Carraets fine, or Standard: If  $\frac{3}{24}$  Parts, it is said to be 21 Carraets fine, or rather one Carraet worse than Standard, and so in Proportion as it happens to be *Better*, or *Worse*.

In the same manner they make their *Affay* on Silver, only they compute its Loss by Penny-weights, &c.

The Author of the *Present State of England*, mention'd before (Page 32) says,

'That



## Chap. 10. Of Metals, Gravities, &c. 119

‘ That the *English* Coin may not want neither the Purity nor Weight required, it is most wisely and carefully provided, that once every Year the chief Officers of the *Mint* appear before the Lords of the Council in the *Star-Chamber* at *Westminster*, with some Pieces of all Sorts of Moneys coined the foregoing Year, taken at Adventure out of the *Mint*, and kept under several Locks, by several Persons, till that Appearance, and then by a Jury of 24 able Goldsmiths, in the Presence of the said Lords, every Piece is most exactly weighed and assay’d.

This if it were constantly practised would keep our Coin to its true Standard, &c.

Many pretty Questions may be started concerning the Fineness of Gold and Silver, &c.

### Example 1.

If an Ingot of Silver weighing 787 oz. 14 pwt. 6 grains, be 11 oz. 6 pwt. fine: How much fine Silver is there in it, and what amounts it to, at 5 s. 1½ d. the Ounce?

This Ingot is better than Standard by 4 pwt. For 11 oz. 2 pwt. = 222 pwt. the fine Silver in 12 oz. of Standard. But 11 oz. 6 pwt. = 226 pwt. the fine Silver in 12 oz. according to the Question.

First 787 oz. 14 pwt. 6 gr. = 378102 Grains.

And 12 oz. = 240 pwt.

Then as 240 : 226 :: 378102 : 356046  $\frac{1}{20}$  = 741 oz. 15 pwt. 6  $\frac{1}{20}$  gr. the fine Silver in that Ingot.

Which at 5 s. 1½ d. the Ounce, amounts to 190 l. 1 s. 6 d. and near a half-penny.

### Example 2.

If an Ingot of Gold weighing 115 oz. 13 pwt. 18 grains; Be  $\frac{1}{4}$  of a grain worse than Standard: How much Standard Gold is there in it, and what comes it to at 3 l. 11 s. an Ounce?

First 115 oz. 13 pwt. 18 gr. = 55530 Grains Troy.

Then 24) 55530 (2313.75 = a Carract of that Quantity.

And 4) 2313.75 (578.4375 = a Grain of that Carract.

Consequently 4) 578.4375 (144.609375 =  $\frac{1}{4}$  of a Grain.

Again, 2313.75 × 22 = 50902.5 ought to be the fine Gold in that Ingot, if it had been Standard:

But

But  $50902.5 - 144.609375 = 50757.890625$  is the Quantity of fine Gold according to the Question.

Therefore  $50925 : 50757.890625 :: 55530 : 55372.244 \&c.$  Grains  $= 115 \text{ oz. } 7 \text{ pwt. } 4.244 \&c.$  Grains Troy, being the Quantity of Standard Gold in that Ingot. As was required.

Next for the Value of it, at  $3 \text{ l. } 11 \text{ s. per Ounce}$ ;  $1 \text{ oz.} = 480$  Grains. And  $3 \text{ l. } 11 \text{ s.} = 71 \text{ s.}$

Consequently  $480 : 71 :: 55372.244 \&c. : 8190.4777 \&c.$   $= 409 \text{ l. } 10 \text{ s. } 5\frac{3}{4} \text{ d.}$  very near, being the Value of that Ingot. As was required.

Or the last Question may be otherwise wrought thus;  $115 \text{ oz. } 13 \text{ pwt. } 18 \text{ grains} = 115.6875$ . And  $\frac{1}{4}$  of a Grain, of a Carraet is  $\frac{1}{18}$  (*viz.* the  $\frac{1}{4}$  of  $\frac{1}{4}$ )

Then  $22 - \frac{1}{18} = 21\frac{17}{18} = 21.9375$ .

Consequently  $22 : 21.9375 :: 115.6875 : 115.358842 \&c. = 115 \text{ oz. } 7 \text{ pwt. } 4.244 \text{ Grains, } \&c.$  As before.

Next for the Value, As  $1 : 3.55 :: 115.358842 : 409.523889 = 409 \text{ l. } 10 \text{ s. } 5\frac{3}{4} \text{ d.}$  very near. As before.

## Sect 2. The Specific Gravity of Metals. &c.

I take an Enquiry made about the different Gravities, or Weights of Metals, and other Bodies, to be (not only a Work of Curiosity, but also) of very good Use upon many Occasions. Therefore several Authors have given us such Proportions, or Difference of their Weights, as they are said to have one to another; supposing every one of them to be of the same Magnitude or Bigness. Some of which I shall here insert.

1. Henry Van Etten, in his *Mathematical Recreations*, Printed Anno 1633, sets down the Proportions of their Weights. Thus;

Gold 1875. Lead 1165. Silver 1040. Copper 910. Iron 810. Tin 750. Water 100.

2. One Alsted, in his *Encyclopædia*, Printed 1649, hath them Thus;

Gold 1875. Quicksilver 1500. Lead 1165. Silver 1040. Copper 910. Iron 806. Tin 750. Honey 150. Water 100. Oil 90. These seem to be taken from those of Van Etten's, with some Additions only.

3. The Ingenious Mr. Oughtred, in his *Circles of Proportions*, Printed Anno 1660, hath their Proportions according to the Experiments of one Marinus Ghetaldi, in his Tract called *Archimedes Promotus*. Thus;

Gold 3990. Quicksilver 2850. Lead 2415. Silver 2170. Brass 1890. Iron 1680. Tin 1554.



4. In the *Philosophical Transactions*, (Number 169 and 199) there is an Account of a great many Experiments of this kind; from whence I collected these following. *Viz.* Gold 18888. Mercury 14019. Lead 11343. Silver 11087. Copper 8843. Hammer'd Brass 8349. Cast Brass 8100. Steel 7852. Iron 7643. Tin 7321. Pump-water 1000.

These last Proportions being approved of, and published by Order of the Royal Society, seem to be unquestionably true: Nevertheless because they differ so much from the before-mentioned (*and those from one another*) I have for my own Satisfaction made several Experiments of that kind: And have (*I presume*) obtained the Proportions of Weight that one Body bears to another, of the same Bulk or Magnitude, as nicely as the Nature of such Matter, as may be contracted or brought into a lesser Body (*viz.* either by Drying, or Hammering, or otherwise) will admit of; which are as followeth.

A Cubic Inch of

- Fine Gold is
- Standard Gold,
- Quicksilver,
- Lead,
- Fine-Silver,
- Standard-Silver,
- Rose-Copper,
- Plate-Brass,
- Cast-Brass,
- Steel,
- Common Iron,
- Block-Tin,
- Fine Marble,
- Common Glass,
- Alabaster,
- Dry Ivory,
- Dry Box wood,
- Sea-Water.
- Common Clear }  
Water,
- Red Wine,
- Proof Spirits or }  
Brandy,
- Sound dry Oak
- Lin-seed Oil,
- Oil-Olive,

Ounces Troy.	Ounces Aver.
10,359273	= 11,365602
9,962625	= 10,930422
7,384411	= 8,101753
5,984010	= 6,553885
5,850035	= 6,418324
5,556769	= 6,096569
4,747121	= 5,208369
4,404273	= 4,832116
4,272409	= 4,630300
4,142127	= 4,544505
4,031361	= 4,422979
3,861519	= 4,236538
1,429411	= 1,568859
1,360841	= 1,493037
0,988456	= 1,084477
0,962083	= 1,055542
0,543282	= 0,596057
0,542742	= 0,594894
0,527458	= 0,578697
0,523766	= 0,574646
0,489268	= 0,536796
0,489008	= 0,536569
0,491591	= 0,539345
0,481569	= 0,528350



In this Table you have the Specific Gravity or Weight of a Cubic Inch, of various Sorts of Bodies, both in Troy Ounces and Averdupois Ounces, and Decimal Parts of an Ounce, which I can assure you required more Charge, Care and Trouble, to find out nicely, than I was at first aware of.

Now from hence it will be easy to determine the Weight of any proposed Quantity, of the same Matter and Kind with those in the Table; its solid Content being given in Cubic Inches. For it is plain, that if the Number of Cubic Inches contained in any given Quantity, be multiplied with the Tabular Weight of one Inch (of the same kind of Matter) the Product will be the Weight of that Quantity in Ounces, &c.

*Example.*

Suppose it were required to find the Weight of a Piece of Marble, containing three Solid Feet, and 40 Cubic Inches.

First  $1728 \times 3 = 5184$  the Cubic Inches in 3 Solid Feet.

And  $5184 + 40 = 5224$  the Number of Cubic Inches in the Piece of Marble.

Then  $5224 \times 1,429411 = 7410,066624$  Ounces Troy.

Or  $5224 \times 1,568859 = 8195,719416$  Ounces Averdupois.

The Weight of that Piece of Marble, in Ounces, &c. which is easily brought into Pounds, &c. The like for any of the rest.

The Converse of this Work is easy; viz. If the Weight of any proposed Quantity be given, thence to find the solid Content of that Quantity in Cubic Inches, &c.

Thus, *Divide the given Weight of the proposed Quantity (it being first reduced into Ounces, &c.) by the Tabular Weight of one Inch (of the same kind of Matter) and the Quotient will be the Number of Cubic Inches contained in that Quantity.*

*Note,* If you would find what Weight any Quantity of those Bodies mentioned in the Table will have, when it is immersed or put into Water, you must subtract the Weight of an equal Quantity of Water (with that of the Body) from the Weight of the proposed Body (if it be heavier than Water) and there will remain the Weight required. As for Instance,

A Cubic Inch of Lead	= 5,984010	} Ounces Troy, &c.
A Cubic Inch of Water	= 0,542742	

their Difference is, = 5,441268 the Weight of a Cubic Inch of Lead in the Water, &c.



C H A P. XI.

Evolution, or Extracting the Roots out of all Single Powers; by one General Method.

Sect. I.

**E**volution is the Unravelling, or as it were the Unfolding and Resolving any proposed Power or Number, into the same Parts of which it was composed, or supposed to be made up. Now in order to perform that, it will be convenient to consider how those Powers are composed, &c.

A Square Number is that which is equally equal; or which is contained under two equal Numbers. *Euclid 7. Def. 18.*

Thus the Square Number 4 is composed of the two equal Numbers 2 and 2. *viz.*  $2 \times 2 = 4$ .

Or the Square Number 9 is composed of the two equal Numbers 3 and 3. *viz.*  $3 \times 3 = 9$ . According to *Euclid*.

That is, if any Number be multiplied into it self; that Product is called a Square Number.

A Cube is that Number, which is equally equally equal, or which is contained under three equal Numbers. *Eucl. 7. Def. 19.*

Thus the Cube Number 8 is composed of the three equal Numbers 2 and 2 and 2. *viz.*  $2 \times 2 \times 2 = 8$ , &c.

That is, if any Number be multiplied into it self, and that Product be multiplied with the same Number; the second Product is called a Cube Number.

These Two, *viz.* the Square, and Cube Numbers borrow their Names from Geometrical Extensions or Figures; as from the three signal Quantities mention'd in *Page 2*.

That is, a Root is represented by a Line or Side, having but one Dimension. *viz.* that of Length only.

The Square is a Plain or Figure of two Dimensions, having equal Length and Breadth. The Cube is a solid Body of three Dimensions; having equal Length, Breadth, and Thickness: But beyond these three, Nature proceeds not, as to local Extension. That is, the Nature of Place or Space, admits no Room for other Ways of Extension, than Length, Breadth and Thickness. Neither is it possible to form, or compose any Figure or Body beyond that of a Solid.

And therefore all the Superior Powers above the Cube or third Power; as the Biquadrat or fourth Power, the Surfsolid or fifth Power, &c. are best explain'd and understood by a Rank or Series of Numbers in Geometrical Proportion.

For Instance:

Suppose any Rank of Geometrical Proportionals, whose first Term and Ratio are the same; And to them let there be assigned a Series

a Series of Numbers in *Arithmetical Progression*, beginning with an Unit or 1. whose common Difference is also 1. as in *Page 79*.

Thus  $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \text{ Indices.} \\ 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64 \cdot 128 \text{ \&c. in } \div \end{array} \right.$

Then are those Numbers in  $\div$  produced by a continued Multiplication of the first Term or Root into it self; And those in *Arithmetical Progression* or *Indices*, do shew what Degree or Power each Term in the *Geometrical Proportion* is of.

For *Example*; In this Series of  $\div$  2 is both the first Term or Root, and common Ratio of the Series.

Then  $2 \times 2 = 4$  the second Term or Square.

And  $2 \times 2 \times 2 = 8$  Or  $4 \times 2 = 8$  the Cube or third Term.

Again  $2 \times 2 \times 2 \times 2 = 16$  Or  $8 \times 2 = 16$  the fourth Term or Biquadrat. And so on for the rest.

*Note*, This is called *Involution*, viz. When any Number is drawn into it self, and afterwards into that Product, &c. 'tis said to be so often involved into it self; And the *Indices* are the Exponents of their respective Powers so involved.

And according to these *Involutions*, is formed the following *Table of Powers*; wherein the Root is only one single Figure.

Root, or single Side.	Square or 2d Power.	Cube, or the 3d Power.	Biquadrat, or Square Squared; being the 4th Power.	Sursolid, or the 5th Power.	Square Cubed, or Cube Squared; the 6th Power.	The second Sursolid, or seventh Power.	The Biquadrat Squared, or the 8th Power.	The Cube Cubed, or the ninth Power, &c.
Index (2)	Index (3)	Index (4)	Index (5)	Index (6)	Index (7)	Index (8)	Index (9)	Index (9)
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

This *Table* plainly shews ( by Inspection ) any Power ( under the tenth ) of all the nine Figures; and from thence may be taken



taken the nearest Root of any Square, Cube, Biquadrat, &c. of any Number whose Root or Side is a single Figure.

But if the Root consist of two, three, or more Places of Figures, then it must be found by Piece-meal, or Figure after Figure, at several Operations.

The Extraction of all Roots, above the Square (*viz.* of the Cube, Biquadrat, Surfolid, &c.) hath heretofore been a very tedious and troublesome Piece of Work: All which is now very much shortned, and rendred easy, as will appear further on.

When any Number is proposed to have its Root extracted, the first Work is to prepare it. by Points set over (or under) their proper Figures; according as the given Power, whose Root is sought, doth require; and that's done by considering the Index of the given Power, which for the Square is 2. for the Cube is 3. for the Biquadrat is 4. &c. (as in the preceding Table). Then allow so many Places of Figures in the given Power, for each single Figure of the Root, as its Index denotes; always beginning those Points over the Place of Unity, and ascend toward the Left Hand if the given Number be Integers, and descend towards the Right Hand in Decimal Parts. As in these following.

Suppose any given Number; As 75640387246 which I all along hereafter call the Resolvend.

Then if it be required to extract any of the following Roots, it must be pointed (according to the foremention'd Consideration) in this Manner.

<i>Viz.</i> For the	{	Square Root Thus	75640387246
		Cube Root	75640387246
		Biquadrat Root	75640387246
		Surfolid Root	75640387246

Or suppose the Number to be 0,674035982

Then for the	{	Square Root Thus	0,6740359820
		Cube Root	0,674035982
		Biquadrat Root	0,674035982000

Now the Reason of Pointing the given Resolvend in this Manner; *viz.* the allowing two Figures in the Square; three Figures

Figures in the Cube, and four Figures in the Biquadrat, &c. For one Figure in the Root, may be made evident several Ways; but I think 'tis easily conceiv'd from the *Table of single Powers*, wherein you may observe that all the Powers of the Figure 9, (which is but a single Figure) have the same Number of Places of Figures, as the Index of those Powers denotes: Therefore so many Places of Figures must be taken or assigned for every single Figure in the Root. Consequently by these Points is known how many Places of Figures there will be in the Root, *viz.* So many Points as there are, so many Figures there must be in the Root, and whether they must be Integers, or Decimal Parts, is easily determined by the respective Places of the Points.

### Sect. 2. To Extract the Square Root.

And first how to extract the Square Root, according to the common Method.

Having pointed the given Resolvend into Periods of two Figures, as before directed; then by the *Table of Powers*, (or otherwise) find the greatest Square that is contained in the first Period towards the Left-Hand; (setting down its Root, like a Quotient Figure in Division) and subtract that Square out of the said Period of the Resolvend: To the Remainder bring down the next Period of Figures, for a Dividend, and double the Root of the first Square for a Divisor; enquiring how oft it may be had in that Dividend; So as when the Quotient Figure is annexed to the Divisor, and that increased Divisor being multiplied with the same Quotient Figure, the Product may be the greatest Number that can be taken out of that Dividend; which subtract from the said Dividend, and to the Remainder bring down the next Period of Figures, for another new Dividend: Then see how often the last increased Divisor, can be had in the new Dividend; (with the same Caution as before, *viz.*) so as that the Quotient Figure being annexed to the Divisor, and that increased Divisor multiplied with the same Quotient Figure, their Product may be the greatest Number that can be subtracted from the new Dividend. (As before). And so proceed on from Period to Period; (*viz.* from Point to Point) in the very same Manner, until all be finished.

An *Example* or two being well observed will render the Work of forming the new Divisors, &c. more plain and easy, than can be expressed in a Multitude of Words.

*Example*



# Chap. 11. Of Extracting Roots, &c. 127

*Example 1.* Let it be required to extract the Square Root out of 572199960721. This Resolvend being prepared or pointed as before directed, will stand

Thus 572199960721 ( 756439 the Root.  
49 = the greatest Square in 57.

1 Divisor	145)	821	
	5	725	= 145 X 5
2 Divisor	1506)	9699	
	6	9036	= 1506 X 6
3 Divisor	15124)	66396	
	4	60496	= 15124 X 4
4 Divisor	151283)	590007	
	3	453849	= 151283 X 3
5 Divisor	1512869)	13615821	
	9	13615821	= 1512869 X 9

Proof 756439 X 756439 = 572199960721 the Resolvend.

*Example 2.* What's the Square Root of 1850701,764025?

Operation 1850701,764025 ( 1360,405

23)	85	
3	69	
266)	1607	
6	1596	
27204)	1101,76	
4	1088,16	
2720805)	13,604025	
5	13,604025	
	(0)	

Hence 1360,405 is the Root required.

*Example 3.* What's the Square Root of 0,06076225 Decimal Parts?

Operation 0,06076225 (0,2465 the Root required.

,44	207
4	176
,486	3162
6	2916
,4925)	24625
5	24625
	(0)

Proof

0,2465 X 0,2465 =  
0,06076225 the  
Resolvend.

What

What is here done in Whole Numbers, Mix'd Numbers and Decimals, may also be done in Vulgar Fractions; if you first change the given Fraction into Decimals. (As in *sect. 5. p. 68.*)

*Example 4.* Let it be required to extract the Square Root of  $\frac{16}{25}$ .  
First  $\frac{16}{25} = 0,64$ .

Then  $0,64$  (,8 the Root required.

$$\begin{array}{r} .64 \\ \hline (0) \end{array}$$

In these four Examples the Resolvend hath been a perfect Square; and therefore the Root hath been extracted without leaving any Remainder: But it very often happens that the Resolvend is not a true figurate Number, according to the proposed Power. That is, 'tis not a perfect Square, Cube, Biquadrat, &c. And then something will remain after the Extraction hath been made throughout all the Points. Such Numbers are called *Surd Numbers*, and their Roots can never be truly found, but will become a continued Series *ad infinitum*: If to the Remainder there be still annexed Cyphers according as the proposed Power requires, viz. by Two's in the Square; Three's in the Cube; Four's in the Biquadrat, &c. And the Operations continued on as before.

*Example 5.* Suppose it were required to extract the Square Root of 6968.

Operation	$6968$ (83,4745, &c.
$163)$	$64$
$\underline{3}$	$\underline{568}$
$1664)$	$489$
$\underline{4}$	$79,00$
$16687)$	$\underline{6656}$
$\underline{7}$	$124400$
$166944)$	$\underline{116809}$
$\underline{4}$	$759100$
$1669485)$	$\underline{667776}$
$\underline{5}$	$9132400$
$1669490$	$\underline{8347425}$
	$784975$ &c.

Thus the Root of any *Surd Number* may be continued on to what Exactness you please, but cannot be truly found.

In my *Compendium of Algebra*, Chap. 9. I have proposed another Way of extracting the Square Root, and there given *Examples* of the Work: which to avoid Prolixity is this;

- Having



# Chap. 11. Of Extracting Roots, &c. 129

Having pointed the given Resolvend, and taken the greatest Square to the first Point from it, as before. Then divide the Remainder of the whole Resolvend by 2 (that is, halve it) and point it a new. (This I call a new Dividend.) Then make the Root of the first Square a Divisor, enquiring how oft it may be found in the new Dividend to the next Figure forward, reserving that Figure under the next Point, for the half Square of the Quotient Figure. Which being found, multiply the Divisor with it, adding to that Product the Tens of the Half Square if there be any; As in plain Division.

Then annex the Quotient Figure to the last Divisor for a new Divisor, with which proceed in all Respects as with the last Divisor; and so until all be finished.

*Example 6.* What's the Square Root of 2990667969

$$\begin{array}{r}
 \text{Operation} \quad \begin{array}{r} \cdot \cdot \cdot \cdot \cdot \\ 2990667969 \\ - 25 \end{array} \quad (5 \text{ The first Single Root.} \\
 \hline
 2) \quad 490667969 \quad \text{The Remainder to be divided by 2.} \\
 \\
 \text{First Root } 5) \quad \begin{array}{r} \cdot \cdot \cdot \cdot \cdot \\ 245333984,5 \end{array} \quad (54687 \\
 \quad + 4 \quad 208 = 5 \times 4 : + \frac{1}{2} \text{ the Square of 4 viz. } \frac{16}{2} = 8. \\
 \hline
 \text{Divisor } \begin{array}{r} 54) \quad 3733 \\ + 6) \quad 3258 = 54 \times 6 : + \frac{1}{2} \text{ the Square of 6.} \end{array} \\
 \hline
 \text{Divisor } \begin{array}{r} 546) \quad 47539 \\ + 8) \quad 43712 = 546 \times 8 : + \frac{1}{2} \text{ the Square of 8} \end{array} \\
 \hline
 \text{Divisor } \begin{array}{r} 5468) \quad 382784,5 \\ + 7 \quad 382784,5 = 5468 \times 7 : + \frac{1}{2} \text{ the Square of 7} \end{array} \\
 \hline
 \quad \quad \quad (0)
 \end{array}$$

Hence the Root is found to be 54687 As was required.

All the Difficulty in this Method is only in the true placing of the half Square of the Quotient Figure, when it happens to be an odd Number; In that Case you must bring down one Figure more of the Dividend; viz. of the next Period; under which, place the odd 5 that will always arise from the half Square of an odd Number: As 7 whose Square is 49; the half of which is 24,5 to be placed as in the last Operation of this Example.

N. B. When the Number of Figures in the Root of any Surd Number are limited; you need not proceed in extracting the whole Root as before; but only to one Figure more than half the designed Number of Figures; for the rest may be obtained by plain Division only.

*Example 7.* Suppose it were required to extract the Square Root of 7 (a Surd Number) to have 12 Places of Figures in it.

7 4	( 2,645751 First Part of the Root.
<hr style="width: 100px; margin: 0 auto;"/>	
Remainder 3 .	
2 )	1,50 = half the Remainder.
+ ,6	1,38 = 2 × ,6 : + $\frac{1}{2}$ the Square of 0,6 = 0,18
<hr style="width: 100px; margin: 0 auto;"/>	
2,6 )	1200
+ 0,4	1048
<hr style="width: 100px; margin: 0 auto;"/>	
2,64 )	152000
+ ,005	132125
<hr style="width: 100px; margin: 0 auto;"/>	
2,645 )	1987500
+ ,0007	1851745
<hr style="width: 100px; margin: 0 auto;"/>	
2,6457 )	13575500
+ ,00005	13228625
<hr style="width: 100px; margin: 0 auto;"/>	
2,64575 )	34687500
+ ,000001	26457505
<hr style="width: 100px; margin: 0 auto;"/>	
2,645751	8229995

Having thus got 7 of the 12 Figures required in the Root; the rest may be easily found by the contracted Way of Division proposed in *Page 68*.

Thus, 2,645751)	8229995	( 2,64575131106
....	7937253	
	292742	
	264575	
	28167	
	26457	
	1710	
	1697	
	(13)	

Hence I find the Root of 7 to be 2,6457131106 As was required.

Thus you have two Ways of extracting the Square Root, either of them may be practised as every one likes best.

Sect.



Sect. To Extract the Cube Root.

The Method that I shall here propose for extracting the Cube Root admits of two Cases; both which are to be very well observed.

Having pointed the given Resolvend, (as before directed) *viz.* into Periods of three Figures; then seek a Cube Number by the Table of Powers (or otherwise) that comes the nearest to the first Period of the Resolvend, whether it be greater or less than that Period.

*Case I.* If the Cube Number so taken, be less than the first Period of the Resolvend.

*Call its Root Less than Just.*

And subtract that Cube from the first Period of the Resolvend.

*Case 2.* But if that Cube be greater than the first Period of the Resolvend.

*Call its Root More than Just.*

And subtract the Resolvend from that Cube, annexing Cyphers to it, that so Subtraction may be made.

To the first Root whether it be less, or more than just, annex so many Cyphers as there are remaining Points over the whole Numbers of the Resolvend, and multiply it with 3; then make that Product a Divisor; by which you must divide the Difference between the Resolvend and the foresaid Cube, then will that Quotient be the Resolvend depressed to a Square; and therefore it must be pointed as such: *viz.* into Periods of two Figures each. That being done, make the first Root (without those Cyphers that were annex'd to it) a Divisor, enquiring how oft it may be found in the first Period of the new Resolvend, (as before in extracting the Square Root) with this Consideration, that if the Root, (now a Divisor) be less than just, as in *Case I.* you must annex the Quotient Figure to it, and then multiply the Root so encreased, into the said Quotient Figure; setting down the Units Place of their Product under the pointed Figure of that Period, subtracting it, as in Division. And so on from one Period to another. As before.

But if the said Root (now a Divisor) be more than just, as in *Case 2.* Then you must subtract the Quotient Figure from a Cypher annexed, or supposed to be annexed to the said Divisor; multiplying the Root so decreased into the Quotient Figure; setting down their Product as before, &c. An Example or two in each Case will render the Work plain and easy.

## Example 1.

What's the Cube Root of 146363183 the given Resolvend, to be pointed thus  $\dot{1}46\dot{3}6\dot{3}18\dot{3}$  (5 the first Root, less than just.  $125 =$  the nearest Cube to 146

$500 \times 3 = 1500$ )  $21363183$  (14242,12 New Resolvend.

First Root 5 )  $14242,12$  (527 the Root required.

$\begin{array}{r} + 2 \\ \hline \end{array}$   $\begin{array}{r} 104 \\ \hline \end{array}$

1 Divisor 52 )  $\begin{array}{r} 3842 \\ \hline \end{array}$

$\begin{array}{r} + 7 \\ \hline \end{array}$   $\begin{array}{r} 3689 \\ \hline \end{array}$

2 Divisor 527 (153) the Remainder to be rejected.

Here the Root 527 is the true Root at the first Operation, as may be easily tried by involving it.

That is  $527 \times 527 \times 527 = 146363183$  the given Resolvend. But if it had not been the true Root; then every thing that hath been here done must have been repeated; only instead of the first single Root (*viz.* 5) you must have taken the encreased Root (*viz.* 527) and this I call a second Operation; which would increase the last Root to nine Places of Figures; *viz.* every Operation triples the Number of Places in the last Root; as will appear further on.

N. B It often happens that four, or sometimes five Places of Figures may be taken into the Root; especially when the second Place proves to be a Cypher. That is, when the first Cube comes very near to the first Period of the Resolvend.

## Example 2.

What's the Cube Root of  $\dot{6}7507824239$  (4000 Root less than just, First nearest Cube = 64

Root  $4000 \times 3 = 12000$  )  $3507824239$  (292318,68

$\begin{array}{r} + 4 \\ \hline \end{array}$   $\begin{array}{r} 292318,68 \\ \hline \end{array}$  (4071,8

1 Divisor 40 )  $\begin{array}{r} 2923 \\ \hline \end{array}$

$\begin{array}{r} + 7 \\ \hline \end{array}$   $\begin{array}{r} 2849 \\ \hline \end{array}$

2 Divisor 407 )  $\begin{array}{r} 7418 \\ \hline \end{array}$

$\begin{array}{r} + 1 \\ \hline \end{array}$   $\begin{array}{r} 4071 \\ \hline \end{array}$

3 Divisor 4071 )  $\begin{array}{r} 3347,68 \\ \hline \end{array}$

$\begin{array}{r} + 8 \\ \hline \end{array}$   $\begin{array}{r} 3257,44 \text{ \&c.} \\ \hline \end{array}$

Root = 40718

In this Example I have taken five Figures into the Root, because the second Place proved to be a Cypher. And in these five



## Chap. 11. Of Extracting Roots, &amp;c. 133

five the Excess is not an Unit in the last Place; for if there were made a second Operation, the Root would be 4071,78 &c. as may be easily tried.

*Example 3.*

Let it be required to extract the Cube Root out of this Number;

*Viz.* 976379602989073960279630298890

The nearest Cube to 976 is 1000 whose Root is 10 being more than just.

[illegible]

+ 976379602989073960279630298890 the Resolvend.

Remains 23620397010926039720369701110

**The First Root** 10000000000  $\times 3 = 30000000000$  **the Divisor**

Then 30000000000) 23620397010926039720369701110 (\*)

First Root 10 )      \*7873̣465̣670̣3̣0867990̣      New Resolvend.  
— 00                  (007929

1. Divisor	<u>100</u>	<u>7873</u>	
	<u>7</u>	<u>6951</u>	
2. Divisor	<u>993</u>	<u>92246</u>	first Root = 10000000000
	<u>9</u>	<u>89289</u>	<u>007929</u>
3. Divisor	<u>9921</u>	<u>295756</u>	<u>9920710000</u>
	<u>2</u>	<u>198416</u>	
4. Divisor	<u>99208</u>	<u>9734070</u>	
	<u>9</u>	<u>8928639</u>	
5 Divisor	<u>992071</u>	<u>66</u>	

At this first Operation I take but 99207 to which I annex 5 Cyphers for the remaining Points, viz. 9920700000 which being involved to the third Power or Cube, for a second Operation,

will be	97639815602274300000000000000000	
—	976379602989073960279630298890	Resolvend
Remains	<u>18553033669039720369701110</u>	

The last Root  $9920700000 \times 3 = 29762100000$  the new Divisor.

Then 29762100000) 18553033669039720369701110 (\*)

\* 623377841921091 The Quotient or New Resolvend.

Last Root 99207 being more than juſt, therefore the new Quo-  
tients muſt be ſubtracted, as in the laſt Operation.

Thus

Thus

	99207 )	623377841	921091	( 62836,45
	— 6	5952384		
Divisor	992064	28139441		
	— 2	19841276		
Divisor	9920638	8298165	92	
	— 8	7936509	76	
(*)	99206372	361656	1610	
	— 3	297619	1151	
	992063717	64037	045991	
	— 6	59523	822984	
	9920637164	4513	22300700	
	— 4	3968	25486544	
	99206371636	544	9681415600	
	— 5	496	0318581775	
	9920637163,55		&c.	
Last Root 9920700000				
— 62836,45 &c.				
9920637163,55				

Note, In this Operation all the Figures on this side the Line, and all the new Divisors after the (\*) are useless, and might have been omitted.

the Root required.

Thus I have obtained the Cube Root to twelve Places of Figures, viz. 9920637163,55 at two Operations; being but an Unit too much in the last Place of it, as may be tried by involving it to a Cube, and comparing that Cube with the given Resolvend.

In the same manner the Cube Roots of Decimal Parts; or of Vulgar Fractions, being first changed into Decimals, may be extracted.

#### Sect. 4. To Extract the Biquadrat Root.

In extracting the Biquadrat Root, or that of the fourth Power; (and indeed the Roots of all even Powers) there are some small Difficulties, not so easily express'd and explain'd in a few Words, as they are by an *Algebraic Theorem* (such as shall be shewed further on) I have therefore in this Place, made choice of extracting such Roots by two several Extractions; and the rather, because I presume the Reader by this Time thoroughly acquainted with the Business of extracting the Square Root, by which this may easily be performed. Thus;

First, Extract the Square Root of the proposed Resolvend, then the Square Root of that first Root will be the Biquadrat Root required.

Example,



# Chap. 11. Of Extracting Roots, &c. 135

*Example 1.* What's the Biquadrat Root of 4857532416?  
First extract its Square Root,

Thus  $\begin{array}{r} 4857532416 \\ - 36 \\ \hline 1257532416 \end{array}$  the greatest Square, whose Root is 6.  
Remainder to be divided by 2.

First Root 6 )  $\begin{array}{r} 628766208 \\ + 9 \\ \hline 69 \\ + 6 \\ \hline 696 \\ + 9 \\ \hline 6969 \end{array}$  ( 69696  
 $\begin{array}{r} 5805 \\ - 4826 \\ \hline 4158 \\ - 668620 \\ \hline 626805 \\ - 418158 \\ \hline 418158 \\ \hline (0) \end{array}$

Then  $\begin{array}{r} 69696 \\ - 4 \\ \hline 29696 \end{array}$  being the first Root, whose Square Root must now be extracted.  
Remainder to be divided by 2.

First Root 2 )  $\begin{array}{r} 14848 \\ + 6 \\ \hline 26 \\ + 4 \\ \hline 264 \end{array}$  ( 264 the Biquadrat Root as was required.  
 $\begin{array}{r} 138 \\ - 1048 \\ \hline 1048 \\ \hline (0) \end{array}$

This is so easy I need not insert any more Examples.

## Sect. 5. To Extract the Surfsolid Root.

Having pointed the given Resolvend according as its *Index* denotes; viz. into Periods of five Figures; seeking such a Surfsolid Number in the *Table of Powers*, or otherwise, as comes the nearest to the first Period of the Resolvend, whether greater or less; and call its respective Root accordingly; viz. *More* than *Just*; or *Less* than *Just*; annexing so many Cyphers to it as there are remaining Periods of whole Numbers in the Resolvend. As before in extracting the Cube Root.

Then find the Difference between the Resolvend, and the Surfsolid Number so taken, by subtracting the lesser from the greater (as before in the Cube). Next find the Cube of the aforesaid Surfsolid

Sur-solid Root with its annexed Cyphers, (which you may also do by the *Table of Powers*) and multiply that Cube with 5 the Index of the Sur-solid, the Product must be a Divisor, by which the Difference between the Resolvend and the Sur-solid Number must be divided; that so it may be depressed to a Square (as before in the Cube (which must be pointed into Periods of two Figures each, calling it the new Resolvend, (as before). Then make the first Root, without its Cyphers, a Divisor enquiring, how oft it may be found in the first Period of the new Resolvend, with this Consideration, if the Root (now a Divisor) be less than just, you must annex twice the Quotient Figure to it; but if it be more than just, you must subtract twice the Quotient Figure from a Cypher either annexed, or suppos'd to be annex'd to that Divisor or Root, multiplying it so encreased, or diminished, with the said Quotient Figure, setting down their Product, &c. as before. An *Example* in each Case will render it plain and easy.

*Example 1.* Suppose it be required to extract the Sur-solid Root out of this Number 12309502009375.

12309502009375 The Resolvend pointed.

The nearest Sur-solid Number to 1230, the first Period of the Resolvend, is 1024, whose Root is 4, being *less* than *just*.

Therefore 12309502009375  
— 1024

2069502009375 their Difference.

Next the Cube of 400 is 64000000 per *Table*, &c.

And 64000000  $\times 5 = 320000000$  the Divisor

Then 320000000 ) 2069502009375 (6467 &c.

First Root 4 ) 6467 ( 15  
+ 1  $\times 2 = 2$  42

First Root = 400  
+ 15

Divisor 42 2267  
+ 5  $\times 2 = 10$  2150

the true Root 415 as required.

Divisor 430 (117) the Remainder to be rejected:

That is, 415 is the Sur-solid Root of the given Resolvend. As may be easily tried by involving it to the fifth Power.

*Viz.*  $415 \times 415 \times 415 \times 415 \times 415 = 12309502009375$  the given Resolvend.

*Example.*



# Chap. I I. Of Extracting Roots, &c. 137

*Example 2.* What's the Surfolid Root of 2327834559873

The nearest Surfolid Number to 232 is 243 whose Root is 3 being more than just.

$$\begin{array}{r} \text{Therefore } 2430000000000 \\ - \quad 2327834559873 \\ \hline \end{array}$$

Remains 102165440127 For a Dividend.

The Cube of 300 is 27000000 And  $27000000 \times 5 = 135000000$   
Then 135000000 ) 102165440127 ( 756,7810 New Resolvend.

First Root 300 )	756,7810	( 2,558
— 02 X 2 = 4	592	
Divisor 296 )	164,78	
— ,5 X 2 = 1.0	147,50	
2. Divisor. 295,0 )	17,2810	
— ,05 X 2 = ,10	14,7450	
294,90	2,53600 &c.	

The first Root was 300, being more than just.  
Therefore it is — 02.558

The next Root, 297,442 And is very near the true  
Root which is 297,436 &c. Now the Reason why  
this Root comes out to so many Places of Figures at the first O-  
peration; is because the first Surfolid Number was so near the  
Resolvend, &c. As before.

## Sect. 6. To Extract the Root of the Square Cubed.

This may be easily performed by two Extractions; according as its Name denotes.

Thus, First extract the Square Root of the given Resolvend; then extract the Cube Root of that Square Root: And it will be the Root required. That is, it will be the Root of the sixth Power.

Or thus, First extract the Cube Root of the Resolvend, then extract the Square Root of that Cube Root: And it will be the Root required.

*Example 1.* Let it be required to extract the Square Cubed Root out of this Number 145220537353515625 the Resolvend.

First I extract the Square Root of this Resolvend, which I take to be the best and easiest Way.

T

Thus

Thus

145220537353515625

— 9

Remains

55220537353515625 To be halved,

$$\begin{array}{r}
 \text{Then } 3 \ ) \quad 27610268676757812,5 \quad ( \quad 381078125 \\
 \underline{+ \quad 8} \quad 272 \\
 \quad 38 \ ) \quad 4102 \\
 \underline{+ \quad 10} \quad 3805 \\
 \quad 3810 \ ) \quad 2976867 \\
 \underline{+ \quad 7} \quad 2667245 \\
 \quad 38107 \ ) \quad 3096226 \\
 \underline{+ \quad 8} \quad 3048592 \\
 \quad 381078 \ ) \quad 47634757 \\
 \underline{+ \quad 1} \quad 38107805 \\
 \quad 3810781 \ ) \quad 95269528 \\
 \underline{+ \quad 2} \quad 76215622 \\
 \quad 38107812 \ ) \quad 1905390612,5 \\
 \underline{+ \quad 5} \quad 1905390612,5 \\
 \quad 381078125 \quad (0)
 \end{array}$$

Having found the Square Root of the given Resolvend, I proceed to extract the Cube Root of that Square Root.

That is, of

381078125

— 343 = the nearest Cube, its Root is 700

Then  $700 \times 3 = 2100$  ) 38078125 ( 18161

First Root 7 ) 18161 ( 25

+ 2 144

1. Divisor 72 ) 3761

+ 5 3625

2. Divisor 725 (136)

First Root 700

+ 25

725

Hence I find 725 to be the Square Cube Root required; as may easily be tried by involving it to the sixth Power.

That is,  $725 \times 725 \times 725 \times 725 \times 725 \times 725$  will be found = 145220537353515625 the given Resolvend.

Sect.



## Sect. 7. To Extract the Root of the Seventh Power.

Having pointed the given Resolvend, as its Index denotes, *viz.* into Periods of 7 Figures, seek out such a Number of the seventh Power, by the *Table of Powers*, as comes nearest to the first Period of the Resolvend; whether it be greater or lesser, calling its respective Root *more* than *just*, or *less* than *just*, annexing its proper Number of Cyphers, &c. as in the Cube and Surfsolid.

Then find the Difference between the given Resolvend, and that Number of the seventh Power (found by the *Table of Powers*) by subtracting the lesser from the greater.

Next find the Surfsolid or fifth Power of that Root with its annexed Cyphers (which you may also do by the *Table of Powers*) and multiply that Surfsolid Number with 7, the Index of the given Resolvend, that Product must be a Divisor, by which the foresaid Difference must be divided; that so it may be depressed to a Square, to be pointed, &c. as before in the Cube, &c. then make the first Root, without its Cyphers, a Divisor; working with it and the new Resolvend, as before, only here you must encrease, or diminish the Divisor with thrice the Quotient Figure.

### Example.

What's the second Surfsolid Root, or that of the seventh Power?

of 373236553955078125 the Resolvend pointed.  
 — 2178 the nearest Number of the seventh Power.

155436553955078125 their Difference.

The first Root is 300 being less than just; and the fifth Power of 300 is 2430000000000 which being multiplied with 7 is 17010000000000 for a Divisor, by which the aforesaid Difference must be divided; which contracted may stand thus,

1701 ) 15543655 ( 9137.95 &c.

First Root 3 ) 9137 ( 25  
 + 2 × 3 = 6 72

First Root = 300  
 + 25

1. Divisor 36 1937  
 + 5 × 3 = 15 1875

True Root 325

375 (62) the Remainder to be rejected as before.

T 2

Hence

Hence I have found 325 to be the true Root required, that is, the true Root of the seventh Power.

I think it needless to proceed farther; *viz.* to insert Examples of higher Powers. For if what is already done be well understood, it will be easy to conceive how to proceed in extracting the Root of any single Power how high soever it be (for the Method is general and alike in all Powers) due Regard being had to their Indices; and to the first single Side or Root. That is, whether it be *more*, or *less*, than *just*, &c.

Yet methinks I hear the young Learner say, 'tis possible to follow the Directions and Examples, as they are here laid down; but still here is not the Reason why they are so, and so, perform'd; and why there should be a Remainder left after the true Root is found; *viz.* when the given Resolvend hath a true Root of its kind.

'Tis true, the Reasons of these are not here laid down; neither indeed can they be rendred so plain and intelligible by Words, as by an *Algebraic Process*, from whence the *Theorems* or *Rules* here given, had their first Invention; as shall be shewed in the next Part, when I come to treat of resolving *Compounded* or *Affected Æquations*; however, take this short and general Account of this Method.

This, and all other of the new Methods of *converging Series* (as they are called) are very different from the former (and still common) Methods of extracting Roots, which requires the first single Side or Root of the first Period (in any Resolvend) to be taken exactly true, and then by involving, and other tedious Ways of ordering it, there is formed a Divisor; which helps to grope out by Trials a second Figure in the Root. And so proceeds on from Point to Point; still repeating the whole Work for every single Figure that comes into the Root. And if by chance there be a Mistake or Error committed in any one Figure (as 'tis possible there may) it spoils the whole Process, which must then be wholly begun anew, or at least from that Part of it where the Error first entred.

But the Nature and Design of the Method which I have here laid down is quite otherwise; it being so contrived, as to gradually lessen the Difference betwixt any proposed Power, and the like Power of another Number assumed; *viz.* it lessens that Difference until it either quite vanishes, or becomes so infinitely small as to be insignificant.

Therefore when any Number is proposed to have its Root extracted; it is here required to take the next nearest Root of the first Period in the Resolvend; that so the Difference betwixt the given Resolvend, and the Homogeneal Power (*viz.* the like Power<sup>3</sup>)



Power) of the Root thus taken, may be less either in Excess, or Defect. Which Difference being reduced, or depressed lower, becomes so prepared, that by plain Division, comparatively, there will arise such Quotient Figures as will both correct and encrease the first Root to three Places of Figures at least, sometimes to four, or five Places of Figures; according as the said first Difference happens to be more or less; (of which you may have observed Instances): But yet there will be a Remainder left, and perhaps an Excess or Defect in the Root so encreased; viz. in the last Figure of it.

Now to rectify the said Excess or Defect in the Root, and to discover whether the given Resolvend be a true figurate Number, or not: That is, whether, it have a true Root of its kind. It will be necessary to make a second Operation; by taking the Root so encreased, and proceeding with it and the given Resolvend, in all respects as in the first Work, like to the third Example of extracting the Cube Root, I say, if the given Resolvend have a true Root, it will appear at this second Operation, and all the aforesaid Differences, &c. will vanish; provided the Root required is not to have more than three, or four, Places of Figures in it.

But if the Root be to have more than three Figures in it; or, that the given Resolvend prove to be a Surd Number. Then there will be a Difference as before; which will afford Quotient Figures to rectify and encrease the Root last taken to three times as many Places of Figures, as it had at the Beginning of that second Operation. As you may see in the aforesaid *Example 3.* of the Cube Root; wherein that Root is encreased to twelve Places of Figures at two Operations: which if it were to be extracted the old, and still common, Way, it would require at least forty times the Number of Figures I have here used.

Again, if there chance to be a Mistake committed in any Operation perform'd by the Method here laid down, that Mistake will not destroy the preceding Work, but will be rectified in the next Operation, although it were not discovered before. And thus you may proceed on to a third Operation, which will afford 27 Places of Figures in the Root. &c. with very little Trouble, if compared with former Methods.

This brief Account, which I have here given, by way of explaining the Nature of this Method of extracting Roots, being well considered and compared with the several Operations of the foregoing Examples, must needs help the Learner to form such an Idea of it, that he cannot, I presume, but understand how to proceed in extracting the Root out of any single Power, how high soever it be; without the Help of an *Algebraic Theorem*.  
Not.



Not, but when that comes to be once understood; the Work will be much readier and easier perform'd: As will appear in the next Part.

I did intend to have here inserted the whole Business of Interest and Annuities; but finding that it would require too large a Discourse, to shew the Grounds and Reasons of the several *Theorems* useful therein, I have therefore reserved that Work for the Close of the next Part. Neither indeed can the raising of those *Theorems* be so well delivered in Words, as by an *Algebraic* Way of *Arguing*; which renders them not only much shorter, but also plainer and easier to be understood.

I have also omitted that Rule in Arithmetick, usually called the Rule of *Position*, or Rule of *False*: Because all such Questions as can be answered by that *Guessing Rule*, are much better done by any one, who hath but a very small Smattering of *Algebra*. I shall therefore conclude this Part of *Numerical Arithmetick*; and proceed to that of *Algebraic Arithmetick*, wherein I would advise the young Learner not to be too hasty in passing from one Rule to another, and then he will find it very easy to be attained.

---

Just Publish'd by S. FULLER, at the  
GLOBE and SCALES in Meath-street.

A MATHEMATICAL MISCELLANY, in Four Parts. I. An Essay towards the probable Solution of the Forty five surprizing PARADOXES in GORDON'S GEOGRAPHY. II. Fifty five New and amazing PARADOXES, some in Verse, some in Prose, with their SOLUTIONS. III. Answers to the Hundred ARITHMETICAL PROBLEMS left unanswer'd in HILL'S ARITHMETICK, and ALEXANDER'S ALGEBRA. IV. Miscellaneous Rules about forming ÆNIGMA'S, QUESTIONS, the Doctrine of ECLIPSES, of PENDULUMS, the ÆQUATION of TIME, &c. Price bound, One Shilling and eight Pence.

The COMPLETE MEASURER; or the whole ART of MEASURING, by Vulgar, Decimal, and Cross MULTIPLICATION; as also instrumentally, by SCALE and COMPASSES; the vulgar and true Ways of Measuring all Sorts of SUPERFICIES, and SOLIDS, and the Works of the several ARTIFICERS, about BUILDINGS; with an APPENDIX of GAGING and SURVEYING, according to both the ENGLISH and IRISH ACRE and GALLON. Very useful for all TRADESMEN, &c. By WILLIAM HAWNEY. Price bound, a British Half Crown.

ANNUITIES upon LIVES: Or, the VALUATION of ANNUITIES upon any Number of LIVES; as also, of REVERSIONS. To which is added, An APPENDIX concerning EXPECTATIONS of LIFE, and PROBABILITIES of SURVIVORSHIP. By A. DE MOIVRE. The Second EDITION Corrected, with large ADDITIONS. Price bound, a British Half Crown.



A N

# INTRODUCTION

T O T H E

# Mathematicks.

## P A R T II.

### P R O E M.

**H**AVING formerly wrote a small Tract of Algebra, perhaps it may seem (to some) very improper to write again upon the same Subject; but only (as the usual Custom is) have referr'd my Reader to that Tract. However, because the following Parts of this Treatise are managed by an Algebraic Method of Arguing; which may fall into the Hands of those who have not seen that Tract, or any other of that kind; I thought it convenient to accommodate the young Geometer with the first Elements, or principal Rules, by which all Operations in this Art are performed: That so he may not be at a Loss as he proceeds farther on: Besides, what I formerly wrote was only a Compendium of that which is here fully handled at large.

The principal Rules are Addition, Subtraction, Multiplication, Division, Involution, and Evolution; as in common Arithmetick (but differently perform'd): And therefore some call it Algebraic Arithmetick. Others call it Arithmetick in Species, because all the Quantities concerned in any Question, remain in their Substituted Letters (howsoever manag'd by Addition, Subtraction, or Multiplication, &c.) without being destroy'd or changed into others, as Figures in common Arithmetick are.

Mr. Harriot call'd it *Logistica Speciosa*, or *Specious Computation*.

C H A P.

## C H A P. I.

*Concerning the Method of Noting down Quantities; and Tracing their Steps.*

## Sect. I. Of Notation.

**T**HE Method of noting down Letters for Quantities, is various, according to every one's Fancy; but I shall here follow the same as in my former Tract: And represent the Quantity sought, be it Line, or Number, &c. by the small (*a*), and if more Quantities than one are sought, represent them by the other small Vowels; *e. u. or y.*

The given Quantities are represented by the small Consonants, *b. c. d. f. g. &c.*

And for Distinction's sake, mark the Points or Ends of Lines in all Schemes, with the Capital, or great Letters, *viz. A. B. C. D. &c.*

When any Quantity, either given or sought, is taken more than once, you must prefix its Number to it; as  $3a$  stands for *a* taken three times, or three times *a*, and  $7b$  stands for seven times *b*, &c.

All Numbers thus prefixt to any Quantity are called Co-efficients or Fellow-Factors; because they multiply the Quantity; and if any Quantity be without a Co-efficient, it is always suppos'd, or understood to have an Unit prefix'd to it; As *a* is  $1a$ , or *b* is  $1b$ , &c.

The Signs by which Quantities are chiefly managed are the same; and have the same Signification, with those in the *First Part, Page 5.* which I here presume the Reader to be very well acquainted with. To them must be here added these three more:

*Viz.*  $\left\{ \begin{array}{l} \textcircled{\circ} \\ w \\ \sqrt{\phantom{x}} \end{array} \right\}$  the Sign of  $\left\{ \begin{array}{l} \text{Involution.} \\ \text{Evolution, or extracting Roots.} \\ \text{Irrationality, or Sign of a Surd Root.} \end{array} \right.$

All Quantities that are express'd by Numbers only, as in Vulgar Arithmetick, are called *Absolute Numbers.*

Those Quantities that are represented by single Letters, as *a. b. c. d. &c.* or by several Letters that are immediately joined together; as *ab. cd. or 7bd. &c.* are called simple or single whole Quantities.

But when different Quantities represented by different or unlike Letters, are connected together by the Signs ( $+$  or  $-$ ); As  $a+b. a-b. or ab+dc. &c.$  they are called *Compound whole Quantities.*

And



# Chap. I. Notation of Quantities. 145

And when Quantities are express'd or set down like Vulgar Fractions, Thus  $\frac{a}{b}$ . Or  $\frac{a+b}{d}$ . Or  $\frac{ab+dc}{b-c}$ . &c.

they are called Fractional or broken Quantities.

The Sign wherewith Quantities are connected, always belongs to that Quantity which immediately follows it; and therefore all the Quantities concern'd in any Question, may stand in any Order at Pleasure, *viz.* the most convenient for the next Operation. As  $a+b-d$  may stand thus  $b-d+a$ . Or thus  $a-d+b$ . Or  $-d+a+b$  &c. these being still the same, tho' differently placed.

That Quantity which hath no Sign before it, as generally the leading Quantity hath not, is always understood to have the Sign  $+$  before it. As  $a$  is  $+a$ . Or  $b-d$  is  $+b-d$  &c. for the Sign  $+$  is the affirmative Sign, and therefore all leading or positive Quantities are understood to have it, as well as those that are to be added.

But the Sign  $-$  being the negative Sign, or Sign of Defect, there is a Necessity of prefixing it before that Quantity to which it belongs, where-ever the Quantity stands.

## Sect. 2. Of Tracing the Steps used in bringing the the Quantities to an Equation.

The Method of Tracing the Steps, used in bringing the Quantities concern'd in any Question to an Equation, is best performed by registering the several Operations, with Figures and Signs placed in the Margin of the Work, according as the several Operations require; being very useful in long and tedious Operations.

For Instance: If it be required to set down, and register the Sum of the two Quantities  $a$ , and  $b$ , the Work will stand,

Thus  $\begin{array}{r|l} 1 & a \\ 2 & b \\ \hline 1+2 & 3 \mid a+b \end{array}$  First set down the proposed Quantities,  $a$  and  $b$  over against the Figures 1. 2. in the small Column, which are here called Steps, and against 3, the third Step, set down their Sum, *viz.*  $a+b$ . Then against that third Step, set down  $1+2$  in the Margin; which denotes that the Quantities against the first and second Steps are added together, and that those in the third Step are their Sum.

To illustrate this in Numbers, suppose  $a=9$  and  $b=6$ . Then it will be,

Thus  $\begin{array}{r|l} 1 & a = 9 \\ 2 & b = 6 \\ \hline 1+2 & 3 \mid a+b = 9+6 = 15 \end{array}$  being the Sum of 9 and 6.

ll

Again,

Again, If it were required to set down the Difference of the same two Quantities, Then it will be,

Thus	1	$a = 9$	
	2	$b = 6$	
$1 - 2$	3	$a - b = 9 - 6 = 3$	the Difference between 9 and 6.

Or if it were required to set down their Product.  
Then it will be,

Thus	1	$a = 9$	
	2	$b = 6$	
$1 \times 2$	3	$a \times b$ or $ab = 9 \times 6 = 54$	the Product of 9 into 6.

*&c.*

*Note*, Letters set or joyn'd immediately together, like a Word, signifie the Rectangle or Product of those Quantities they represent. As in the last *Example*, wherein  $ab = 54$  is the Product of  $a = 9$  and  $b = 6$ . *&c.*

### Axioms.

1. If *Equal Quantities* be Added to *Equal Quantities*, the *Sum* of those *Quantities* will be equal.
2. If *Equal Quantities* be Taken from *Equal Quantities*, the *Quantities Remaining* will be equal.
3. If *Equal Quantities* be Multiplied with *Equal Quantities*, their *Products* will be equal.
4. If *Equal Quantities* be Divided by *Equal Quantities*, their *Quotients* will be equal.
5. Those *Quantities*, that are *Equal* to one and the same *Thing*, are *Equal* to one another.

*Note*, I advise the Learner to get these five *Axioms* perfectly by Heart.

These things being premised, and a perfect Knowledge of the Signs, and their Significations being gained, the young *Algebraist* may proceed to the following Rules. But first I must make bold to advise him here, as I have formerly done, that he be very ready in one Rule before he undertakes the next.

That is, he should be expert in Addition, before he meddles with Subtraction; and in Subtraction, before he undertakes Multiplication, *&c.* because they have a Dependency one upon another.



## C H A P. II.

Concerning the Six Principal Rules of Algebraic Arithmetick, of whole Quantities.

### Sect. I. Addition of whole Quantities.

Addition of whole Quantities admits of Three Cases.

*Case 1.* If the Quantities are like, and have like Signs; Add the Co-efficients or prefixt Numbers together; and to their Sum adjoyn the Quantities with the same Sign.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
	1	$a$	$- a$	$5b$	$- 7bc$
	2	$a$	$- a$	$3b$	$- 8bc$
1 + 2	3	$2a$	$- 2a$	$8b$	$- 15bc$

Thus		Exam. 5.	Exam. 6.	Exam. 7.
	1	$3a + 5b$	$3a - 5b$	$6ab + 12$
	2	$2a + 7b$	$2a - 7b$	$3ab + 24$
1 + 2	3	$5a + 12b$	$5a - 12b$	$9ab + 36$

The Reason of these Additions is evident from the Work of common Arithmetick. For suppose  $a$ , to represent one Crown, to which if I add one Crown, the Sum will be two Crowns, or  $2a$ . As in Example 1.

Or if we suppose  $-a$ , to represent the Want or Debt of one Crown, to which if another Want or Debt of one Crown be added, the Sum must needs be the Want or Debt of two Crowns, or  $-2a$ ; as in Example 2. And so for all the rest.

*Case 2.* If the Quantities are alike, and have unlike Signs, subtract the Co-efficients from each other, and to their Difference joyn the Quantities with the Sign of the Greater.

		Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
	1	$+ 5a$	$- 5a$	$7bc$	$- 9abd$
	2	$- 3a$	$+ 3a$	$- 6bc$	$+ 7abd$
1 + 2	3	$+ 2a$	$- 2a$	$bc$	$- 2abd$

		Exam. 12.	Exam. 13.
	1	$7a - 5b$	$- 8ab - 7bc + 15$
	2	$- 5a + 7b$	$+ 12ab + 7bc - 24$
1 + 2	3	$2a + 2b$	$4ab - 9$

The Reason of the Operations in this Case may be easily understood by any one that duly considers the comparing of Stock and Debts together; or the Ballancing of Accompts betwixt Debtor and Creditor.

That is, the Affirmative Quantities represent the Stock or Creditor: The Negative Quantities represent the Debts; and their Sum represents the Ballance, &c.

*Case 3.* When the Quantities are unlike, set them all down without altering their Signs; and thence will arise compound Quantities, which can be no otherwise added but by their Signs.

Thus	1	$a$	$a$	$5b + 7dc$
	2	$b$	$-b$	$4a - 20$
$1 + 2$	3	$a + b$	$a - b$	$5b + 7dc + 4a - 20$

Here follow some few Examples wherein all the 3 Cases are promiscuously concerned.

	1	$aa + 2ab + bb$	$8ab + bc - 37$
	2	$-4ab$	$-7ab - bc + 42 - 6d$
$1 + 2$	3	$aa - 2ab + bb$	$ab + 5 - 6d$

	1	$aa - 2ab + bb$	$9bc + 7ab - 45$
	2	$+ 4ab + bb$	$4d - 6bc - 7ab + da$
$1 + 2$	3	$aa + 2ab + bb$	$3bc + 4d - 45 + da$

	1	$5a$	$a + b - ab$
	2	$-7a$	$7c - d$
	3	$+3a$	$4e + f$
$1 + 2 + 3$	4	$a$	$a + b - ab + 7c - d + 4e + f$

	1	$3aa + 4abc - bb + 30$
	2	$2bb - 3aa + 2abc - 25$
	3	$dd + 2aa - 5abc - 3$
$1 - 2 + 3$	4	$bb + dd + 2aa + abc + 2$

### Sect. 2. Subtraction of whole Quantities.

Subtraction of whole Quantities is perform'd by one general Rule.

#### Rule.

*Change all the Signs of the Subtrahend; viz. of those Quantities which are to be subtracted, or suppose them in your Mind to be changed. Then add all the Quantities together, as before in Addition, and their Sum will be the true Remainder or Difference required.*

This



# Chap.2. Subtraction of Quantities. 149

This *General Rule* is deduced from these evident Truths.

To subtract an Affirmative Quantity, from an Affirmative; is the same as to add a Negative Quantity to an Affirmative.

That is,  $+2a$  taken from  $+3a$ , is the same with  $-2a$  added to  $+3a$ .

Consequently, To subtract a Negative Quantity from an Affirmative; will be the same as to add an Affirmative Quantity to an Affirmative.

That is,  $-2a$  taken from  $+3a$  will be the same with  $+2a$  added to  $+3a$ .

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
I — 2	1	$2a$	$-2a$	$8b$	$-15bc$
	2	$a$	$-a$	$3b$	$-8bc$
	3	$a$	$-a$	$5b$	$-7bc$

		Exam. 5.	Exam. 6.	Exam. 7.
I — 2	1	$5a + 12b$	$5a - 12b$	$9ab + 36$
	2	$2a + 7b$	$2a - 7b$	$3ab + 24$
	3	$3a + 5b$	$3a - 5b$	$6ab + 12$

		Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
I — 2	1	$+2a$	$-2a$	$bc$	$-2abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
	3	$+5a$	$-5a$	$+7bc$	$-9abd$

		Exam. 12.	Exam. 13.
I — 2	1	$2a + 2b$	$4ab - 9$
	2	$-5a + 7b$	$-8ab - 7bc + 15$
	3	$7a - 5b$	$12ab + 7bc - 24$

If these 13 Examples be compared with those in Addition; the Work will appear very evident, these being only the Converse or Proof of those; according to the Nature of Addition and Subtraction, in common Arithmetick.

More Examples in Subtraction.

I — 2	1	$a + b$	$5bc + 3da$	$8a + 5bd + 25$
	2	$a - b$	$5bc - 4da$	$7a - 3bd - 12$
	3	$+2b$	$+7da$	$a + 8bd + 37$

$$\begin{array}{r|l|l|l|l} \text{I} & 1 & c + 13 & a & 0 \\ \text{---} & 2 & 3a - b - 2c & b & 2a - 4b \\ \text{2} & 3 & 3c + 13 - 3a + b & a - b & -2a + 4b \end{array}$$

$$\begin{array}{r|l|l|l} \text{I} & 1 & a + b - 54 & 76 \\ \text{---} & 2 & d - 3b - bc - 75 & a - b - 5d + 7c \\ \text{2} & 3 & a + 4b + bc + 21 - d & 76 - a + b + 5d - 7c \end{array}$$

That,  $a - b$  taken from  $a + b$  leaves  $+ 2b$  for the Remainder; as in the first of these Examples, may be thus proved.

$$\begin{array}{r|l|l} \text{Let} & 1 & a + b = z \\ \text{And} & 2 & a - b = x \\ \text{2} + b & 3 & a = x + b. \text{ per Axiom 1.} \\ \text{I} - 3 & 4 & b = z - x - b. \text{ per Axiom 2.} \\ \text{4} + b & 5 & 2b = z - x. \text{ which was to be proved.} \end{array}$$

The Truth of all Operations in Subtraction, where any Doubt arises, may be proved, by adding the Subtrahend to the Remainder; as in common Arithmetick.

### Examples.

$$\begin{array}{r|l|l|l|l|l} \text{From} & 1 & + 5a & 0 & - 9bc & \\ \text{Take} & 2 & - 2a & + 3b & - 6da & \text{Subtrahend.} \\ \text{I} - 2 & 3 & + 7a & - 3b & + 6da - 9bc & \text{Remainder.} \\ \text{2} + 3 & 4 & + 5a & 0 & - 9bc & \text{Proof.} \end{array}$$

### Sect. 3. Multiplication of whole Quantities.

Multiplication of whole Quantities admits of *Three Cases*.

*Case 1.* When the Quantities have like Signs, and no Co-efficients, set or joyn them together; and prefix the Sign  $+$  Before them, and that will be their Product.

$$\begin{array}{r|l|l|l|l|l} \text{Thus } \{ & 1 & \text{Exam 1.} & \text{Exam. 2.} & \text{Exam. 3.} & \text{Exam. 4.} \\ & 2 & a & - a & a + b & - a - b \\ & 3 & b & - b & d & - d \\ \text{I} + 2 & 3 & ab & + ab & ad + bd & + ad + bd \end{array}$$

*Case. 2.* If there be Co-efficients; multiply them, and to their Product adjoyn the Quantities set together as before.

Thus



# Chap. 2. Multiplicat. of Quantities. 151

Thus	{	1 2 3	Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
			$5a$	$-6d$	$3a + 2b$	$a + b$
			$3b$	$-7b$	$6$	$5b$
1	x	2	$15ab$	$+ 42db$	$18a + 12b$	$5ab + 5bb$

*Case 3.* When the Quantities have unlike Signs; joyn them and the Product of their Co-efficients together (as before) but prefix the Sign — before them;

Thus	{	1 2 3	Exam. 9.	Exam. 10.	Exam. 11.	Exam. 12.
			$+ a$	$- 6d$	$4a - 7b$	$4a - 7b$
			$- b$	$+ 7b$	$3f$	$- 3f$
1	x	2	$- ab$	$- 42db$	$12af - 21bf$	$- 12af + 21bf$

That is, + into +, or — into —, gives {+} in the Product.  
But + into —, or — into +, gives {—} in the Product.

That + into +, will produce + in the Product, is evident from Multiplication in common Arithmetick.

viz. + 5 into + 7 will give + 35, &c.

But that + into —, Or — into + should produce the Sign —, As in the four last Examples.

And that — into — should produce the Sign + as in the second, fourth, and sixth Examples, may perhaps seem somewhat hard to be conceived; and requires a Demonstration.

First to prove that — 7b into + 3f = — 21bf. As in Ex. 11.

Suppose	1	$4a - 7b = 0$	
Then will	2	$4a = 7b.$	per Axiom 1.
But	3	$+ 3f = + 3f$	
2 x 3	4	$12af = 21bf.$	per Axiom. 3.
4 — 21bf	5	$12af - 21bf = 0$	per Axiom 2.

Consequently + into —, Or — into + produces —, which was the thing to be proved.

Secondly, to prove — 7b into — 3f gives + 21bf as in Exam. 12.

Let	1	$4a - 7b = 0$	} as before.
Then	2	$4a = 7b$	
But	3	$- 3f = - 3f$	
the 2 x 3 is.	4	$- 12af = - 21bf$	by what is proved above.
4 + 21bf	5	$- 12af + 21bf = 0.$	per Axiom 1.

Consequently — into — gives + which was to be proved.

Or

Or these may be otherwise proved by Numbers.

Thus, Suppose  $\begin{cases} a = 20 \\ b = 14 \end{cases}$  and  $\begin{cases} c = 12 \\ d = 8 \end{cases}$  } or any other Numbers.

Then  $a - b = 6$   $c - d = 4$  per Axiom 2.

Consequently,  $a - b \times c - d = 6 \times 4 = 24$  per Axiom 3. but  $a - b \times c - d$  according to the Preceding Rules, will be,  $ac - cb + bd - da$ , which if true must be equal to 24.

Proof  $\begin{cases} ac = 20 \times 12 = 240 \\ bd = 14 \times 8 = 112 \end{cases}$   $\begin{cases} cb = 12 \times 14 = 168 \\ da = 8 \times 20 = 160 \end{cases}$

Hence,  $ac + bd = 352$  per Axiom 1.

And  $cb + da = 328$  which being subtracted,

Leaves  $ac + bd - cb - da = 352 - 328 = 24$  which plainly shews,

That  $+$  into  $-$  Produces  $-$   
And  $-$  into  $-$  Produces  $+$  } in the Product.

Q. E. D.

Note. If the Multiplier consists of several Terms, then every one of those Terms must be multiplied into all the Terms of the Multiplicand: And the Sum of those particular Products, will be the Product requir'd. As in common Arithmetick.

Examples.

$\begin{array}{l} 1 \times a \\ 1 \times b \\ 3 + 4 \end{array}$	1	$a + b - d$	$\begin{array}{l} 7b + 5d \\ 3a - 5f \\ \hline 21ba + 15da \\ \hline - 35bf - 25df \\ \hline 21ba + 15da - 35bf - 25df \end{array}$
	2	$a - b$	
	3	$aa + ba - da$	
	4	$-ba - bb + db$	
	5	$aa - da - bb + db$	

$1 \times 2$	1	$aa - ba$	$2c - 3d$
	2	$a + b$	$3a - 4b$
	3	$aaa - abb$	$6ca - 9da - 8bc + 12db$

$1 \times 2$	1	$aa + 2a + 4$	$aa - ba + bb$
	2	$a - 2$	$a + b$
	3	$aaa + 2aa + 4a$	$aaa - baa + bba$
	4	$- 2aa - 4a - 8$	$+ baa - bba + bbb$
	5	$aaa - 8$	$aaa + bbb$



## Sect. 4. Division of whole Quantities.

Division of Species, is the converse or direct contrary to that of Multiplication, and consequently perform'd by converse Operations. (As in common Arithmetick) and admits of four Cases.

*Case 1.* When the Quantities in the Dividend, have like Signs to those in the Divisor, and no Co-efficients in either; cast off or expunge all the Quantities in the Dividend, that are like those in the Divisor; and set down the other Quantities with the Sign + for the Quotient required.

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} ab \\ b \end{array} \right. \left| \begin{array}{l} -ab \\ -b \end{array} \right. \left| \begin{array}{l} ad+bd \\ d \end{array} \right. \left| \begin{array}{l} -ad-bd \\ -d \end{array} \right. \\ 1 \div 2 \left| 3 \right| \left| \begin{array}{l} - \\ + \end{array} \right. \left| \begin{array}{l} a \\ a \end{array} \right. \left| \begin{array}{l} a+b \\ a+b \end{array} \right. \left| \begin{array}{l} a+b \end{array} \right. \end{array}$$

*Case 2.* When the Quantities in the Dividend have unlike Signs to those in the Divisor; then set down the Quotient Quantities found as before, with the Sign - before them.

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} +ab \\ -b \end{array} \right. \left| \begin{array}{l} -ab-bd \\ +b \end{array} \right. \left| \begin{array}{l} abc+bcd+bcf \\ -bc \end{array} \right. \\ 1 \div 2 \left| 3 \right| \left| \begin{array}{l} - \\ - \end{array} \right. \left| \begin{array}{l} a \\ -a-d \end{array} \right. \left| \begin{array}{l} -a-d-f \end{array} \right. \end{array}$$

*Case 3.* If the Quantities in the Dividend and Divisor, have Co-efficients; Divide the Numbers (as in common Arithmetick) and to their Quotients adjoin the Quotient Quantities.

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} 15ab \\ 3b \end{array} \right. \left| \begin{array}{l} 42db \\ -7b \end{array} \right. \left| \begin{array}{l} 12af-21bf \\ 3f \end{array} \right. \\ 1 \div 2 \left| 3 \right| \left| \begin{array}{l} 5a \\ -6d \end{array} \right. \left| \begin{array}{l} 4a-7b \end{array} \right. \end{array}$$

*Note.* When the Quantities and Co-efficients in the Divisor and Dividend are all the same, the Quotient will be an Unit or 1.

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} 1 \\ 2 \end{array} \right. \left| \begin{array}{l} ab \\ ab \end{array} \right. \left| \begin{array}{l} 9bc \\ -9bc \end{array} \right. \left| \begin{array}{l} 7ab+5bc \\ 7ab+5bc \end{array} \right. \left| \begin{array}{l} 8ab+4d \\ -8ab-4d \end{array} \right. \\ 1 \div 2 \left| 3 \right| \left| \begin{array}{l} 1 \\ -1 \end{array} \right. \left| \begin{array}{l} 1 \\ -1 \end{array} \right. \left| \begin{array}{l} 1 \\ -1 \end{array} \right. \left| \begin{array}{l} -1 \end{array} \right. \end{array}$$

*Case 4.* When the Quantities in the Divisor cannot be exactly found in the Dividend; then set them both down like a Vulgar Fraction. As in common Arithmetick.

$$\begin{array}{r|l|l|l|l|l} \text{Thus } \{ & 1 & a & 6bc & 5b+aa & 8adc \\ & 2 & b & 3d & 5d+7b & 4abc \\ & & \hline & & a & 2bc & 5b+aa & 2d \\ 1 \div 2 & 3 & b & d & 5d+7b & b \end{array}$$

N. B. In Division one thing must be very carefully observed; viz. that like Signs give + and unlike Signs give — in the Quotient; which needs no other Proof than that already laid down in the last Section, if duly compared with what hath been said concerning Multiplication and Division, in Vulgar Arithmetick.

### Examples of Division at large.

$$\begin{array}{r|l|l} 2 \times 3a & 1 & 21ba + 15da - 35bf - 25df \quad (+ 3z \\ 1 - 3 & 2 & 7b + 5d \\ 2 \times -5f & 3 & \hline 4 - 5 & 4 & 21ba + 15da \\ 1 \div 2 & 5 & 0 \quad 0 - 35bf - 25df \quad (- 5f \\ & 6 & \quad - 35bf - 25df \\ & 7 & 0 \quad 0 \\ & & 3a - 5f \text{ the Quotient collected from the 3. and 5. Steps} \end{array}$$

Or Division of Quantities may stand as Numbers in common Arithmetick do; Thus

$$\begin{array}{r} 3a - 6 \ ) \ 6aaaa - 96 \quad ( 2aaa + 4aa + 8a + 16 \\ \underline{6aaaa - 12aa} \\ \quad 0 + 12aaa - 96 \\ \quad \underline{+ 12aaa - 24aa} \\ \qquad 0 + 24aa - 96 \\ \qquad \underline{+ 24aa - 48a} \\ \qquad \qquad 0 + 48a - 96 \\ \qquad \qquad \underline{+ 48a - 96} \\ \qquad \qquad \qquad 0 \quad 0 \end{array}$$

That is,  $6aaaa - 96 \div 3a - 6$  gives  $2aaa + 4aa + 8a + 16$  for the Quotient, as may easily be proved by Multiplication, viz.  $2aaa + 4aa + 8a + 16 \times 3a - 6$  will produce  $6a^4 - 96$  and so for the rest.

### Sect. 6. Involution of whole Quantities.

Involution is the raising or producing of Powers, from any proposed Root, and is performed in all respects like Multiplication save only in this; Multiplication admits of any different Factors, but Involution still retains the same.

Exam.



Examples.

	I	1	$a$	$-a$	the Root, or single Power.
I ⊗ 2	2	2	$aa$	$+aa$	Square, or Second Power.
I ⊗ 3	3	3	$aaa$	$-aaa$	Cube, or third Power.
I ⊗ 4	4	4	$aaaa$	$+aaaa$	Biquadrat. or 4th Power.
I ⊗ 5	5	5	$aaaaa$	$-aaaaa$	Surfsolid, or 5th Power, &c.

Note, The Figures placed in the Margin, after the Sign (⊗) of Involution; shew to what Height the Root is involved; and are called Indices of the Power; and are usually placed over the involved Quantities, in order to contract the Work, especially when the Powers are any thing high.

Thus  $\begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases}$  And  $\begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5 b^6 = aaaaaabbbbbbb \\ a^3 b^2 d^4 = aaabbbddd \end{cases}$

If the Quantities have Co-efficients, the Co-efficients must be involved along with the Quantities. As in these,

Thus	I	1	$2a$	$-3a$	$5bc$
I ⊗ 2	2	2	$4aa$	$+9aa$	$25bbcc$
I ⊗ 3	3	3	$8aaa$	$-27aaa$	$125bbbccc$
I ⊗ 4	4	4	$16aaaa$	$+81aaaa$	$625b^4c^4$
I ⊗ 5	5	5	$32aaaaa$	$-243a^5$	$3125b^5c^5$ &c.

Involution of compound Quantities is performed in the same manner, due Regard being had to their Signs and Co-efficients, if there be any.

As for Instance, Suppose  $a + b$  were given to be involved to the 5th Power.

Thus  $\begin{array}{l|l|l} \text{I} & 1 & a + b \text{ called a Binomial Root.} \\ & & \hline \text{I} \times a & 2 & aa + ab \\ \text{I} \times b & 3 & + ab + bb \\ \hline \text{I} \otimes 2 & 4 & aa + 2ab + bb \text{ the Square of } a + b \\ & & \hline & & a + b \\ \hline 4 \times a & 5 & aaa + 2aab + abb \\ 4 \times b & 6 & + aab + 2abb + bbb \\ \hline \text{I} \otimes 3 & 7 & aaa + 3aab + 3abb + bbb \text{ the Cube of } a + b. \end{array}$

	7	$aaa + 3aab + 3abb + bbb$ $a + b$
$7 \times a$	8	$a^4 + 3a^3b + 3aabb + abbb$
$7 \times b$	9	$+ a^3b + 3aabb + 3abbb + b^4$
$I \odot 4$	10	$a^4 + 4a^3b + 6aabb + 4abbb + b^4$ $a + b$
$10 \times a$	11	$a^5 + 4a^4b + 6a^3bb + 4aabb^3 + ab^4$
$10 \times b$	12	$+ a^4b + 4a^3bb + 6aabb^3 + 4ab^4 + b^5$
$I \odot 5$	13	$a^5 + 5a^4b + 10a^3bb + 10aabb^3 + 5ab^4 + b^5$ $a + b \&c.$

Again, Let  $a - b$ , called a Residual Root, be given.

Then	I	$a - b$ $a - b$
$I \times a$	2	$aa - ab$
$I \times -b$	3	$- ab + bb$
$I \odot 2$	4	$aa - 2ab + bb$ the Square of $a - b$ $a - b$
$4 \times a$	5	$aaa - 2aab + abb$
$4 \times -b$	6	$- aab + 2abb - bbb$
$I \odot 3$	7	$aaa - 3aab + 3abb - bbb$ the Cube of $a - b$ $a - b$
$7 \times a$	8	$aaaa - 3aaab + 3aabb - abbb$
$7 \times -b$	9	$- aaab + 3aabb - 3abbb + bbbb$
$I \odot 4$	10	$aaaa - 4aaab + 6aabb - 4abbb + bbbb$ $a - b$
$10 \times a$	11	$a^5 - 4a^4b + 6a^3bb - 4aabb^3 + ab^4$
$10 \times -b$	12	$- a^4b + 4a^3bb - 6aabb^3 + 4ab^4 - b^5$
$I \odot 5$	13	$a^5 - 5a^4b + 10a^3bb - 10aabb^3 + 5ab^4 - b^5$ $a - b \&c.$

By comparing these two Examples together, you may make the following Observations.

1. That the Powers raised from a Residual Root (*viz.* the Difference of two Quantities) are the same with their like Powers raised from a Binomial Root (or the Sum of two Quantities) save only in their Signs; *viz.* the Binomial Powers have the Sign  $+$  to every Term; but the Residual Powers have the Signs  $+$  and  $-$  interchangeably to every other Term.

2. The Indices of the Powers of the leading Quantity ( $a$ )



ty (*a*) continually decrease in Arithmetical Progression; viz. in the Square it is *aa, a*, In the Cube *aaa, aa, a*.

In the Biquadrat it is, *aaaa, aaa, aa, a, &c.*

3. The Indices of the other Quantity *b* do continually encrease in Arithmetical Progression; viz. In the Square it is *b, bb*. In the Cube *b, bb, bbb*. In the Biquadrat it is *b, bb, bbb, bbbb*. &c.

4. The first and last Terms, are always pure Powers of the single Quantities, and are both of the same Height.

5. The Sum of the Indices of any two Letters joined together in the intermediate Terms, are always equal to the Index of the highest Power, viz. of the first or last Term.

These Observations being duly considered, it will be easy to conceive how the Terms of any proposed Power raised from a Binomial or Residual Root must stand, without their *Unciæ* or Numeral Figures.

For Instance, suppose it were required to raise the Binomial Root  $a + b$  to the seventh Power; then the Terms of that Power will stand without their *Unciæ* in this Order.

$$\text{Viz. } a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7.$$

And because the *Uncia* (not only of any single Letter, but also) of every single Power, how high soever it be, is an Unit or 1 (which neither multiplies nor divides) and all the Powers of any Binomial, or Residual Root are naturally raised by multiplying of the preceding Power into its original Root, which is done by only joining each Letter in the Root to the preceding Power, with its *Uncia*, and then removing the said Power, when it is so join'd to the second Letter, one Place forwards (either to the left, or right Hand) it must needs follow,

That the *Unciæ* of the second Terms (in any such Power) will always be the Sum of so many Units added together more one, as there have been Multiplications of the first Root; which will always be determined by the Index of the first Term in the Power.

And because the *Unciæ* of all the intermediate Terms, are only removed along with their Letters, it also follows; that if they are added together, their respective Sums will produce the true *Unciæ* of the intermediate Terms in the new raised Power. As doth plainly appear from the following Numbers so removed without their Letters; which both shews and demonstrates an easy Way of producing the *Unciæ* of any ordinary Power (viz. of one not very high) raised from either a Binomial, or Residual Root.

Thus

	Thus																																								
Add {	<table border="0"> <tr><td>I</td><td>.</td><td>I</td><td>.</td></tr> <tr><td></td><td></td><td>I</td><td>.</td></tr> <tr><td></td><td></td><td></td><td>I</td></tr> </table>	I	.	I	.			I	.				I	The two <i>Unciæ</i> of the first Root.																											
I	.	I	.																																						
		I	.																																						
			I																																						
Add {	<table border="0"> <tr><td>I</td><td>.</td><td>2</td><td>.</td><td>I</td></tr> <tr><td></td><td></td><td>I</td><td>.</td><td>2</td></tr> <tr><td></td><td></td><td></td><td>.</td><td>I</td></tr> </table>	I	.	2	.	I			I	.	2				.	I	The <i>Unciæ</i> of the Square.																								
I	.	2	.	I																																					
		I	.	2																																					
			.	I																																					
Add {	<table border="0"> <tr><td>I</td><td>.</td><td>3</td><td>.</td><td>3</td><td>.</td><td>I</td></tr> <tr><td></td><td></td><td>I</td><td>.</td><td>3</td><td>.</td><td>3</td></tr> <tr><td></td><td></td><td></td><td>.</td><td>3</td><td>.</td><td>I</td></tr> </table>	I	.	3	.	3	.	I			I	.	3	.	3				.	3	.	I	The <i>Unciæ</i> of the Cube.																		
I	.	3	.	3	.	I																																			
		I	.	3	.	3																																			
			.	3	.	I																																			
Add {	<table border="0"> <tr><td>I</td><td>.</td><td>4</td><td>.</td><td>6</td><td>.</td><td>4</td><td>.</td><td>I</td></tr> <tr><td></td><td></td><td>I</td><td>.</td><td>4</td><td>.</td><td>6</td><td>.</td><td>4</td></tr> <tr><td></td><td></td><td></td><td>.</td><td>4</td><td>.</td><td>6</td><td>.</td><td>I</td></tr> </table>	I	.	4	.	6	.	4	.	I			I	.	4	.	6	.	4				.	4	.	6	.	I	The <i>Unciæ</i> of the 4th Power.												
I	.	4	.	6	.	4	.	I																																	
		I	.	4	.	6	.	4																																	
			.	4	.	6	.	I																																	
Add {	<table border="0"> <tr><td>I</td><td>.</td><td>5</td><td>.</td><td>10</td><td>.</td><td>10</td><td>.</td><td>5</td><td>.</td><td>I</td></tr> <tr><td></td><td></td><td>I</td><td>.</td><td>5</td><td>.</td><td>10</td><td>.</td><td>10</td><td>.</td><td>5</td></tr> <tr><td></td><td></td><td></td><td>.</td><td>5</td><td>.</td><td>10</td><td>.</td><td>5</td><td>.</td><td>I</td></tr> </table>	I	.	5	.	10	.	10	.	5	.	I			I	.	5	.	10	.	10	.	5				.	5	.	10	.	5	.	I	<i>Unciæ</i> of the 5th Power.						
I	.	5	.	10	.	10	.	5	.	I																															
		I	.	5	.	10	.	10	.	5																															
			.	5	.	10	.	5	.	I																															
Add {	<table border="0"> <tr><td>I</td><td>.</td><td>6</td><td>.</td><td>15</td><td>.</td><td>20</td><td>.</td><td>15</td><td>.</td><td>6</td><td>.</td><td>I</td></tr> <tr><td></td><td></td><td>I</td><td>.</td><td>6</td><td>.</td><td>15</td><td>.</td><td>20</td><td>.</td><td>15</td><td>.</td><td>6</td></tr> <tr><td></td><td></td><td></td><td>.</td><td>6</td><td>.</td><td>15</td><td>.</td><td>20</td><td>.</td><td>15</td><td>.</td><td>I</td></tr> </table>	I	.	6	.	15	.	20	.	15	.	6	.	I			I	.	6	.	15	.	20	.	15	.	6				.	6	.	15	.	20	.	15	.	I	<i>Unciæ</i> of 6th Power.
I	.	6	.	15	.	20	.	15	.	6	.	I																													
		I	.	6	.	15	.	20	.	15	.	6																													
			.	6	.	15	.	20	.	15	.	I																													
	<table border="0"> <tr><td>I</td><td>.</td><td>7</td><td>.</td><td>21</td><td>.</td><td>35</td><td>.</td><td>35</td><td>.</td><td>21</td><td>.</td><td>7</td><td>.</td><td>I</td></tr> </table>	I	.	7	.	21	.	35	.	35	.	21	.	7	.	I	<i>Unciæ</i> of 7 Power.																								
I	.	7	.	21	.	35	.	35	.	21	.	7	.	I																											
And so on in this manner <i>ad infinitum</i> .																																									

Now if these Numbers are prefix'd to the aforesaid Letters, all the Terms will be completed with their respective *Unciæ*, and will stand thus;

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

But that the Business of finding these *Unciæ*, may be rendred yet more easy for Practice, it will be convenient to consider what Series or Progression, the *Unciæ* of each Term do make, from the aforesaid Additions.

Uncia of the First Term.	Uncia of the Second Term.	Uncia of the Third Term.	Uncia of the Fourth Term.	Uncia of the Fifth Term.	Uncia of the Sixth Term.	Uncia of the Seventh Term.	Uncia of the Eighth Term. &c.
I	I						Uncia of the single Quantities.
I	2	I					Uncia of the Square.
I	3	3	I				Uncia of the Cube.
I	4	6	4	I			Uncia of the Fourth Power.
I	5	10	10	5	I		Uncia of the Fifth Power.
I	6	15	20	15	6	I	Uncia of the Sixth Power.
I	7	21	35	35	21	7	I Uncia of the Seventh Power, &c.

The *Unciæ* of the first Term are only a Series of Units, whose Sum is every where the *Unciæ* of the second Term.

The *Uncia* of the second Term, are a Series of Numbers in Arithmetical Progression; whose Sum is every where the *Uncia* of the next superior Power in the third Term, and may be found by *Proposition 1. Chap. 6. Part 1.* That



## Chap. 2. Involution of Quantities. 159

That is, in the 7th Power it will be the  $\frac{6 + 1 \times 6}{2} = 21$   
*Uncia* of the third Term.

The rest of the *Unciæ* are a compounded Series, whose respective Sums may be obtained from the *Unciæ* of their preceding Terms.

Thus  $\frac{21 \times 5}{3} = 35$ . Then  $\frac{35 \times 4}{4} = 35$ . Again  $\frac{35 \times 3}{5} = 21$

And  $\frac{21 \times 2}{6} = 7$ , &c.

From hence may be deduced this General Rule.

### Rule.

*If the Index of the First Letter of any Term, be multiplied into its own Uncia, and that Product be divided by the Number of Terms to that Place; the Quotient will be the Uncia of the next succeeding Term forward.*

That is, by the Help of those Indices that belong to the several Powers of the first or leading Letter only (as *a*) the true *Uncia* of every Term may be easily found.

### Examples.

Let it be required to complete all the Terms of the aforesaid several Powers, viz.  $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7$ . with their proper *Uncia*.

1. The Index of  $a^7$  the first Term will be the *Uncia* of the second Term. Thus  $a^7 + 7a^6b$ .

2. Then half the Second Term's Index into its *Uncia*,  
 viz.  $\frac{7 \times 6}{2} = 21$  will be the third Term's *Uncia*.

Thus  $a^7 + 7a^6b + 21a^5b^2$  will be the three first Terms.

3. Again  $\frac{21 \times 5}{3} = 35$  is the *Uncia* of the fourth Term.

Then it will be  $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3$ .

3. And  $\frac{35 \times 4}{4} = 35$  will be the *Uncia* of the fifth Term.

Then  $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4$ , &c. until all the Terms are completed with their respective *Unciæ*; and then they will stand

Thus.  $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$ .

Now

Now here it may be further observ'd, that the *Unciæ* do only encrease until the Indices of the two Letters become equal, or change Places; and then the rest of the *Unciæ* will return or decrease in the same Order. That is, wherever the Indices of the Letters are alike, there the *Unciæ* will be alike.

And therefore one needs to find the *Unciæ* (as before) but to half the Number of Terms in any Power.

If what hath been here said, and the Work of the Example be well understood, I presume it will be found very easy to raise any Power from a binomial or residual Root, to what Height you please; without the Trouble of a continued Involution; and without the Help of such a Table of Powers as is proposed by Mr. Oughtred in his *Key to the Mathematicks*, Page 40. and since by others.

Now from these Considerations it was, that I proposed this Method of raising Powers in my *Compendium of Algebra*, Page 57. as wholly new (viz. so much of it as was there useful) having then, (I profess) neither seen the Way of doing it, nor so much as heard of its being done. But since the writing of that *Traçt*, I find in Dr. Wallis's *History of Algebra*, Page 319 and 331, that the learned Sir Isaac Newton had discovered it long before; which the *Doctor* sets down in this Manner.

Let  $m$  be the Exponent of the Power.

$$\text{Then } \S \quad 1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \text{ \&c.}$$

Will be the Series of the *Unciæ* required; but he doth not tell us how they first came to be found out, nor have I ever met with the least Hint of it in any Author.

## Sect. 6. Evolution of whole Quantities.

Evolution is the extracting of Roots from any given Power. That is, it is the converse Work to that of Involution. and in single Quantities 'tis easy, if the given Power hath such a Root as is required, which may be thus known.

If the given Power hath no Numbers prefix'd to it, and its Index can be divided by the Index of the Root required, the Quotient will be the Index of the Root sought.

Thus, if the Cube Root of *aaaaaa* viz.  $a^6$  were required (the Index of the Cube is 3) then  $3 \mid 6$  (2. That is, 3)  $a^6$  ( $a^2$  the Root required. And such Operations are usually set down

Thus



# Chap. 2. Evolution of Quantities. 161

Thus	I	$a^6$	$a^6b^6$	$a^6b^6d^6$
I w 2	2	$a^3$	$a^3b^3$	$a^3b^3d^3$
I w 3	3	$aa$	$aabb$	$aabbdd$
3 w 2	4	$a$	$ab$	$abd$

Note, The Figures placed in the Margin after the Sign (w) of Evolution; denote the Index of the Root to be extracted.

If the given Powers have Co-efficients: viz. Numbers prefix'd to them ) then you must extract the respective Roots as in Vulgar Arithmetick.

Thus	I	$81a^4$	$1296a^8b^8$	$20736a^4b^4c^4$
I w 2	2	$9aa$	$36a^4b^4$	$144aabbcc$
I w 4	3	$3a$	$6aabb$	$12abc$
Or 2 w 2	4	$3a$	$6aabb$	$12abc$

But if the Root required cannot be truly extracted out of both the Co-efficients and Indices of the given Power; then it is a Surd, and must have the Sign of the Root required prefix'd to it.

Thus	I	$a^5$	$67aaaa$	$216bbbdd$
I w 2	2	$\sqrt{a^5}$	$\sqrt{67aaaa}$	$\sqrt{216bbbdd}$
I w 3	3	$\sqrt[3]{a^5}$	$\sqrt[3]{67aaaa}$	$\sqrt[3]{6bd}$

Evolution of Compound Quantities or Powers, is a little more troublesom than that of single Powers; and would require a great many Words to explain the Manner, and Reason of forming the several Canons, that are commonly used in extracting the Roots of Compound Quantities; especially if the Powers be very high, &c. I shall therefore for Brevity's sake omit them, and instead thereof propose an easy Method of discovering the Roots of all compound Powers in general; and in order to that, it will be necessary to premise; that if either the Sum or Difference of several Quantities be involved to any Power, there will arise so many single Powers of the same Height, as there are different Quantities.

As for Instance, if  $a + b + d$  be squared; that is, be involved to the second Power, it will be  $aa + 2ab + 2ad + bb + 2bd + dd$ , here you have  $aa$ ,  $bb$ , and  $dd$ .

Y

Again,

Again, if  $a + b + d$  were cubed, viz. involved to the third Power, then you will have  $aaa$ ,  $bbb$ , and  $ddd$ , in it, &c.

Whence it follows, that in extracting the Roots of all compound Quantities, there must be consider'd,

1. How many different Letters, or Quantities, there are in the given Power.

2. Whether the single Powers of each of those Letters be of an equal Height, and have in them such a single Root as is required: Which if they have, extract it as before.

3. Connect those single Roots together with the Sign  $+$  and involve them to the same Height with the given Power; that being done, compare the new raised Power with the given Power, and if they are alike in all their respective Terms, then you have the Root required; or if they differ only in their Signs, the Root may be easily corrected with the Sign  $-$  as Occasion requires.

*Example 1.* Let it be required to extract the Square Root of  $cc + 2cb - 2cd + bb - 2bd + dd$ .

In this compound Square there are three distinct Powers, viz.  $bb$ ,  $cc$ ,  $dd$ , whose single Roots are  $b$ ,  $c$ ,  $d$ , wherefore I suppose the Root sought to be  $b + c + d$ , or rather  $b + c - d$ , because in the given Power there is  $-2cd$ , and  $-2bd$ , therefore I conclude it is,  $-d$ , then  $b + c - d$  being squared, produces  $bb + 2bc - 2bd + cc - 2cd + dd$  which I find to be the same in all its Terms with the given Power, although they stand in a different Position; consequently  $b + c - d$  is the true Root required.

*Example 2.* 'Tis required to extract the Square Root of  $a^4 - 2aabb + b^4$ . Here are but two single Powers, viz.  $a^4$  and  $b^4$ , whose Square Roots are  $aa$ , and  $bb$ . And because in the given Power there is  $-2aabb$ , therefore I conclude it must either be  $aa - bb$ , or  $bb - aa$ . Both which being involved, will produce  $a^4 - 2aabb + b^4$  consequently the Root sought may either be  $aa - bb$  or  $bb - aa$  according to the Nature or Design of the Question, from whence the given Power was produced.

*Example 3.* Let it be required to extract the Square Root of  $36aaaa + 108aa + 81$ . Here the two single Powers are  $36aaaa$ , and  $81$ , whose Roots are  $6aa$  and  $9$ . And because the Signs are all  $+$  therefore I suppose the Root to be  $6aa + 9$ , the which being involved doth produce  $36a^4 + 108aa + 81$ , consequently  $6aa + 9$  is the true Root required.

*Example*



## Chap. 2. Evolution of Quantities. 163

*Example 4.* Suppose it were required to extract the Cube Root of  $125aaa + 300aae - 450aa + 240aee - 720ae + 64eee + 540a - 288ee + 432e - 216$ .

In this Example there are three distinct Powers, *viz.*  $125aaa$ ,  $64eee$ , and  $-216$ .

The Cube Root of  $125aaa$  is  $5a$ , of  $64eee$  is  $4e$ , and the Cube Root of  $-216$  is  $-6$ .

Wherefore I suppose the Root sought to be  $5a + 4e - 6$ , which being involved to the third Power, does produce the same with the given Power, consequently  $5a + 4e - 6$  is the Cube Root required.

But if the new Power raised from the supposed Root (being involved to its due Height) should not prove the same with the given Power, *viz.* if it hath either more or fewer Terms in it, &c. Then you may conclude the given Power to be a Surd, and must have its proper Sign prefix'd to it, and cannot be otherwise express'd, until it come to be evolved in Numbers.

*Example 5.* Suppose it were required to extract the Cube Root of  $27aaa + 54baa - 8bbb$ .

Here are two distinct and perfect Cubes, *viz.*  $27aaa$ , and  $8bbb$ , whose Cube Roots are  $3a$  and  $2b$ .

Wherefore one may suppose the Root sought to be  $3a + 2b$  which being involved to the third Power, is  $27aaa + 54baa + 36bba + 8bbb$ . Now this new raised Power hath one Term (*viz.*  $36bba$ ) more in it than the given Power hath; but this being a perfect Cube, one may therefore conclude the given Power is not so, *viz.* it is a Surd, and hath not such a Root as was required, but must be express'd or set down,

$$\text{Thus } \sqrt[3]{27aaa + 54baa + 8bbb}.$$

If these Examples be well understood, the Learner will find it very easy by this Method of proceeding to discover the true Root of any given Power whatsoever.

## C H A P. III.

### Of Algebraic Fractions, or Broken Quantities.

#### Sect. 1. Notation of Fractional Quantities.

Fractional Quantities are express'd or set down like Vulgar Fractions in common Arithmetick,

$$\text{Thus } \left\{ \frac{a}{b} \quad \frac{2bc}{d} \quad \frac{5b-4a}{4d+7b} \right. \quad \begin{array}{l} \text{Numerators.} \\ \text{Denominators.} \end{array}$$

How they come to be so, see *Cas.* 4. in the last Chapter of Division. These Fractional Quantities are managed in all respects like Vulgar Fractions in common Arithmetick.

**Sect. 2.** *To Alter or Change different Fractions into one Denomination, retaining the same Value.*

**Rule.**

*Multiply all the Denominators into each other for a new Denominator: and each Numerator into all the Denominators but its own, for new Numerators.*

**Examples.**

Let it be required to bring  $\frac{a}{b}$  and  $\frac{d}{c}$  into one Denomination.

First,  $a \times c$ , and  $d \times b$ , will be the Numerators, and  $b \times c$ , will be the common Denominator, viz.  $\frac{ca}{bc}$  and  $\frac{bd}{bc}$  are the two Fractions required. That is,  $\frac{ca}{bc} = \frac{a}{b}$  and  $\frac{bd}{bc} = \frac{d}{c}$

Again, let  $\frac{b+c}{a+b}$  and  $\frac{d-c}{b-d}$  be brought in one Denomination.

And they will be  $\frac{bb+bc-bd-dc}{ba+bb-da-bd}$  and  $\frac{ad-ac+bd-bc}{ba+bb-da-bd}$  &c.

**Sect. 3.** *To Bring whole Quantities into Fractions of a given Denomination.*

**Rule.**

*Multiply the whole Quantities into the given Denominator, for a Numerator: under which subscribe the given Denominator, and you will have the Fraction required.*

**Examples.**

Let it be required to bring  $a+b$  into a Fraction, whose Denominator is  $d-a$ . First  $a+b \times d-a$  is  $da+bd-aa-ba$ .

Then  $\frac{da+bd-aa-ba}{d-a}$  is the Fraction required.

Again  $b+\frac{a}{d}$  will be  $\frac{db+a}{d}$ . And  $\frac{aa}{d}-a$

will be  $\frac{aa-da}{d}$ . Also  $a+b+\frac{aa+bb}{a-b}$  will be  $\frac{2aa}{a-b}$

When



# Chap. 3. Of Fractional Quantities. 165

When whole Quantities are to be set down Fraction-wise, subscribe an Unit for the Denominator.

Thus  $ab$ , is  $\frac{ab}{1}$  And  $aa - bb$ , is  $\frac{aa - bb}{1}$  &c.

## Sect. 4. To Abbreviate, or Reduce Fractional Quantities into their lowest Denomination.

### Rule.

Divide both the Numerator and Denominator by their greatest common Divisor, viz. by such Quantities as are found in both; and their Quotients will be the Fraction in its lowest Term.

Thus  $\frac{aac}{dc}$ , is  $\frac{aa}{d}$ . And  $\frac{abbb}{abc}$ , is  $\frac{bb}{c}$ . Also  $a + \frac{bdc}{bc} = a + d$ .

In such single Fractions as these, the common Divisors (if there be any) are easily discover'd by Inspection only; but in compound Fractions it often proves very troublesom, and must be done either by dividing the Numerator by the Denominator, until nothing remains, when that can be done: Or else finding their common Measure; by dividing the Denominator by the Numerator, and the Numerator by the Remainder, and so on as in Vulgar Fractions (Sect. 4. Page 51.)

### Examples.

Suppose  $\frac{aac - aad}{cd - dd}$  were to be reduced lower.

Then  $cd - dd$ )  $\frac{aac - aad}{aac - aad}$  ( $\frac{aa}{d}$  the Fraction required.

In this Example it so happens that the Numerator is divided just off by the Denominator; but in the next 'tis otherwise, and requires a double Division to find out the common Measure. Viz. Let it be required to reduce  $\frac{aaa - abb}{aa + 2ab + bb}$  to its lowest Terms.

First  $aa + 2ab + bb$ )  $\frac{aaa - abb}{aaa + 2aab + abb}$

Then  $- 2aab - 2abb$ )  $\frac{aa + 2ab + bb}{aa + ab}$  the Remainder.

$\frac{ab + bb}{ab + bb}$   
 $\frac{ab + bb}{ab + bb}$

$(-\frac{1}{2b} - \frac{1}{2a}$

Hence

Viz. —  $2aab - 2abb$ )  $aaa - abb$   $\left( -\frac{a}{2b} + \frac{1}{2} \right)$   
 $aaa + aab$   


---

 $-aab - abb$   
 $-aab - abb$   


---

 $0 \quad 0$

Conse.



# Chap. 3. Of Fractional Quantities. 167

Consequently  $bd - bb$ )  $dd - bb$   $\left(\frac{d}{b} + 1\right)$  is the new Numerator.

$$\begin{array}{r} dd - bb \\ dd - db \\ \hline + db - bb \\ db - bb \\ \hline 0 \quad 0 \end{array}$$

And  $bd - bb$ )  $ddd - bbb$   $\left(\frac{dd}{b} + d + b\right)$  the new Denominator.

$$\begin{array}{r} ddd - bbb \\ ddd - ddb \\ \hline + ddb - bbb \\ ddb - bbd \\ \hline + bbd - bbb \\ bbd - bbb \\ \hline 0 \quad 0 \end{array}$$

Let both be multiplied with  $b$ , and then you will have  $\left\{ \begin{array}{l} d + b \\ dd + bd + bb \end{array} \right.$  the Numerator } of the Fraction required.

But if after all Means used (as above) there cannot be found one common Measure to both the Numerator and Denominator, then is that Fraction in its least Terms already.

Note, These Operations will be well understood by a Learner after he hath past through Multiplication, and Division of Fractions.

## Sect. 5. Addition and Subtraction of Fractional Quantities.

The given Fractions being of one Denomination, or if they are not, make them so, per Sect. 4. Then

### Rule.

Add or subtract their Numerators, as Occasion requires, and to their Sum, or Difference, subscribe the common Denominator: As in Vulgar Fractions.

### Examples in Addition.

1	$\frac{bb}{c}$	$\frac{a + b}{d}$	$\frac{2a - b}{d + c}$	$\frac{a - b + d}{d + a}$
2	$\frac{aa}{c}$	$\frac{2a + c}{d}$	$\frac{2b - a}{d + c}$	$\frac{a + b - d}{d + a}$
3	$\frac{bb + aa}{c}$	$\frac{3a + b + c}{d}$	$\frac{a + b}{d + c}$	$\frac{2a}{d + a}$
1 + 2				
	$\frac{bb + aa}{c}$	$\frac{3a + b + c}{d}$	$\frac{a + b}{d + c}$	$\frac{2a}{d + a}$

Exam.

## Examples in Subtraction.

1 — 2	1	$\frac{bb + aa}{c}$	$\frac{a + b}{d + c}$	$\frac{3a + b + c}{d}$	$\frac{2b}{d + a}$
	2	$\frac{bb}{c}$	$\frac{2b - a}{d + c}$	$\frac{2a + c}{d}$	$\frac{a + b - d}{d + a}$
		$\frac{c}{aa}$	$\frac{2a - b}{d + c}$	$\frac{a + b}{d}$	$\frac{-a + b + d}{d + a}$
	3	$\frac{c}{aa}$	$\frac{2a - b}{d + c}$	$\frac{a + b}{d}$	$\frac{-a + b + d}{d + a}$
	3	$\frac{c}{aa}$	$\frac{2a - b}{d + c}$	$\frac{a + b}{d}$	$\frac{-a + b + d}{d + a}$
		$\frac{c}{aa}$	$\frac{2a - b}{d + c}$	$\frac{a + b}{d}$	$\frac{-a + b + d}{d + a}$

## Sect. 6. Multiplication of Fractional Quantities.

First, prepare *mix'd Quantities* (if there be any) by making them *improper Fractions*, and *whole Quantities* by subscribing an *Unit* under them. As per Sect. 3. Then

## Rule.

Multiply the Numerators together for a new Numerator: And the Denominators together for a new Denominator. As in *Vulgar Fractions*.

Thus

1 × 2	1	$\frac{ab}{c}$	$\frac{3a - 2b}{2d + c}$
	2	$\frac{d}{f}$	$\frac{4a + 2b}{d}$
		$\frac{abd}{cf}$	$\frac{12aa - 2ab - 4bb}{2dd + dc}$
	3	$\frac{abd}{cf}$	$\frac{12aa - 2ab - 4bb}{2dd + dc}$
	3	$\frac{abd}{cf}$	$\frac{12aa - 2ab - 4bb}{2dd + dc}$
		$\frac{abd}{cf}$	$\frac{12aa - 2ab - 4bb}{2dd + dc}$

Suppose it were required to multiply  $2a + \frac{b}{c} - 25$  with  $3b + 4c$ . These prepared for the Work (per Sect. 3.) will stand

Thus

1 × 2	1	$\frac{2ac + b - 25c}{c}$
	2	$\frac{3b + 4c}{1}$
		$\frac{6bac + 3bb - 75bc + 8acc + 4bc - 100cc}{c}$
	3	$\frac{6bac + 3bb - 75bc + 8acc + 4bc - 100cc}{c}$
	3	$\frac{6bac + 3bb - 75bc + 8acc + 4bc - 100cc}{c}$
		$\frac{6ba - 71b + 8ac - 100c + \frac{3bb}{c}}{c}$ per Sect. 4.



# Chap. 3. Of Fractional Quantities. 169

N. B. Any Fraction is multiplied with its Denominator by casting off, or taking the Denominator away.

Thus  $\frac{b}{a} \times a$  gives  $b$ . For  $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$  &c.

## Sect. 7. Division of Fractional Quantities.

The Fractional Quantities being prepar'd, as directed in the last section. Then

### Rule.

Multiply the Numerator of the Dividend, into the Denominator of the Divisor, for a new Numerator; and multiply the other two together for a new Denominator. As in Vulgar Fractions.

### Examples.

Let  $\frac{abd}{cf}$  be divided by  $\frac{ab}{c}$  the Work may stand

Thus  $\frac{ab}{c} ) \frac{abd}{cf} \left( \frac{abdc}{abcf} = \frac{d}{f} \text{ per Sect. 4.} \right.$

Or thus	1	$\frac{abd}{cf}$	$\frac{a+b}{d}$	$\frac{aaa - bbb}{a+b}$
	2	$\frac{ab}{c}$	$\frac{c-b}{a}$	$\frac{aa - ab + bb}{c}$
	3	$\frac{d}{f}$	$\frac{aa + ba}{dc - db}$	$\frac{aaac - bbbc}{aaa + bbb}$

Suppose it were required to divide  $aa + \frac{3abb}{a+4b}$

By  $a + b$ . Then the Work prepared will stand

Thus  $\frac{a+b}{1} ) \frac{aaa + 4aab + 3aabb}{a+4b} \left( \frac{aaa + 4aab + 3aabb}{aa + 5ba + 4bb} \right.$

But  $\frac{aaa + 4aab + 3aabb}{aa + 5ba + 4bb} = \frac{aa + 3ba}{a + 4b} \text{ (per Sect 4.)}$

When Fractions are of one Denomination, cast off the Denominators, and divide the Numerators.

Thus, if  $\frac{ab^3}{c}$  were to be divided by  $\frac{bb}{c}$  it will be

$bb ) ab^3 \text{ ( } ab \text{ the Quotient required.}$

Z

For

For  $\frac{bb}{c} \bigg) \frac{ab^3}{c} \left( \frac{ab^3c}{bbc} \right.$  But  $\frac{ab^3c}{bbc} = ab$  (per Sect. 4.)

Again, Suppose it were required to divide  $\frac{a^3 - abb}{c - d}$

By  $\frac{aa + 2ab + bb}{c - d}$  Casting off  $c - d$  in both, it will be

$$aa + 2ab + bb \bigg) aaa - abb \quad \left( \frac{aa - ba}{a + b} \&c. \right.$$

### Sect. 8. Involution of Fractional Quantities.

#### Rule.

Involve the Numerator into it self, for a new Numerator; and the Denominator into it self, for a new Denominator; each as often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b + d}{a - c}$
I ⑥ 2	2	$\frac{bb}{aa}$	$\frac{9bbcc}{4aadd}$	$\frac{bb + 2bd + dd}{aa - 2ac + cc}$
I ⑥ 3	3	$\frac{bbb}{aaa}$	$\frac{27bbbccc}{8aaaddd}$	$\frac{bbb + 3bbd + 3bdd + ddd}{aaa - 3aac + 3acc - ccc}$

### Sect. 9. Evolution of Fractional Quantities.

If the Numerator and Denominator of the given Fraction have each of them such a Root as is required; (which very rarely happens.) Then evolve them; and their respective Roots will be the Numerator, and Denominator of the new Fraction required.

Thus		$\frac{9aabb}{4dd}$	$\frac{aa + 2ab + bb}{aa - 2ab + bb}$
I w 2		$\frac{3ab}{2d}$	$\frac{a + b}{a - b}$

Again	1	$\frac{27aaabbb}{8ddd}$	$\frac{aaa + 3aab + 3abb + bbb}{aaa - 3aab + 3abb - bbb}$
I w 3	2	$\frac{3ab}{2d}$	$\frac{a + b}{a - b}$

Sometimes it so falls out, that the Numerator may have such a Root as is required, when the Denominator hath not; or the Deno-



Denominator may have such a Root, when the Numerator hath not. In those Cases the Operations may be set down

$$\begin{array}{l|l|l} \text{Thus} & 1 & \frac{aabb}{ddd} \\ \hline \text{I w 2} & 2 & \frac{ab}{\sqrt{ddd}} \end{array} \quad \begin{array}{l} \frac{aaa+4bb-dd}{aa+2ab+bb} \\ \hline \frac{\sqrt{aaa+4bb-dd}}{a+b} \end{array}$$

But when neither the Numerator, nor the Denominator have just such a Root as is required; prefix the Radical Sign of the Root to the Fraction; and then it becomes a Surd, as in the last Step, which brings me to the Business of managing Surds.

## C H A P. IV.

### Of Surd Quantities.

The whole Doctrine of Surds (as they call it) were it fully handled, would require a very large Explanation (to render it but tolerably intelligible) even enough to fill a Treatise it self; if all the various Examples that may be of Use to make it easy should be inserted; without which 'tis very intricate and troublesome for a Learner to understand. But now those tedious Reductions of Surds, which were heretofore thought useful to fit *Æquations* for such a Solution, as was then understood, are wholly laid aside as useless: Since the new Methods of resolving all Sorts of *Æquations* renders their Solution equally easy, although their Powers are never so high.

Nay, ever since the true Use of Decimal Arithmetick hath been well understood, the Business of Surd Numbers has been managed that Way; as appears by several Instances of that kind, in Dr. Wallis's *History of Algebra*. from Page 23 to 29.

I shall therefore, for Brevity's sake, pass over those tedious Reductions, and only shew the young Algebraist how to deal with such Surd Quantities as may arise in the Solution of hard Questions.

#### Sect. I. Addition and Subtraction of Surd Quantities.

Case I. When the Surd Quantities are Homogeneous (*viz.* they and their Indices are alike) Add, or Subtract the

the Rational Part, if are joined to any, and to their Sum, or Difference, adjoin the Irrational or Surd.

*Examples in Addition.*

$$\begin{array}{r|l} 1 & 5 \sqrt{bc} \quad | \quad 6b \sqrt{ac} \quad | \quad b \sqrt{aa+cc} \\ 2 & 7 \sqrt{bc} \quad | \quad 4b \sqrt{ac} \quad | \quad 3b \sqrt{aa+cc} \\ \hline 1+2 & 3 \quad | \quad 12 \sqrt{bc} \quad | \quad 10b \sqrt{ac} \quad | \quad 4b \sqrt{aa+cc} \end{array}$$

$$\begin{array}{r|l} 1 & 4d^3 \sqrt{aa} \quad | \quad b+c^3 \sqrt{aa+cc} \quad | \quad bc^5 \sqrt{aa+d} \\ 2 & d^3 \sqrt{aa} \quad | \quad c-^3 \sqrt{aa-cc} \quad | \quad 2bc^5 \sqrt{aa+d} \\ \hline 1+2 & 3 \quad | \quad 5d^3 \sqrt{aa} \quad | \quad b+c \quad | \quad 4bc^5 \sqrt{aa+d} \end{array}$$

*Examples in Subtraction.*

$$\begin{array}{r|l} 1 & 12 \sqrt{bc} \quad | \quad 10b \sqrt{ac} \quad | \quad 4b \sqrt{aa+cc} \\ 2 & 7 \sqrt{bc} \quad | \quad 4b \sqrt{ac} \quad | \quad 3b \sqrt{aa+cc} \\ \hline 1-2 & 3 \quad | \quad 5 \sqrt{bc} \quad | \quad 6b \sqrt{ac} \quad | \quad b \sqrt{aa+cc} \end{array}$$

$$\begin{array}{r|l} 1 & 5d^3 \sqrt{aa} \quad | \quad b+c \quad | \quad 4bc^5 \sqrt{aa+d} \\ 2 & 4d^3 \sqrt{aa} \quad | \quad c-^3 \sqrt{aa-cc} \quad | \quad 3bc^5 \sqrt{aa+d} \\ \hline 1-2 & 3 \quad | \quad d^3 \sqrt{aa} \quad | \quad b+^3 \sqrt{aa-cc} \quad | \quad bc^5 \sqrt{aa+d} \end{array}$$

*Case 2.* When the Surd Quantities are Heterogeneous (*viz.* they or their Indices are unlike) they are only to be added, or subtracted by their Signs, *viz.* + or - And from thence will arise Surds, either Binomial, or Residual.

*Examples in Addition.*

$$\begin{array}{r|l} 1 & ^3 \sqrt{ba} \quad | \quad 4d \sqrt{a} \quad | \quad ^2 \sqrt{ac-ba} \\ 2 & \sqrt{ba} \quad | \quad 3b \sqrt{ac} \quad | \quad ^3 \sqrt{ac+ba} \\ \hline 1+2 & 3 \quad | \quad ^3 \sqrt{ba} + \sqrt{ba} \quad | \quad 4d \sqrt{a} + 3b \sqrt{ac} \quad | \quad ^2 \sqrt{ac-ba} + ^3 \sqrt{ac+ba} \end{array}$$

*Examples in Subtraction.*

$$\begin{array}{r|l} 1 & \sqrt{bc} \quad | \quad b-d \sqrt{aaa+ca} \\ 2 & \sqrt{ba} \quad | \quad d-2a \sqrt{bd+dd} \\ \hline 1-2 & 3 \quad | \quad \sqrt{bc} - \sqrt{ba} \quad | \quad b-d \sqrt{aaa+ca} - d + 2a \sqrt{bd+dd} \end{array}$$



## Sect. 2. Multiplication of Surd Quantities.

*Case 1.* When the Quantities are Pure Surds of the same kind; Multiply them together, and to their Product prefix their radical Sign.

*Examples.*

I	x	2	3	I	$\sqrt{b}$	$\sqrt{ba+da}$	$\sqrt{aa+bb}$
				2	$\sqrt{a}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
				3	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$

*Case 2.* If the Surd Quantities of the same kind (as before) are joined to rational Quantities, then multiply the Rational into the Rational; and the Surd into the Surd, and join their Products together.

*Examples.*

I	x	2	3	I	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$
				2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
				3	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$

## Sect. 3. Division of Surd Quantities.

*Case 1.* When the Quantities are pure Surds of the same kind, and can be divided off (*viz.* without leaving a Remainder) divide them, and to their Quotient prefix their radical Sign.

*Examples.*

I	÷	2	3	I	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$
				2	$\sqrt{b}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
				3	$\sqrt{a}$	$\sqrt{ba+da}$	$\sqrt{aa-bb}$

*Case 2.* If Surd Quantities of the same kind, are joined to Rational Quantities; then divide the Rational by the Rational, if it can be, and to their Quotient, join the Quotient of the Surd, divided by the Surd, with its first radical Sign.

*Examples.*

I	÷	2	3	I	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$
				2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
				3	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$

*Note*

Note, If any Square be divided by its Root, the Quotient will be its Root.

*Examples.*

I	÷ 2	I	a	bb + 2bc + cc	aaaa - 2bbaa + bbbb
		2	$\sqrt{a}$	$\sqrt{bb + 2bc + cc}$	$\sqrt{a^4 - 2bbaa + b^4}$
		3	$\sqrt{a}$	$\sqrt{bb + 2bc + cc}$	$\sqrt{a^4 - 2bbaa + b^4}$

Sect. 4. Involution of *Surd Quantities*.

*Case 1.* When the Surds are not joined to Rational Quantities; they are involved to the same Height as their Index denotes, by only taking away their Radical Sign.

*Examples.*

I	⊙ 2	I	$\sqrt{a}$	$\sqrt{bca}$	$\sqrt{aa - bb}$	$\sqrt{5a - da}$
		2	a	bca	aa - bb	5a - da

*Case 2.* When the Surds are joined to Rational Quantities; Involve the Rational Quantities to the same Height as the Index of the Surd denotes; then multiply those involved Quantities into the Surd Quantities, after their Radical Sign is taken away. As before.

*Examples.*

I	⊙ 2	$b\sqrt{a}$	$5d\sqrt{ca}$	$3b\sqrt{aa - dd}$
		bba	25ddca	9bbaa - 9bbdd

I	⊙ 3	I	$a^3\sqrt{bc}$	$3d^3\sqrt{aa + bb}$	$da^3\sqrt{b}$
		2	aaabc	$27ddd\sqrt{aa + bb}$	ddd\sqrt{aab}

The Reason of only taking away the Radical Sign, as in *Case 1.* is easily conceived, if you consider that any Root, being involved into it self, produces a Square, &c.

And from thence the Reason of those Operations performed by the second *Case* may be thus stated.

Suppose  $b\sqrt{a} = x$ . Then  $\sqrt{a} = \frac{x}{b}$  per *Axiom 4.* and both

Sides of the Equation being equally involved, it will be  $a = \frac{xx}{bb}$ .

Then multiplying both Sides of the Equation into  $bb$ , it will become  $bba = xx$ . per *Axiom 3.*

Which was to be proved.

Again



And  $ca = \frac{xx}{25dd}$  Consequently  $25ddca = xx$

Suppose	$\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right\}$	1	$\sqrt{b} = z$	$\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\}$ Example 1. Case 1.
		2	$\sqrt{a} = x$	
1		3	$b = zz$	
2		4	$a = xx$	
3		5	$ba = z z x x.$ per Axiom 3.	
5	2	6	$\sqrt{ba} = z x.$ which was to be proved.	

Let  $\left. \begin{array}{l} 1 \\ 2 \end{array} \right\}$   $\left. \begin{array}{l} d \sqrt{bc} = z \\ 3b \sqrt{a} = x \end{array} \right\}$  Example 1. Case 1.

$1 \div d$   $3$   $\sqrt{bc} = \frac{z}{d}$

$2 \div 3b$   $4$   $\sqrt{a} = \frac{x}{3b}$

$4 \times 3$   $5$   $\sqrt{bca} = \frac{zx}{3bd}$  from what is proved above

$5 \times 3bd$   $6$   $3bd \sqrt{bca} = zx$ , &c. for the rest.

Division being the Converse to Multiplication, needs no other Proof.

## CHAP. V.

Concerning the Nature of Equations, and how to prepare them for a Solution.

When any Problem or Question is proposed to be analytically resolved; it is very requisite that the true Design or Meaning thereof, be fully and clearly comprehended (in all its Parts) that so it may be truly abstracted from such ambiguous Words as Questions of this kind are often disguised with; otherwise it will be very difficult, if not impossible, to state the Question right in its substituted Letters, and ever to bring it to an  $\text{\AA}$ equation, by such various Methods of ordering those Letters as the Nature of the Questions may require.

Now the Knowledge of this difficult Part of the Work is only to be obtained by Practice, and a careful minding the Solution of such leading Questions as are in themselves very easy.

And for that Reason I have inserted a Collection of several Questions ; wherein there is great Variety.

Having got so clear an Understanding of the Question proposed, as to place down all the Quantities concerned in their due Order, *viz.* all the substituted Letters, in such Order as the Nature of the Question requires; the next thing must be to consider whether it be limited or not. That is, whether it admits of more Answers than one. And to discover that, observe the two following Rules.

### Rule.

*When the Number of the Quantities sought, exceed the Number of the given Æquations, the Question is capable of innumerable Answers.*

### Example.

Suppose a Question were proposed thus; There are three such Numbers, that if the first be added to the second, their Sum will be 22. And if the second be added to the third, their Sum will be 46. What are those Numbers?

Let the three Numbers be represented by three Letters, thus, call the first *a*, the second *e*, and the third *y*.

Then  $\left\{ \begin{array}{l} a + e = 22 \\ e + y = 46 \end{array} \right\}$  according to the Question.

Here the Number of Quantities sought are three; *viz.* *a*, *e*, *y*, and the Number of the given Æquations are but two. Therefore this Question is not limited, but admits of various Answers; because for any one of those three Letters, you may take any Number at Pleasure, that is less than 22. Which with a little Consideration will be very easy to conceive.

### Rule 2.

*When the Number of the given Æquations (not depending upon one another) are just as many as the Number of the Quantities sought; then is the Question truly limited, viz. each Quantity sought hath but one single Value.*

As for Instance, Let the aforesaid Question be proposed thus. There are three Numbers (*a*, *e*, and *y*, As before) if the first be added to the second, their Sum will be 22; if



## Chap. 5. Of Reducing Equations. 177

if the Second be added to the Third, their Sum will be 46; and if the First be added to the Third, their Sum will be 36. What are the Numbers?

That is,  $a + e = 22$ .  $e + y = 46$ . and  $a + y = 36$ .

Now the Question is perfectly limited, each single Quantity having but one single Value, to wit  $a = 6$ ,  $e = 16$ , and  $y = 30$ .

N. B. If the Number of the given *Æquations* exceeds the Number of the Quantities sought; they not only *limit* the Question, but oftentimes *render* it impossible, by being propos'd inconsistent one to another.

Having truly stated the Question in its substituted Letters, and found it limited to one Answer, (or at least so bounded as to have a certain determinate Number of Answers) then let all those substituted Letters be so ordered or compared together, either by Adding, Subtracting, Multiplying, or Dividing them, &c. according as the Nature of the Question requires, until all the unknown Quantities, except One, are cast off or vanished; but therein great Care must be taken to keep them to an exact Equality; and when that unknown Quantity, or some Power of it (as *Square*, *Cube*, &c.) is found Equal to those that are known; then the Question is said to be brought to an Equation, and consequently to a *Solution*, viz. fitted for an Answer.

But no particular Rules can be prescribed for the casting off, or getting away Quantities out of an *Æquation*; that Part of the Art is only to be obtained by Care and Practice. And when that is done, it generally happens so, that the unknown Quantity which is retained in the *Æquation*, is so mix'd and entangled with those that are known; that it often requires some Trouble and Skill to bring it (or its Powers, &c.) to one Side of the *Æquation*, and those that are known to the other Side; (still keeping them to a just Equality) which the Ingenious *VAN SCHOTEN* in his *Principia Matheseos Universalis*, calls *Reduction* of *Æquations*.

The Business of Reducing *Æquations* (as of most, if not all *Algebraic* Operations) is grounded and depends upon a right Application of the five *Axioms* proposed in Page 146. and therefore, if those *Axioms* be well understood, the Reason of such Operations must needs appear very plain, and the Work be easily performed; as in the following *Sections*.

## Sect. 1. Of Reduction by Addition.

Reduction by Addition is grounded upon Axiom 1. and is only the transposing (*viz.* the removing) of any negative Quantity from either Side of an Equation to the other Side, with the Sign + before it; as in these.

Examples.

Suppose	1	$a - b = d$	Again,	1	$aa - d = c - aa$
Then	2	$a = d + b$	Let	2	$aa = c - aa + d$
For	3	$b = b$	1 + d	3	$2aa = c + d$
1 + 3	4	$a = d + b$	2 + aa		

Let	1	$3a - 4 = 6 - a$	} Note, When any absolute Number is register'd in the Margin, you must draw a Line over it, to distinguish it from the other Numbers. As $\overline{4}$ in the 2d Step of this Example.
1 + $\overline{4}$	2	$3a = 6 + 4 - a$	
2 + a	3	$4a = 6 + 4 = 10$	

Let	1	$aa - dc - b = dd - 2ba$
1 + b	2	$aa - dc = dd - 2ba + b$
2 + dc	3	$aa = dd - 2ba + b + dc$
3 + 2ba	4	$aa + 2ba = dd + b + dc$

Suppose	1	$2da - d = cc - 3baa - aaa$
1 + aaa	2	$aaa + 2da - d = cc - 3baa$
2 + 3baa	3	$aaa + 3baa + 2da - d = cc$
3 + d	4	$aaa + 3baa + 2da = cc + d. \&c.$

## Sect. 2. Of Reduction by Subtraction.

Reduction by Subtraction is grounded upon Axiom 2. and is perform'd by transposing (or removing) any affirmative Quantity from either Side of the Equation, to the other Side, with the Sign - before it. As in these

Examples.

Suppose	1	$a + b = d$	Let	1	$3a + 4 = 6 + a$
And	2	$b = b$	1 - a	2	$2a + 4 = 6$
1 - 2	3	$a = d - b$	2 - $\overline{4}$	3	$2a = 6 - 4 = 2$

Suppose	1	$aa + dc + b = dd + 2ba$
1 - 2ba	2	$aa - 2ba + dc + b = dd$
2 - dc	3	$aa - 2ba + b = dd - dc$
3 - b	4	$aa - 2ba = dd - dc - b$

Let



Let	1	$aaa + d = cc + 3baa + 2da$
1 — 3baa	2	$aaa - 3baa + d = cc + 2da$
2 — 2da	3	$aaa - 3baa - 2da + d = cc$
3 — d	4	$aaa - 3baa - 2da = cc - d$

## Sect. 3. Of Reduction by Multiplication.

Fractional Quantities in any Equation, are brought into whole Quantities; by multiplying every Term in the Equation with the Denominators of the Fractions, *per* Axiom 3. As in these

*Examples.*

Suppose	1	$\frac{a}{5} = 6$
Then	2	$a = 6 \times 5 = 30.$ For $\frac{a}{5} \times 5 = \frac{5a}{5} = a$

Let	1	$3a = \frac{dc}{2b}$	Suppose	1	$a = \frac{dd}{a-b}$
1 × 2b	2	$6ba = dc$	1 × a — b	2	$aa - ba = dd$

Suppose	1	$\frac{aa}{b} + c + f = \frac{dx}{a}$
1 × b	2	$aa + bc + bf = \frac{dxb}{a}$
2 × a	3	$aaa + bca + bfa = dxb$

Suppose	1	$\frac{aaa}{aa-bb} = \frac{ba-bb}{a+b}$
1 × aa — bb	2	$aaa = \frac{baaa-bbaa-bbba+bbbb}{a+b}$
1 × a + b	3	$aaaa + baaa = baaa - bbaa - bbba + bbbb$

## Sect. 4. Of Reduction by Division.

When any Quantity (either known, or unknown) is in every Term of an Equation; if the whole Equation be divided by that Quantity, it will be reduc'd into lower Terms, *per* Axiom 4. As in these

A 2 2

*Examples.*

Examples.

Suppose	1	$baa + bca = bcd$	Let	1	$aa = 7a$
$1 \div b$	2	$aa + ca = cd$	$1 \div 1a$	2	$a = 7$

Let	1	$ffaa + ffcaa - ffa = ffd a + ffd a$
$1 \div ff$	2	$aa + caa - a = da + dda$
$2 \div a$	3	$a + ca - 1 = d + dd$

Or when the unknown Quantity is multiplied (*viz.* join'd) with any that is known; let the whole Æquation be divided by the known Quantity, that so the unknown may be cleared.

As in these

Examples.

Suppose	1	$ba - ca = d$	Let	1	$caa - daa = cd - dd$
$1 \div b - c$	2	$a = \frac{d}{b - c}$	$1 \div c - d$	2	$aa = \frac{cd - dd}{c - d} = d$

Suppose	1	$bbaaa - 2bbaa = bda + cba$
$1 \div ba$	2	$baa - 2ba = d + c$
$2 \div b$	3	$aa - 2a = \frac{d + c}{b}$

Let	1	$49daa + 42aa = 7bca + 21ca$
$1 \div 7$	2	$7daa + 6aa = bca + 3ca$
$2 \div a$	3	$7da + 6a = bc + 3c$
$3 \div a$	4	$a = \frac{bc + 3c}{7d + 6}$

### Sect. 5. Of Reduction by Involution.

When there happens to be an Æquation, between any Homogeneous or like Surds; take away the radical Signs from the Quantities, and they will become Rational.

As in these

Examples.

Suppose	1	$\sqrt{a} = \sqrt{d + c}$	Let	1	$\sqrt{aa} = \sqrt{db + bc}$	} per Sect. 4. Ch. 3.
$1 \odot 2$	2	$a = d + c$	$1 \odot 3$	2	$aa = db + bc$	

Or if one Side of the Æquation consists of Surd Quantities, and the other Side be rational; then involve the rational Quantities



# Chap. 5. Of Reducing Equations. 181

Quantities to the same Power (or Height) with the Index of the Surd, and take away the Radical Sign.

As in these

Examples.

Let	1	$\sqrt{a=36}$	Suppose	1	$\sqrt{a=b+c}$
I $\odot$ 2	2	$a=36$	I $\odot$ 2	2	$a=bb+2bc+cc.$

Suppose	1	$\sqrt[3]{aa-ba=d}$	Let	1	$\sqrt[5]{aa=7}$
I $\odot$ 3	2	$aa-ba=ddd$	I $\odot$ 5	2	$aa=16807.$

## Sect. 6. Of Reduction by Evolution.

When any single Powers of the unknown Quantity is on one Side of an Equation; Evolve both Sides of the Equation, according as the Index of that Power denotes, and their Roots will be equal. As in these

Examples.

Suppose	1	$aa=36$	Let	1	$aaa=27$
I w 2	2	$a=\sqrt{36}=6$	I w 3	2	$a=\sqrt[3]{27}=3, \&c.$

Suppose	1	$aa=bb-dd$	Let	1	$aaa=b^3+3bbc+3bcc+c^3$
I w 2	2	$a=\sqrt{bb-dd}$	I w 3	2	$a=b+c$

Or if any compound Power of the unknown Quantity be at one Side of the Equation (that hath a true Root of its Kind) Evolve both Sides of the Equation, and it will be deprefs'd into lower Terms. As in these

Examples.

Suppose	1	$aa+2ba+bb=dd$	$aa-2ba+bb=ddcc$
I w 2	2	$a+b=d$	$a-b=dc.$

Here follow a few Examples of Equations, wherein all the foregoing Reductions are promiscuously used.

As Occasion requires.

Example 1.

Suppose	1	$\frac{aa+c-d}{4} = \frac{g-aa}{b}$	What is $a$ = to
I $\times$ 4	2	$aa+c-d = \frac{4g-4aa}{b}$	

2 x 1

$2 \times b$	3	$baa + bc - bd = 4g - 4aa$
$3 + 4aa$	4	$baa + 4aa + bc - bd = 4g$
$4 + bd$	5	$baa + 4aa + bc = 4g + bd$
$5 - bc$	6	$baa + 4aa = 4g + bd - bc$
$6 \div b + 4$	7	$aa = \frac{4g + bd - bc}{b + 4}$
$7 \text{ w } 2$	8	$a = \sqrt{\frac{4g + bd - bc}{b + 4}}$ . As was required.

## Example 2.

Suppose	1	$\frac{a + 354}{a} = \frac{3a}{354 - a}$ What is the Value of $a$ .
$1 \times a$	2	$a + 354 = \frac{3aa}{354 - a}$
$2 \times 354 - a$	3	$125316 - a^2 = 3aa$
$3 + aa$	4	$4aa = 125316$
$4 \div 4$	5	$aa = 31329$
$5 \text{ w } 2$	6	$a = \sqrt{31329} = 177$ the Value of $a$ requir'd.

## Example 3.

Suppose	1	$\sqrt{\frac{aa + 3bb}{4}} - \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}} : a = ?$
$1 \odot 2$	2	$\left\{ \begin{array}{l} \frac{aa + 3bb}{4} - 2 \sqrt{\frac{aa + 3bb}{4}} \times \sqrt{\frac{aa - 3bb}{4}} \\ : + \frac{aa - 3bb}{4} = \frac{baa}{c} \end{array} \right.$
That is	3	$\frac{aa}{2} - \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}$
For		$\frac{aa + 3bb}{4} + \frac{aa - 3bb}{4} = \frac{2aa}{4} = \frac{aa}{2}$
And		$2 \sqrt{\frac{aa + 3bb}{4}} = \sqrt{\frac{4aa + 12bb}{4}} = \sqrt{aa + 3bb}$
Then		$\sqrt{aa + 3bb} \times \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{a^4 - 9b^4}{4}}$
$3 + \sqrt{\&c.}$	4	$\frac{aa}{2} = \frac{baa}{c} + \sqrt{\frac{a^4 - 9b^4}{4}}$



4 $-\frac{baa}{c}$	5 $\frac{aa}{2} - \frac{baa}{c} = \sqrt{\frac{a^4 - 9b^4}{4}}$
5 $\odot 2$	6 $\frac{a^4}{4} - \frac{ba^4}{c} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4}$
6 $+\frac{ba^4}{c}$	7 $\frac{a^4}{4} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4} + \frac{ba^4}{c}$
7 $\pm$	8 $\frac{bba^4}{cc} + \frac{9b^4}{4} = \frac{ba^4}{c}$
8 $\div b$	9 $\frac{ba^4}{cc} + \frac{9b^3}{4} = \frac{a^4}{c}$
9 $\times cc$	10 $ba^4 + \frac{9ccb^3}{4} = ca^4$
10 $\times 4$	11 $4ba^4 + 9ccb^3 = 4ca^4$
11 $\leftarrow 4ba^4$	12 $9ccb^3 = 4ca^4 - 4ba^4$
12 $\div$	$aaaa = \frac{9ccb^3}{4c - 4b}$
For	13 $4c - 4b \times a^4 = 4ca^4 - 4ba^4$
13 $w 2$	14 $aa = \sqrt{\frac{9ccb^3}{4c - 4b}}$
14 $w 2$	15 $a = \sqrt{\frac{9ccb^3}{4c - 4b}}$ As was required.

By help of these Reductions (properly applied) the unknown Quantity, (*a*) or its Powers, are cleared and brought to one Side of an *Æ*quation; and if the unknown Quantity (*a*) chance to be equal to those that are known, the Question is answered.

As in the first Example of *Seçt.* 1. and 2.

Or if any single Power of the unknown Quantity (*a*) is found equal to those that are known, then the respective Root of the known Quantities, is the Answer; as in the first four *Examples* of *Seçt.* 6, &c.

But when the Powers of the unknown Quantity are either mixed with their Root, As  $aa + ba = dd$ , &c. Or do consist of different Powers; As  $aaa + baa = dd$ , &c. Then they are called affected, or adaffected *Æ*quations, which require other Methods to resolve them, *viz.* to find out the Value of (*a*) as shall be shewed further on.

## C H A P. VI

## Of Proportional Quantities; both Arithmetical, Geometrical and Musical.

What hath been said of Numbers in Arithmetical Progression, Chap. 6. Part 1. may be easily applied to any Series of homogeneal or like Quantities.

## Sect. 1. Of Quantities in Arithmetical Progression.

Those Quantities are said to be in the most simple or natural Progression, that begin their Series of Increase or Decrease with a Cypher.

Thus  $\begin{matrix} 0 : a : 2a : 3a : 4a : 5a : 6a : \&c. & \text{Increasing.} \\ 0 : -a : -2a : -3a : -4a : -5a : -6a : \&c. & \text{Decreasing.} \end{matrix}$

Or Universally, putting  $a$  the first Term in the Progression, and  $e$  the common Excess, or Difference.

Then  $\begin{matrix} a : a+e : a+2e : a+3e : a+4e : a+5e : a+6e : \&c. \\ a : a-e : a-2e : a-3e : a-4e : a-5e : a-6e : \&c. \end{matrix}$

In the first of these Series it's evident, that if there be but three Terms; the Sum of the Extremes will be double to the Mean.

As in these.  $0 : a : 2a$ : Or,  $a : 2a : 3a$ : Or,  $2a : 3a : 4a$ , &c. viz.  $2a + 0 = a + a$ : Or,  $a + 3a = 2a + 2a$ , &c.

Also, in the second Series, either Increasing, or Decreasing, it is evident, that if the Terms be  $a : a+e : a+2e$ , &c. Increasing; Then  $a + a+2e$ , viz.  $2a+2e$  the Sum of the Extremes, is double to  $a+e$  the Mean; Or if they be  $a : a-e : a-2e$ , &c. Decreasing, then  $a + a-2e$ : viz.  $2a-2e$  the Sum of the Extremes, is double to  $a-e$  the Mean. And so it will be in any other three of the Terms.

Secondly, If there are four Terms; then the Sum of the two Extremes, will be equal to the Sum of the two Means. As in these.  $a : a+e : a+2e : a+3e$ , in the Series Increasing; Here  $a + a+3e = a+e + a+2e$ .

Also in these,  $a : a-e : a-2e : a-3e$  in the Series Decreasing; here  $a + a-3e = a-e + a-2e$ , &c. in any other four Terms.

Consequently, If there are never so many Terms in the Series, the Sum of the two Extremes, will always be equal to the Sum of



## Ch. 6. Of Proportional Quantities. 185

of any two Means, that are equally distant from those Extremes. As in these,  $a : a + e : a + 2e : a + 3e : a + 4e : a + 5e : \&c.$  Here  $a + a + 5e = a + e + a + 4e = a + 2e + a + 3e, \&c.$  And if the Number of Terms be odd; the Sum of the two Extremes will be double to the middle Term, &c. As in Corol. 1. Chap. 6. before mention'd.

### CONSECTARY 1.

Whence it follows, (and is very easy to conceive) that if the Sum of the two Extremes be multiplied into the Number of all the Terms in the Series, the Product will be double the Sum of all the Series.

Now for the easier resolving such Questions as depend upon these ProgreSSIONal Quantities.

Let  $\begin{cases} a = \text{the first Term, as before.} \\ y = \text{the last Term.} \\ e = \text{the Common Excess, \&c. as before.} \\ N = \text{the Number of all the Terms.} \\ S = \text{the Sum of all the Series, viz. of all the Terms.} \end{cases}$

Then will  $a + y \times N = 2S$ , by the precedent Consectary: That is,  $Na + Ny = 2S$ . Consequently  $\frac{Na + Ny}{2} = S$ , the Sum of all the Series, be the Terms never so many. Thirdly, In these Series, it is easy to perceive, that the common Difference ( $e$ ) is so often added to the last Term of the Series; as are the Number of Terms, except the first; That is, the first Term ( $a$ ) hath no Difference added to it, but the last Term hath so many times ( $e$ ) added to it, as it is distant from the first.

Consequently, the Difference betwixt the two Extremes, is only the common Difference ( $e$ ) multiplied into the Number of all the Terms less Unity or 1.

That is,  $N - 1 \times e = y - a$ , the Difference betwixt the two Extremes, viz.  $Ne - e = y - a$ .

### CONSECTARY 2.

Whence it follows, that if the Difference betwixt the two Extremes be divided by the Number of Terms Less 1. the Quotient will be the common Difference of the Series.

To wit,  $\frac{y - a}{N - 1} = e$ .

Now by the help of these two *Consecutaries*, if any three of the aforesaid five Parts (*viz.* *a.* *y.* *e.* *N.* *S.*) be given; the other two may be easily found.

Thus,	I	$\frac{Na + Ny}{2} = S$	} <i>As before.</i>
And	2	$\frac{y - a}{N - 1} = e$	
$2 \times N - 1$	3	$y - a = Ne - e$	
$3 + e$	4	$y - a + e = Ne$	
$4 \div e$	5	$\frac{y - a + e}{e} = N$	<i>The Number of Terms.</i>
$1 \times 2$	6	$Na + Ny = 2S$	
$6 - Na$	7	$Ny = 2S - Na$	
$7 \div N$	8	$\frac{2S - Na}{N} = y$	<i>The last Term.</i>
$6 - yN$	9	$Na = 2S - Ny$	
$9 \div N$	10	$\frac{2S - Ny}{N} = a$	<i>The first Term.</i>
$6 \div a + y$	11	$\frac{2S}{a + y} = N$	<i>The Number of Terms.</i>
5, and 11	12	$\frac{y - a + e}{e} = \frac{2S}{a + y}$	<i>Per Axiom 5.</i>
$12 \times a + y$	13	$\frac{yy - aa}{e} + a + y = 2S$	
$13 \div 2$	14	$\frac{yy - aa}{2e} + \frac{a + y}{2} = S$	<i>The Sum of all the Series.</i>
$14 \times 2e$	15	$yy - aa + ae + ye = 2Se$	
$15 - ae$	16	$yy - aa + ye = 2Se - ae$	
$16 - ye$	17	$yy - aa = 2Se - ae - ye$	
$17 \div$	18	$\frac{yy - aa}{2S - a - y} = e$	<i>The Common Difference.</i>
$3 + a$	19	$Ne - e + a = y$	<i>The last Term.</i>
$19 + e$	20	$Ne + a = y + e$	
$20 - Ne$	21	$y + e - Ne = a$	<i>The first Term.</i>
			<i>&amp;c.</i>

In like Manner you may proceed to find out any of the five Quantities (*a.* *e.* *y.* *N.* *S.*) otherways, *viz.* by varying or comparing of these *Æquations* one with another, you may produce new *Æquations* with other *Data* in them; the which  
I shall



# Ch. 6. Of Proportional Quantities. 187

I shall here omit pursuing, and leave them for the Learner's Practice.

## Sect. 2. Of Quantities in Geometrical Proportion.

*Geometrical Proportion continued* has been already defined in Sect. 2. Chap. 6. Part 1. And what is there said concerning Numbers in  $\div$  may easily be applied to any Sort of Homogeneous Quantities that are in  $\div$ .

The most natural and simple Series of Geometrical Proportionals, is when it begins with Unity or 1.

As 1 . a . aa . aaa . aaaa . a<sup>5</sup> . a<sup>6</sup>, &c. in  $\div$

For 1 : a :: a : aa :: aa : aaa :: aaa : aaaa, &c.

Or a . b .  $\frac{bb}{a}$  .  $\frac{bbb}{aa}$  .  $\frac{bbbb}{aaa}$  .  $\frac{b^5}{a^4}$ . &c. are Terms in  $\div$

For a : b :: b :  $\frac{bb}{a}$  ::  $\frac{bb}{a}$  :  $\frac{bbb}{aa}$  ::  $\frac{bbb}{aa}$  :  $\frac{bbbb}{a^3}$  ::  $\frac{bbbb}{a^3}$  :  $\frac{b^5}{a^4}$ . &c.

That is, when all the middle Terms betwixt the two Extremes are both *Consequents* and *Antecedents*, that Series is in *Geometrical Proportion continued*.

Therefore in every Series of Quantities in  $\div$  all the Terms except the Last are *Antecedents*; and all the Terms except the First are *Consequents*.

But Universally putting a the first Term in the Series, and e the Ratio, viz. the common Multiplier, or Divisor, then it will be

a . ae . aee . aeee . aeeee . ae<sup>5</sup> . ae<sup>6</sup> . &c. in  $\div$

Or a .  $\frac{a}{e}$  .  $\frac{a}{ee}$  .  $\frac{a}{eee}$  .  $\frac{a}{eeee}$  .  $\frac{a}{e^5}$ . &c. are in  $\div$  Decr.

For a : ae :: ae :  $\frac{aee}{a}$  = aee, &c.

And a :  $\frac{a}{e}$  ::  $\frac{a}{e}$  :  $\frac{aa}{aee}$  =  $\frac{a}{ee}$  a :  $\frac{a}{e}$  ::  $\frac{a}{e}$  :  $\frac{a}{eee}$  &c.

I. In any of these Series it is evident, that if three Quantities are in  $\div$  the Rectangle of the two Extremes will be equal to the Square of the Mean.

As in these, a : ae : aee here a × aee = ae × ae. = aae. &c.

B b 2

Or

Or  $a \cdot \frac{a}{e} \cdot \frac{a}{ee}$  here also,  $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$  &c.

II. If four Quantities are in  $\div$  the Rectangle of the Extremes will be equal to the Rectangle of the Means.

As in these,  $a \cdot ae \cdot aee \cdot aeee$ . here  $a \times ae^3 = ae \times aee$ .

Or  $a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee}$  here also  $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{aa}{eee}$  &c.

Consequently, If there are never so many Terms in the Series of  $\div$  the Rectangle of the Extremes will be equal to the Rectangle of any two Means that are equally distant from those Extremes.

As in these,  $a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5$  &c.

viz.  $ae^5 \times a = ae^4 \times ae$ . Or  $ae^5 \times a = aeee \times aee = aae^5$ .

III. If never so many Quantities are in  $\div$  it will be, As any one of the *Antecedents* is to its *Consequent*; So is the Sum of all the *Antecedents*, to the Sum of all the *Consequents*.

As in these,  $\left\{ \begin{array}{l} a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5, \text{ \&c. Increasing.} \\ a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee} \cdot \frac{a}{eeee} \cdot \frac{a}{e^5} \text{ \&c. Decreasing.} \end{array} \right.$

$a : ae :: a + ae + aee + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + ae^5$

Or  $a : \frac{a}{e} :: a + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e} + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e^5}$

viz.  $a \times ae + aee + ae^3 + ae^4 + ae^5 = ae \times a + ae + aee + ae^3 + ae^4$

That is, the Rectangle of the Extremes is equal to the Rectangle of the Means; *per Second* of this *Seçt.*

*Note*, The Ratio of any Series in  $\div$  increasing is found by dividing any of the *Consequents* by its *Antecedent*.

Thus,  $a) ae (e$  Or  $ae) aee (e$  &c.

But if the Series be decreasing, then the Ratio is found by dividing any of the *Antecedents* by its *Consequent*.

Thus,  $\frac{a}{e}) a (e$  Or  $\frac{a}{ee}) \frac{a}{e} (e$  &c.

CON-



CONSECTARY.

These Things being premised, such  $\mathcal{A}$ equations may be deduced from them, as will solve all such Questions as are usually proposed about Quantities in Geometrical Proportion  $\div$  In Order to that

Let  $\left\{ \begin{array}{l} a = \text{The First Term.} \\ e = \text{The Common Ratio.} \\ y = \text{The Last Term.} \\ S = \text{The Sum of all the Terms.} \end{array} \right\}$  as before.

Then  $S - y =$  the Sum of all the *Antecedents*.  
And  $S - a =$  the Sum of all the *Consequents*.

Analogy.	I	$a : ae :: S - y : S - a$ Per the III. of this Sect.
I $\therefore$	2	$Sa - aa = aeS - aey$
2 $\div a$	3	$S - a = eS - ey$
3 $\div ey$	4	$S \div ey - a = eS$
4 $- S$	5	$ey - a = eS - S$
5 $\div e - I$	6	$\frac{ye - a}{e - I} = S$ The Sum of all the Series.
3 $\div S - y$	7	$\frac{S - a}{S - y} = e$ The common Ratio.
5 $\div a$	8	$ey = eS \div a - S$
8 $\div e$	9	$\frac{eS \div a - S}{e} = y$ The last Term.
4 $\div a$	10	$S \div ey = eS \div a$
10 $- Se$	11	$S \div ey - eS = a$ The first Term.

Note, The  $\therefore$  set in the Margin at the second Step, is instead of *Ergo*; and imports that the Rectangle of the two Extremes in the first Step, is equal to the Rectangle of the Means. And so for any other Proportion.

Sect. 3. Of Harmonical Proportion.

Harmonical or Musical Proportion is, when of three Quantities (or rather Numbers) the First hath the same Ratio to the Third, As the Difference between the First and Second, hath to the Difference between the Second and Third. As in these following.

Suppose  $a, b, c$  in Musical Proportion.

Then  $\left\{ \begin{array}{l} 1 \mid a : c :: b - a : c - b \\ 1 \therefore \mid 2 \mid cb - ca = ac - ba \end{array} \right.$

$2 \div ca$

$2 + ca$	3	$cb = 2ac - ba$
$3 \div 2c - b$	4	$\frac{cb}{2c - b} = a$ The first Term.
$3 + ba$	5	$2ac = cb + ba$
$5 \div c + a$	6	$\frac{2ac}{c + a} = b$ The second Term.
$5 - cb$	7	$2ac - cb = ba$
$7 \div 2a - c$	8	$\frac{ba}{2a - c} = c$ The third Term.

If there are four Terms in Musical Proportion, the first hath the same Ratio to the Fourth, as the Difference between the First and Second hath to the Difference between the Third and Fourth.

That is, Let  $a. b. c. d$  be the four Terms, &c.

Then	1	$a : d :: b - a : d - c$
$1 \div$	2	$db - da = da - ca$
$2 + da$	3	$db = 2da - ca$
$3 \div 2d - c$	4	$\frac{db}{2d - c} = a$
$3 \div d$	5	$b = \frac{2da - ca}{d}$
$3 + ca$	6	$db + ca = 2da$
$6 - db$	7	$ca = 2da - db$
$7 \div a$	8	$c = \frac{2da - db}{a}$
$7 \div 2a - b$	9	$\frac{ca}{2a - b} = d$

## C H A P. VII.

Of Proportion Disjunct, and how to turn *Æquations* into Analogies, &c.

*Proportion Disjunct*, or the Rule of Three in Numbers is already explain'd in Chap. 7. Part I. And what hath been there said is applicable to all Homogeneous Quantities, viz. of Lines to Lines, &c.

### Sect. I.

If four Quantities (viz. either Lines, Superficies, or Solids) be Proportional: The Rectangle comprehended under the Extremes,



# Ch. 7. Of Proportional Quantities. 191

is equal to the Rectangle comprehended under the two Means, (16 Euclid 6.)

For Instance, Suppose,  $a . b . c . d .$  to represent the four Homogeneous Quantities in Proportion.

viz.  $a : b :: c : d$ . Then will  $ad = bc$ .

For suppose  $b = 2a$  then will  $d = 2c$

And it will be  $a : 2a :: c : 2c$ . Here the Ratio is 2.

but  $a \times 2c = 2a \times c$ . viz.  $2ca = 2ac$

Or suppose  $b = 3a$  then will  $d = 3c$

And it will be  $a : 3a :: c : 3c$ . Here the Ratio is 3.

but  $a \times 3c = 3a \times c$ . viz.  $3ac = 3ac$ .

Or Universally putting  $e$  for the Ratio of the Proportion, viz. making  $b = ae$ , then will  $d = ce$

And it will be  $a : ae :: c : ce$

but  $a \times ce = ae \times c$  viz.  $ace = aec$ .

Consequently,  $ad = bc$  which was to be proved.

Whence it follows, that if any Three of the four Proportional Quantities be given, the Fourth may be easily found.

Thus,

Let	1	$a : b :: c : d$ . Then
1 $\therefore$	2	$ad = bc$ as before
2 $\div d$	3	$a = \frac{bc}{d}$
2 $\div c$	4	$b = \frac{ad}{c}$
2 $\div b$	5	$c = \frac{ad}{b}$
2 $\div a$	6	$d = \frac{bc}{a}$
2 $\div bd$	7	$\frac{a}{b} = \frac{c}{d}$
Or 2 $\div ac$	8	$\frac{b}{a} = \frac{d}{c}$

Note, In this Manner Euclid in his 5th Book expresses the Ratio of Proportionals, viz. the Ratio of  $a$  to  $b$  is  $\frac{a}{b}$

If four Quantities are Proportional, they will also be Proportional in Alternation, Inversion, Composition, Division, Conversion, and Mixtly. Euclid 5. Def. 12, 13, 14, 15, 16.

That

That is, if	1	$a : b :: c : d$ be in <i>Direct Proportion</i> , as before.
Then	2	$a : c :: b : d$ <i>Alternate</i> . For $ad = bc$
And	3	$b : a :: d : c$ <i>Inverted</i> . For $ad = bc$
Also	4	$a + b : b :: c + d : d$ <i>Compounded</i> .
4 ∴	5	$da + bd = bc + bd$ That is, $ad = bc$ , as before.
Or	6	$a + c : c :: b + d : d$ <i>Alternately Compounded</i> .
6 ∴	7	$ad + cd = bc + cd$ That is $ad = bc$ .
Again,	8	$a - b : b :: c - d : d$ <i>Divided</i> .
8 ∴	9	$ad - bd = bc - bd$ That is, $ad = bc$ .
Or	10	$a - c : c :: b - d : d$ <i>Alternately Divided</i> .
10 ∴	11	$ad - cd = bc - cd$ That is, $ad = bc$ .
And	12	$a : b \pm a :: c : d \pm c$ <i>Converted</i>
12 ∴	13	$ad \pm ac = bc \pm ac$ That is $ad = bc$ .
Lastly	14	$a + b : a - b :: c + d : c - d$ <i>Mixtly</i> .
14 ∴	15	$ac - ad + bc - bd = ac + ad - bc - bd$
15 ±	16	$2bc = 2ad$ ; That is, $ad = bc$ As at first.

*Note*, What has been here done about whole Quantities in *Simple Proportion*, may be easily perform'd in *Fractional Quantities*; And *Surds*, &c.

For Instance, If  $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{c}$  and it be required to find the fourth Term.

it will be  $\frac{dd - cc}{fc}$  the Rectangle of the Means; which being divi-

ded by the first Extreme  $\frac{ab}{c}$  it will become

$$\left(\frac{ab}{c}\right) \frac{dd - cc}{fc} \left(\frac{ddc - ccc}{abfc} = \frac{dd - cc}{abf}\right) \text{ the fourth Term.}$$

Or if  $b : \sqrt{bd + bc} :: \sqrt{bd + bc} : \text{to a fourth Term.}$

Then is,  $\sqrt{bd + bc} \times \sqrt{bd + bc} = bd + bc$  the Rectangle of the Means, And  $b) bd + bc (d + c$  the fourth Term.

That is,  $b : \sqrt{bd + bc} :: \sqrt{bd + bc} : d + c$  &c.

## Sect. 2. Of Duplicate and Triplicate Proportion.

The Proportions treated of in the last Section, are to be understood when Lines are compared to Lines, and Superficies to Su-



## Ch. 7. Of Proportional Quantities. 193

Superficies ; or Solids to Solids; *viz.* when each is compared to that of its like Kind, which is only called Simple Proportion.

But when Lines are compared to Superficies, or Lines are compared to Solids, such Comparisons are distinguished from the former, by the Names of Duplicate and Triplicate, &c. Proportions; so that Simple, Duplicate and Triplicate, &c. Proportions are to be understood in a different Sense from Single, Double, Triple, &c. Proportions, which are only as 1, 2, 3, &c. to 1; but those of Simple, Duplicate, Triplicate, &c. Proportion, is that of  $a . aa . aaa .$ , &c. to 1. Or if the Simple Proportions be that of  $a$  to  $b$ , whose Ratio or Exponent is  $\frac{a}{b}$  or  $\frac{b}{a}$  according to *Euclid's* Way.

Then  $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$  is the Exponent of the Duplicate  
 And  $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$  is the Exponent of the Tr. Prop. } of  $\frac{a}{b}$   
 &c.

And if there are Three, Four, or more Quantities, in  $\therefore$  as 1.  $a . aa . aaa : a^4 . a^5$ , &c. (As in the first Series *sect.* 2. of the last Chapter.) Then, that of the First to the Third, Fourth and Fifth, &c. (*viz.* 1 To  $aa . aaa . aaaa . a^5$ ) is Duplicate, Triplicate, Quadruplicate, &c. of the First to the Second. (*viz.* of 1 To  $a$ ;) And by Inversion, that of the Third, Fourth, Fifth, is Duplicate, Triplicate, &c. of that of the Second to the First ( $a$  To 1) per *Def.* 10. *Eucl.* 5. But the Nature of these Proportions will appear more evident, and be easier understood when they are applied to Practice, and illustrated by Geometrical Figures. Further on.

### *Sect.* 3. How to turn *Æ*quations into Analogies.

From the first *Section* of this *Chap.* it will be easy to conceive how to turn or dissolve *Æ*quations into Analogies or Proportions.

For if the Rectangle of two (or more) Quantities, be equal to the Rectangle of two (or more) Quantities; then are those four (or more) Quantities Proportional. By the 16 *Euclid* 6.

That is, if  $ab = dc$ . Then is  $a : c :: d : b$ .

Or  $c : a :: b : d$  &c.

From whence there arises this general Rule for turning *Æ*quations into Analogies.

C c

Rule.

## Rule.

*Divide either Side of the given Equation, (if it can be done) into two such Parts, or Factors, as being multiplied together, will produce that Side again; and make those two Parts the two Extremes. Then divide the other Side of the Equation (if it can be done) in the same Manner as the First was, and let those two Parts or Factors, be the two Means.*

For Instance, Suppose  $ab + ad = bd$ .

Then  $a : b :: d : b + d$ . Or  $b : a :: b + d : d$  &c.

Or taking  $ad$  from both Sides of the Equation, and It will be  $ab = bd - ad$ . Then  $a : d :: b - a : b$ .

Or,  $b : d :: b - a : a$  &c.

Again, Suppose  $aa + 2ae = 2by + yy$ .

Here  $a$  and  $a + 2e$  are the two Factors of the first Side in this Equation; for  $a + 2e \times a = aa + 2ae$ .

Again,  $y$  and  $2b + y$  are the two Factors of the other Side.

Therefore,  $a : y :: 2b + y : a + 2e$ .

Or  $2b + y : a + 2e :: a : y$  &c.

When one Side of any Equation can be divided into two Factors, as before; and the other Side cannot be so divided, then make the Square Root of that Side either the two Extremes or the two Means.

For Instance, Suppose  $bc + bd = da + g$ .

Then  $b : \sqrt{da + g} :: \sqrt{da + g} : c + d$ .

Or  $\sqrt{da + g} : b :: c + d : \sqrt{da + g}$ . &c.

## C H A P. VIII.

## Of Substitution, and the Solution of Quadratic Equations.

## Sect. I. Of Substitution.

When new Quantities not concerned in the first stating of any Question, are put instead of some that are engag'd in it, that is called Substitution.

For Instance, If instead of  $\sqrt{bc - dc}$  you put  $z$ , or any other Letter.

That is, make  $z = \sqrt{bc - dc}$ .

Or



## Chap. 8. Of Quadratic Equations. 195

Or, Suppose  $aa + ba - ca + da = dc$ , instead of  $b - c + d$  put  $s$ , or any other Letter not engaged in the Question,

Viz.  $s = b - c + d$  Then  $aa + sa = dc$

That is, if  $c$  be greater than  $b + d$ , it's  $aa - sa = dc$ .

But if  $b + d$  be greater than  $c$ , Then it's  $aa + sa = dc$ .

And this Way of Substituting or putting of new Quantities instead of others, may be found very useful upon several Occasions; viz. in order to make some following Operations in the Question more easy, and perhaps much shorter than they would be without it, as you may observe in some Questions hereafter proposed in this Tract.

And when those Operations, in which the substituted Quantities were assisting or useful, are performed according as the Nature of the Question required, you may then (if there be Occasion) bring the Original or first Quantities into the  $\mathcal{A}$ equation, in the Place (or Places) of those substituted Quantities, which is called Restitution, as you will see further on.

### Sect. 2. The Solution of Quadratic Equations.

When the Quantity sought is brought to an Equality with those that are known, and is on one Side of the  $\mathcal{A}$ equation, in no more than two different Powers whose Indices are double one to another, those  $\mathcal{A}$ equations are called Quadratic  $\mathcal{A}$ equations Adfected; and do fall under the Consideration of three Forms or Cases.

Case 1.  $aa + 2ba = dc.$

Case 2.  $aa - 2ba = dc.$

Case 3.  $2ba - aa = dc.$

And

$\left\{ \begin{array}{l} a^4 + 2ba^2 = dc. \\ a^4 - 2ba^2 = dc. \\ 2ba^2 - a^4 = dc. \end{array} \right.$

Also  $\left\{ \begin{array}{l} a^6 + 2ba^3 = dc. \\ a^6 - 2ba^3 = dc. \\ 2ba^3 - a^6 = dc. \end{array} \right.$

And

$\left\{ \begin{array}{l} a^8 + 2ba^4 = dc. \\ a^8 - 2ba^4 = dc. \\ 2ba^4 - a^8 = dc. \end{array} \right.$

&c.

When there happen to be more Terms in one of these Kind of  $\mathcal{A}$ equations than two, and the highest Power of the unknown Quantity is multiplied into some known Co-efficients; you must reduce them by Division; as in Sect. 4. of Chap. 5. and for the Fractional Quantities that may arise by those Divisions, Substitute another Quantity doubled.

For Instance, Let  $baa + caa - ca - da = dc + cb$ .

Then  $aa - \frac{ca - da}{b + c} = \frac{dc + cb}{b + c}$  make  $\frac{c - d}{b + c} = 2x$ .

C c 2

And

And if you please, for  $\frac{dc + cb}{b + c}$  put  $z$ .

Then will  $aa - 2xa = z$  be the new  $\mathcal{A}$ equation, equal to the other, being now fitted for a Solution.

Now any of these three Forms of  $\mathcal{A}$ equations being thus prepared for a Solution, may be reduced to simple Powers by casting off the second or lowest Term of the unknown Quantity; which is done by Substitution, thus always take half the known Co-efficient, and add it to (in Case 1.) or subtract it from (in Case 2.) its fellow Factor; and for their Sum, or Difference, substitute another Letter. As in these.

Let	1	$aa + 2ba = dc$	Case 1.
Put	2	$a + b = e$	
2 $\odot$ 2	3	$aa + 2ba + bb = ee$	
3 $-$ 1	4	$bb = ee - dc$	
4 $+ dc$	5	$ee = bb + dc$	
5 $\sqrt{\phantom{x}}$ 2	6	$e = \sqrt{bb + dc}$	
2 and 6	7	$a + b = \sqrt{bb + dc}$	Per Axiom 5.
7 $- b$	8	$a = \sqrt{bb + dc} - b$	

Again.

Let	1	$aa - 2ba = dc$	Case 2.
Put	2	$a - b = e$	
2 $\odot$ 2	3	$aa - 2ba + bb = ee$	
3 $-$ 1	4	$bb = ee - dc$	
4 $+ dc$	5	$ee = dc + bb$	
5 $\sqrt{\phantom{x}}$ 2	6	$e = \sqrt{dc + bb}$	
2, 6	7	$a - b = \sqrt{dc + bb}$	
7 $+ b$	8	$a = b + \sqrt{dc + bb}$	

In Case 3. From half the known Co-efficient subtract its fellow Factor.

Thus, Let	1	$2ba - aa = dc$	
Put	2	$b - a = e$	
2 $\odot$ 2	3	$bb - 2ba + aa = ee$	
1 $+ 3$	4	$bb = dc + ee$	
4 $- dc$	5	$ee = bb - dc$	

5  $\sqrt{\phantom{x}}$  2



# Chap. 8. Of Quadratic Equations. 197

5 w 2	6	$e = \sqrt{bb - dc}$
(2, 6	7	$b - a = \sqrt{bb - dc}$
7 + a	8	$b = a + \sqrt{bb - dc}$
8 - $\sqrt{\&c.}$	2	$a = b - \sqrt{bb - dc}$

And this Method holds good in those other  $\mathcal{A}$ equations, where in the highest Powers are  $a^4, a^6, a^8, \&c.$  As for Instance.

Let	1	$a^6 + 2ba^3 = dc$ Case 1.
Put	2	$a^3 + b = e$
2 $\odot$ 2	3	$a^6 + 2ba^3 + bb = ee$
3 - 1	4	$bb = ee - dc$
4 + dc	5	$ee = bb + dc$
5 w 2	6	$e = \sqrt{bb + dc}$
2, 6	7	$a^3 + b = \sqrt{bb + dc}$
7 - b	8	$a^3 = \sqrt{bb + dc} : -b$
8 w 3	9	$a = \sqrt[3]{\sqrt{bb + dc} : -b}$

The same may be done with all the rest, Care being taken to add, or subtract, according as the Case requires.

But all Quadratic  $\mathcal{A}$ equations may be more easily resolved by completing the Square which is grounded upon the Consideration of raising a Square from any Binomial, or Residual Root, (See Sect. 5. Chap. 1.

*Viz.* If  $a + b$  be involved to a Square, it will be,  $aa + 2ba + bb$

And if  $a - b$  be so involved, it will be,  $aa - 2ba + bb$

Whence it is easy to observe, that  $aa + 2ba = dc$ . Case 1.

And  $aa - 2ba = dc$ . Case 2. are imperfect Squares, wanting only  $bb$  to make them complete. And therefore it is that if half the known Co-efficient be involved to the second Power, and the Square be added to both Sides of the  $\mathcal{A}$ equation, the unknown Side will become a complete Square.

Thus, Let	1	$aa + 2ba = dc$ .	} Here half the Co-efficient $2b$ is $b$ , which being squared, is $bb$ .
But	2	$bb = bb$	
1 + 2	3	$aa + 2ba + bb = dc + bb$ Case 1.	
3 w 2	4	$a + b = \sqrt{dc + bb}$ As before.	

Again.

Again.

Let	1	$aa - 2ba = dc$	Case 2.
But	2	$bb = bb$	
1 + 2	3	$aa - 2ba + bb = dc + bb$	
3 w 2	4	$a - b = \sqrt{dc + bb}$ &c.	As before.

But in Case 3. you must change the Signs of all the Terms in the  $\mathcal{A}$ equation,

Thus	1	$2ba - aa = dc$	Case 3.
1 +	2	$aa - 2ba = -dc$	
Then	3	$aa - 2ba + bb = bb - dc$ &c.	

And this Method of completing the Square holds true in those other  $\mathcal{A}$ equations.

Viz.	1	$aaaa + 2baa = dc$	Case 1.
For	2	$bb = bb$	As before.
1 + 2	3	$aaaa + 2baa + bb = dc + bb$	
3 w 2	4	$aa + b = \sqrt{dc + bb}$	
4 - b	5	$aa = \sqrt{dc + bb} : -b$	
5 w 2	6	$a = \sqrt{\sqrt{dc + bb} : -b}$	And so on for the rest.

Or let	1	$a^6 + 2baaa = dc$	As before, Case 1.
And	2	$bb = bb$	
1 + 2	3	$a^6 + 2baaa + bb = dc + bb$	
1 w 2	4	$aaa + b = \sqrt{dc + bb}$	
4 - b	5	$aaa = \sqrt{dc + bb} : -b$	
5 w 3	6	$a = \sqrt[3]{\sqrt{dc + bb} : -b}$ &c.	

### COROLLARY.

Hence it is evident, that whatsoever Method is used in solving these (or indeed any other)  $\mathcal{A}$ equations, the Result will still be the same, if the Work be true; as you may observe from the Operations of this Section: For both these Methods here proposed, give the same Theorems in their respective Cases for the Value of (a).

Thus



# Chap. 8. Of Quadratic Equations. 199

Thus, when  $aa + 2ba = dc$  Then

Theorem 1.  $a = \sqrt{dc + bb} : -b$

And when  $aa - 2ba = dc$  Then

Theorem 2.  $a = b + \sqrt{dc + bb}$

Again, when  $2ba - aa = dc$  Then

Theorem 3.  $a = b - \sqrt{bb - dc}$

The like Theorems may be easily raised for the rest.

If the known Co-efficients (of the second or lowest Term) be any single Quantity, as  $aa + ba = dc$ , &c. Then is  $\frac{1}{2}b$  its half, and  $\frac{1}{4}bb$  will be the Square of that half: That is,  $\frac{1}{2}b \times \frac{1}{2}b = \frac{1}{4}bb$ . And then the Work will stand

Thus,	1	$aa + ba = dc$	
1 C □	2	$aa + ba + \frac{1}{4}bb = dc + \frac{1}{4}bb$	
2 w 2	3	$a + \frac{1}{2}b = \sqrt{dc + \frac{1}{4}bb}$	
3 - $\frac{1}{2}b$	4	$a = \sqrt{dc + \frac{1}{4}bb} : -\frac{1}{2}b$ .	And so for the rest.

Note, C □ placed in the Margin against the second Step, signifies that the imperfect Square  $aa + ba$  in the first Step, is there compleated, viz. in the second Step.

Now by the help of these Theorems, it will be easy to calculate or find the Value of the unknown Quantity, (a) in Numbers.

## Example 1.

Suppose  $aa + 2ba = z$ . Let  $b = 16$  And  $z = 4644$ .

Then  $a = \sqrt{z + bb} : -b$  Per Theorem 1.

But  $z + bb = 4644 + 256 = 4900$ . And  $\sqrt{4900} = 70$

Consequently  $a = 70 - 16$ , viz.  $a = 54$ .

But every Adfected Equation, hath as many Roots (or rather Values of the unknown Quantity) either real or imaginary, as are the Dimensions (viz. the Index) of its highest Power; and therefore the Quantity  $a$ , in this Equation, hath another Value either affirmative or negative; which may be thus found.

The given Equation is  $aa + 32a = 4644$ , and its Root  $a = 54$

Let these two Equations be made equal or Equated to 0, viz. to Nothing.

Thus,

Thus,  $aa + 32a - 4644 = 0$  And  $a - 54 = 0$ .

Then divide the given Equation by its first Root, and the Quotient will shew the second Value of  $a$ .

Thus,  $a - 54 = 0$   $aa + 32a - 4644 = 0$  ( $a + 86 = 0$ )

$$\begin{array}{r} aa - 54a \\ \hline + 86a - 4644 \\ 86a - 4644 \\ \hline (0) \end{array}$$

Hence the second Value of  $a$  is  $= -86$ , Or  $86 = -a$  which seems impossible viz. that an affirmative Quantity should be equal to a negative Quantity; yet even by this second Value of  $a$ , and the same Co-efficient, the true (or first) Equation may be formed.

Thus, Let

1	⊖ 2	2	$aa = +7396$ , viz. $-86 \times -86 = +7396$
1	× 32	3	$32a = -2752$
2	+ 3	4	$aa + 32a = 4644$ As at first.

#### Example 2.

Suppose

1	⊖	2	$aa - 7a = 948,75$ Then per Theorem 2.
1	⊖	2	$aa - 7a + \frac{49}{4} = 948,75 + \frac{49}{4} = 961$
2	√	3	$a - \frac{7}{2}$ (Or 3, 5) $= \sqrt{961} = 31$
3	+ 3,5	4	$a = 31 + 3,5 = 34,5$

Again, for the second Value of  $a$ ,

Let  $aa - 7a - 948,75 = 0$ . And  $a - 34,5 = 0$

Then,  $a - 34,5 = 0$   $aa - 7a - 948,75 = 0$  ( $a + 27,5 = 0$ )

Consequently this second Value is  $a = -27,5$

which will form the original Equation,  $aa - 7a = 948,75$  if it be ordered as the last was.

#### Example 3.

Suppose  $36a - aa = 243$  Then per Theorem 3.

$a = 18 - \sqrt{324 - 243}$  viz half Squared is 324 &c.

That is,  $a = 18 - \sqrt{81}$  but  $\sqrt{81} = 9$

Therefore  $a = 18 - 9 = 9$ . Now this third Form is called an ambiguous Equation, because it hath two affirmative Values of the unknown Quantity ( $a$ ), both which may be found without such Division, as was used before,

For



Chap. 8. Of Quadratic Equations. 201

For in this Case,  $a = 18 \pm \sqrt{81}$ , viz.  $a = 18 + 9 = 27$ . Or,  $a = 18 - 9 = 9$ , As before; And both these Values of  $a$  are equally true, as to forming the given Equation;

Viz.  $36a - aa = 243$ . For if  $a = 9$ , then  $aa = 81$ , and  $36a = 324$ ; but  $324 - 81 = 243$ , therefore  $a = 9$ .

Again, If  $a = 27$ , then will  $aa = 729$  and  $36a = 972$ : But  $972 - 729 = 243$ , consequently it may be,  $a = 27$ .

Now either of these Values of  $a$  may be found by Division, as those were in the other two Cases, one of them being first found by the Theorem.

Thus, Let  $36a - aa - 243 = 0$  And  $9 - a = 0$   
Then  $9 - a = 0$  )  $36a - aa - 243 = 0$  ( $a - 27 = 0$

$$\begin{array}{r} 9a - aa \\ \hline 27a - 0 - 243 \\ 27a \quad \quad - 243 \\ \hline (0) \quad \quad (0) \end{array}$$

Hence, if  $a - 27 = 0$  Then  $a = 27$  As before.  
Notwithstanding all Quadratic Equations of this third Form have two Affirmative Roots, (as in this) yet but one of those Roots will give a true Answer to the Question, and that is to be chosen according to the Nature and Limits of the Question, as shall be shewed further on.

SCHOLIUM.

From the Work of the three last Examples, it may be observed; that the Sum of both the Roots, will always be equal to the Co-efficient of their respective Equations, with a contrary Sign.

Thus. In Example 1.  $aa + 32a = 4644$   
Here  $a = 54$  } Add  
And  $a = -86$  }

In Example 2.  $aa - 7a = 948.75$   
Here  $a = 34.5$  } Add  
And  $a = -27.5$  }

In the last Example  
Which was changed into  $36a - aa = 243$   
Here  $a = 9$  } Add  
And  $a = 27$  }

D d Hence

Hence it is evident, that if either of the Roots be found, the other may be easily had without Division.

If the Contents of this Section be well understood, it will be easy to give a Numerical Solution to any Quadratic Equation, that happens to arise in resolving of Questions, &c. And as for giving a Geometrical Construction of them, I think it not proper in this Place; because I here suppose the Learner wholly ignorant of the first Principles of Geometry, therefore I shall refer that Work to the next Part.

## C H A P. IX.

*Of Analysis, or the Method of Resolving Problems; Exemplified by Variety of Numerical Questions.*

N. B. Here I advise the young Learner to make use always of the same Letters, to represent the same Data in all Questions.

*Viz.* { If  $a$  represent any Number } or other Quantity.  
 { And  $e$  represent a less Number }

$$\left\{ \begin{array}{ll} a + e = s & \text{Their Sum.} \\ a - e = d & \text{Their Difference.} \\ ae = p & \text{Their Product.} \end{array} \right.$$

$$\text{Then let } \left\{ \begin{array}{ll} a & \\ \frac{a}{e} = q & \text{Their Quotient.} \\ e & \end{array} \right.$$

$$\left\{ \begin{array}{ll} aa + ee = z & \text{The Sum of their Squares.} \\ aa - ee = x & \text{The Difference of their Squares.} \end{array} \right.$$

Any two of these six, ( $s, d, p, q, z, x$ ) being given, thence to find the Rest; which admits of fifteen Variations, or Questions.

**Question 1.** Suppose  $s$  and  $d$  were given, and it were required by them to find  $a, e, p, q, z$ , and  $x$ .

$$\begin{array}{lcl} \text{Let } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \left\{ \begin{array}{l} a + e = s \\ a - e = d \end{array} \right\} & \text{and suppose } \left\{ \begin{array}{l} s = 240 \\ d = 192 \end{array} \right\} \text{ Then} \\ 1 + 2 & 3 & 2a = s + d = 432 \\ 3 \div 2 & 4 & a = \frac{s + d}{2} = 216 \text{ Here } a \text{ is found.} \\ 1 - 2 & 5 & 2e = s - d = 48. \end{array}$$

$$5 \div 2$$



# Chap. 9. Of Numerical Questions. 203

$5 \div 2$	6	$e = \frac{s-d}{2} = 24$ Here $e$ is found.
$4 \times 6$	7	$ae = \frac{ss-dd}{4} = p = 5184$ . Here $p$ is found.
$4 \div 6$	8	$\frac{a}{e} = \frac{s+d}{s-d} = q = 9$ Here $q$ is found.
$4 \odot 2$	9	$aa = \frac{ss+2sd+dd}{4} = 46656$ .
$6 \odot 2$	10	$ee = \frac{ss-2sd+dd}{4} = 576$ .
$9 + 10$	11	$aa + ee = \frac{ss+dd}{2} = z = 47232$ . $z$ found.
$9 - 10$	12	$aa - ee = sd = x = 46080$ . $x$ found.

Question 2. Let  $s$  and  $p$  be given; To find the Rest.

That is,  $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right.$

	1	$a + e = s = 240$	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\}$ Quere $a . e . d . q . z . x$ .
	2	$ae = p = 5184$	
$1 \odot 2$	3	$aa + 2ae + ee = ss = 57600$	
$2 \times 4$	4	$4ae = 4p = 20736$	
$3 - 4$	5	$aa - 2ae + ee = ss - 4p = 36864$	
$5 \div 2$	6	$a - e = \sqrt{ss - 4p} = d = 192$	
$1 + 6$	7	$2a = s + \sqrt{ss - 4p}$	
$7 \div 2$	8	$a = \frac{s + \sqrt{ss - 4p}}{2}$ Hence $a = 216$ .	
$1 - 6$	9	$2e = s - \sqrt{ss - 4p}$	
$9 \div 2$	10	$e = \frac{s - \sqrt{ss - 4p}}{2}$ Hence $e = 24$ .	
$8 - 10$	11	$\frac{a}{e} = \frac{s + \sqrt{ss - 4p}}{s - \sqrt{ss - 4p}} = q = 9$	
$8 \odot 2$	12	$aa = \frac{ss + s \sqrt{ss - 4p}}{2} : - p$	
$10 \odot 2$	13	$ee = \frac{ss - s \sqrt{ss - 4p}}{2} : - p$	

$$12 + 13 \quad | \quad 14 \quad | \quad aa + ee = ss - 2p = z = 47232$$

$$12 - 13 \quad | \quad 15 \quad | \quad aa - ee = s \sqrt{ss - 4p} = x = 46080$$

Question 3. Suppose  $s$  and  $q$  are given; To find the Rest.

$$\text{Viz. } \left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} a + e = s. = 240 \\ \frac{a}{e} = q = 9 \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \end{array}} \right\} \text{Quere } a.e.d.p.z.x.$$

$$2 \times e$$

$$1 - 3$$

$$4 + qe$$

$$5 - q + 1$$

$$1 - 6$$

$$6 \times 7$$

$$7 - 6$$

$$7 \odot 2$$

$$6 \odot 2$$

$$10 + 11$$

$$10 - 11$$

$$3$$

$$4$$

$$5$$

$$6$$

$$7$$

$$8$$

$$9$$

$$10$$

$$11$$

$$12$$

$$13$$

$$a + e = s. = 240$$

$$\frac{a}{e} = q = 9$$

$$a = qe$$

$$e = s - qe$$

$$qe + e = s$$

$$e = \frac{s}{q + 1} \text{ For } q + 1 \times e = qe + e$$

$$a = s - \frac{s}{q + 1} = \frac{qs}{q + 1}$$

$$ae = \frac{qs}{qq + 2q + 1} = p.$$

$$a - e = \frac{qs - s}{q + 1} = d$$

$$aa = \frac{qqss}{qq + 2q + 1}$$

$$ee = \frac{ss}{qq + 2q + 1}$$

$$aa + ee = \frac{qqss + ss}{qq + 2q + 1} = z$$

$$aa - ee = \frac{qqss - ss}{qq + 2q + 1} = x$$

Question 4. Let  $s$  and  $z$  be given; To find the Rest.

$$\text{Viz. } \left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} a + e = s. = 240 \\ aa + ee = z. = 47232 \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \end{array}} \right\} \text{Quere } a.e.d.p.q.x.$$

$$1 \odot 2$$

$$3 - 2$$

$$2 - 4$$

$$5 \text{ w } 2$$

$$3$$

$$4$$

$$5$$

$$6$$

$$aa + 2ae + ee = ss$$

$$2ae = ss - z$$

$$aa - 2ae + ee = 2z - ss$$

$$a - e = \sqrt{2z - ss} = d$$



1 + 6	7	$2a = s + \sqrt{2z - ss}$	} The Rest are found just as in the 2d Question; the 8 and 10 Steps here, being the very same with the 8 and 10 Steps there.
$7 \div 2$	8	$a = \frac{s + \sqrt{2z - ss}}{2}$	
1 - 6	9	$2e = s - \sqrt{2z - ss}$	
$9 \div 2$	10	$e = \frac{s - \sqrt{2z - ss}}{2}$	

Question 5. When  $s$  and  $x$  are given; To find the Rest.

Viz. {	1	$a + e = s = 240$	} Quere $a.e.d.p.q.z.$
	2	$aa - ee = x = 46080$	
$2 \div 1$	3	$a - e = \frac{x}{s} = d$ viz. $a + e$ ) $aa - ee$ ( $a - e$	
1 + 3	4	$2a = s + \frac{x}{s} = \frac{ss + x}{s}$	
$4 \div 2$	5	$a = \frac{ss + x}{2s}$	
1 - 3	6	$2e = s - \frac{x}{s} = \frac{ss - x}{s}$	
$6 \div 2$	7	$e = \frac{ss - x}{2s}$	
$5 \times 7$	8	$ae = \frac{ssss - xx}{4ss} = p$	
$5 \div 7$	9	$\frac{a}{e} = \frac{ss + x}{ss - x} = q$	
$5 \odot 2$	10	$aa = \frac{s^4 - 2ssx + xx}{4ss}$	
$7 \odot 2$	11	$ee = \frac{s^4 - 2ssx + xx}{4ss}$	
10 + 11	12	$aa + ee = \frac{s^4 + xx}{2ss} = z$	

Question 6. Suppose  $d$  and  $p$  are given; To find the Rest.

Viz. {	1	$a - e = d = 192$	} Quere $a.e.s.q.z.x.$
	2	$ae = p = 5184$	

1 ⊖ 2	3	$aa - 2ae + ee = dd$
2 × 4	4	$4ae = 4p$
3 + 4	5	$aa + 2ae + ee = dd + 4p$
5 w. 2	6	$a + e = \sqrt{dd + 4p} = s$
6 + 1	7	$2a = d + \sqrt{dd + 4p}$
7 ÷ 2	8	$a = \frac{d + \sqrt{dd + 4p}}{2}$
6 - 1	9	$2e = \sqrt{dd + 4p} - d$
9 ÷ 2	10	$e = \frac{\sqrt{dd + 4p} - d}{2}$
8 ÷ 10	11	$\frac{a}{e} = \frac{d + \sqrt{dd + 4p}}{\sqrt{dd + 4p} - d} = q$
8 ⊖ 2	12	$aa = \frac{dd + 2p + d\sqrt{dd + 4p}}{2}$
10 ⊖ 2	13	$ee = \frac{dd + 2p - d\sqrt{dd + 4p}}{2}$
12 + 13	14	$aa + ee = dd + 2p = z$
12 - 13	15	$aa - ee = d\sqrt{dd + 4p} = x$

Question 7. Let  $d$  and  $q$  be given; To find the Rest.

Viz. {	1	$a - e = d$	192 } Quere $a . e . s . p . z . x .$
	2	$\frac{a}{e} = q = 9$	
2 × e	3	$a = qe$	
1 + e	4	$a = d + e$	
3, 4	5	$qe = d + e$	
5 - e	6	$qe - e = d$	
		$d$	
6 ÷ q - 1	7	$e = \frac{d}{q - 1}$	For $q - 1 \times e = qe - e$
1 + 7	8	$a = d + \frac{d}{q - 1} = \frac{qd}{q - 1}$	
7 + 8	9	$a + e = \frac{qd + d}{q - 1} = s$	



$7 \times 8$	10	$ae = \frac{qdd}{qq - 2q + 1} = p$
$8 \odot 2$	11	$aa = \frac{qqdd}{qq - 2q + 1}$
$7 \odot 2$	12	$ee = \frac{dd}{qq - 2q + 1}$
$11 + 12$	13	$aa + ee = \frac{qqdd + dd}{qq - 2q + 1} = z$
$11 - 12$	14	$aa - ee = \frac{qqdd - dd}{qq - 2q + 1} = x$

Question 8. Suppose  $d$  and  $z$  given; To find the Rest.

Viz. {	1	$a - e = d = 192$	} Quere $a . e . s . p . q . x .$
	2	$aa + ee = z = 47232$	
$1 \odot 2$	3	$aa - 2ae + ee = dd$	
$2 - 3$	4	$2ae = z - dd$	
$2 + 4$	5	$aa + 2ae + ee = 2z - dd$	
$5 \div 2$	6	$a + e = \sqrt{2z - dd} = s$	
$1 + 6$	7	$2a = d + \sqrt{2z - dd}$	
$7 \div 2$	8	$a = \frac{d + \sqrt{2z - dd}}{2}$	
$6 - 1$	9	$2e = \sqrt{2z - dd} - d$	
$9 \div 2$	10	$e = \frac{\sqrt{2z - dd} - d}{2}$	
$8 \times 10$	11	$ae = \frac{z - dd}{2} = p$	
$8 \odot 2$	12	$aa = \frac{z + d \sqrt{2z - dd}}{2}$	
$10 \odot 2$	13	$ee = \frac{z - d \sqrt{2z - dd}}{2}$	
$12 - 13$	14	$aa - ee = d \sqrt{2z - dd} = x$	
$8 \div 10$	15	$\frac{a}{e} = \frac{d + \sqrt{2z - dd}}{\sqrt{2z - dd} - d} = q$	

Question 9. Let  $d$  and  $x$  be given; To find the Rest.

<i>Viz.</i> {	I	$a - e = d = 240$	} Quere $a.e.s.p.q.z.$
	2	$aa - ee = x = 46080$	
$2 \div 1$	3	$a + e = \frac{x}{d} = s$ viz. $a - e$ ) $aa - ee$ ( $a + e$	
$1 + 3$	4	$2a = \frac{dd + x}{d}$	
$4 \div 2$	5	$a = \frac{dd + x}{2d}$	
$3 - 5$	6	$e = \frac{x - dd}{2d}$	
$5 \times 6$	7	$ae = \frac{xx - d^4}{4dd} = p$	
$5 \div 6$	8	$\frac{a}{e} = \frac{dd + x}{x - dd} = q$	
$5 \odot 2$	9	$aa = \frac{d^4 + 2ddx + xx}{4dd}$	
$6 \odot 2$	10	$ee = \frac{xx - 2ddx + d^4}{4dd}$	
$9 + 10$	11	$aa + ee = \frac{d^4 + xx}{2dd} = z$	

Question 10. Let  $p$  and  $q$  be given; To find the Rest.

<i>Viz.</i> {	I	$ae = p = 5184$	} Quere $a.e.s.d.z.x.$
	2	$\frac{a}{e} = q = 9$	
$1 \times 2$	3	$aa = qp$ For $\frac{ae}{1} \times \frac{a}{e} = \frac{aae}{e} = aa$	
$3 \text{ w } 2$	4	$a = \sqrt{qp}$	
$1 \div 2$	5	$ee = \frac{p}{q}$ For $\frac{a}{e} \div \frac{ae}{1} = \frac{aee}{a} = ee$	
$5 \text{ w } 2$	6	$e = \sqrt{\frac{p}{q}}$	
$4 + 6$	7	$a + e = \sqrt{qp} + \sqrt{\frac{p}{q}} = s$	



# Chap. 9. Of Numerical Questions. 209

4 — 6	8	$a - e = \sqrt{qp} - \sqrt{\frac{p}{q}} = d$
3 + 5	9	$aa + ee = qp + \frac{p}{q} = z$
3 — 5	10	$aa - ee = qp - \frac{p}{q} = x$

Question 11. Let  $p$  and  $z$  be given; To find the Rest.

Viz. {	1	$ae = p = 5184$	} Quere $a, e, \&c.$
	2	$aa + ee = z = 47232$	
$1 \times 2$	3	$2ae = 2p$	
$2 + 3$	4	$aa + 2ae + ee = z + 2p$	
$4 \div 2$	5	$a + e = \sqrt{z + 2p} = s$	
$2 - 3$	6	$aa - 2ae + ee = z - 2p$	
$6 \div 2$	7	$a - e = \sqrt{z - 2p} = d$	
$5 + 7$	8	$2a = \sqrt{z + 2p} + \sqrt{z - 2p}$	
$8 \div 2$	9	$a = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{2}$	
$5 - 7$	10	$2e = \sqrt{z + 2p} - \sqrt{z - 2p}$	
$10 \div 2$	11	$e = \frac{\sqrt{z + 2p} - \sqrt{z - 2p}}{2}$	
$9 \div 11$	12	$\frac{a}{e} = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{\sqrt{z + 2p} - \sqrt{z - 2p}}$	
$9 \odot 2$	13	$aa = \frac{z + \sqrt{zz - 4pp}}{2}$	
$11 \odot 2$	14	$ee = \frac{z - \sqrt{zz - 4pp}}{2}$	
$aa - ee$	15	$aa - ee = \sqrt{zz - 4pp} = x$	

Question 12. Let  $p$  and  $x$  be given; To find the Rest.

Viz. {	1	$ae = p = 5184$	} Quere $a, e, \&c.$
	2	$aa - ee = x = 46080$	
$1 \odot 2$	3	$aaee = pp$	

E e

2  $\odot$  2

2 ⊙ 2	4	$aaaa - 2aade + eeee = xx.$
3 × 4	5	$4aade = 4pp$
4 + 5	6	$aaaa + 2aade + eeee = xx + 4pp$
6 w 2	7	$aa + ee = \sqrt{xx + 4pp} = z$
2 + 7	8	$2aa = x + \sqrt{xx + 4pp}$
8 ÷ 2	9	$aa = \frac{x + \sqrt{xx + 4pp}}{2}$
9 w 2	10	$a = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}}$
7 - 2	11	$2ee = \sqrt{xx + 4pp} : - x$
11 ÷ 2	12	$ee = \frac{\sqrt{xx + 4pp} : - x}{2}$
12 w 2	13	$e = \sqrt{\frac{\sqrt{xx + 4pp} : - x}{2}}$
10 + 13	14	$a + e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = s$
10 - 13	15	$a - e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = d$
9 + 12	16	$aa + ee = \sqrt{xx + 4pp} = z$

Question 13. Having  $q$  and  $z$  given; To find the Rest.

$\text{viz. } \left\{ \begin{array}{l} 1 \times e \\ 3 \odot 2 \\ 2 - 4 \\ 4 + qqee \\ 6 \div qq + 1 \end{array} \right.$	1	$\frac{a}{e} = q = 9$	} Quere $a, e, \&c.$
	2	$aa + ee = z = 47232$	
	3	$a = qe$	
	4	$aa = qqee$	
	5	$ee = z - qqee$	
	6	$qqee + ee = z$	
	7	$ee = \frac{z}{qq + 1}$	For $qq + 1 \times ee = qqee + ee$



2 — 7	8	$aa = z - \frac{z}{qq + 1} = \frac{qqz}{qq + 1}$
8 w 2	9	$a = \sqrt{\frac{qqz}{qq + 1}}$
7 w 2	10	$e = \sqrt{\frac{z}{qq + 1}}$
9 + 10	11	$a + e = \sqrt{\frac{qqz}{qq + 1}} : + \sqrt{\frac{z}{qq + 1}} = s$
9 — 10	12	$a - e = \sqrt{\frac{qqz}{qq + 1}} : - \sqrt{\frac{z}{qq + 1}} = d$
9 × 10	13	$ae = \sqrt{\frac{qqzz}{q^4 + 2qq + 1}} = p$
8 — 7	14	$aa - ee = \frac{qqz - z}{qq + 1} = x$

Question 14. When  $q$  and  $x$  are given; To find the Rest.

viz. {	1	$\frac{a}{e} = q = 9$	} Quere $a, e, \&c.$
	2	$aa - ee = x = 46080$	
1 × e	3	$a = qe$	
3 ⊙ 2	4	$aa = qqe$	
2 + ee	5	$aa = x + ee$	
4, 5	6	$qqee = x + ee$	
6 — ee	7	$qqee - ee = x$	
7 ÷ qq — 1	8	$ee = \frac{x}{qq - 1}$	
2 + 8	9	$aa = x + \frac{x}{qq - 1} = \frac{qqx}{qq - 1}$	
9 w 2	10	$a = \sqrt{\frac{qqx}{qq - 1}}$	
8 w 2	11	$e = \sqrt{\frac{x}{qq - 1}}$	
10 + 11	12	$a + e = \sqrt{\frac{qqx}{qq - 1}} : + \sqrt{\frac{x}{qq - 1}} = s$	

10 — 11	13	$a - e = \sqrt{\frac{qqx}{qq - 1}} = \sqrt{\frac{x}{qq - 1}} = d$
10 × 11	14	$ae = \sqrt{\frac{qqxx}{qqq - 2qq + 1}} = p$
8 + 9	15	$aa + ee = \frac{qqx + x}{qq - 1} = z$

Question 15. When  $z$  and  $x$  are given; To find the Rest.

Viz. {	1	$aa + ee = z = 47232$	} Quere $a, e, \&c.$
	2	$aa - ee = x = 46080$	
1 + 2	3	$2aa = z + x$	
3 ÷ 2	4	$aa = \frac{z + x}{2}$	
1 — 2	5	$2ee = z - x$	
5 ÷ 2	6	$ee = \frac{z - x}{2}$	
4 w 2	7	$a = \sqrt{\frac{z + x}{2}}$	
6 w 2	8	$e = \sqrt{\frac{z - x}{2}}$	
7 + 8	9	$a + e = \sqrt{\frac{z + x}{2}} + \sqrt{\frac{z - x}{2}} = s$	
7 — 8	10	$a - e = \sqrt{\frac{z + x}{2}} - \sqrt{\frac{z - x}{2}} = d$	
7 × 8	11	$ae = \sqrt{\frac{zz - xx}{4}} = p$	
7 ÷ 8	12	$\frac{a}{e} = \frac{\sqrt{z + x}}{\sqrt{z - x}} = q$	

These fifteen Questions are proposed in Doctor Pell's Algebra; but he pursues only the first Question throughout, and breaks off in the other Fourteen, after the Values of what I call  $a$  and  $e$  are found. But I have proceeded in every one of them, to find the Values of all the unknown Quantities, because they afford such Variety, as being well observed by a Learner, will be found very useful in the Solution of most Questions.

Note,



# Chap. 9. Of Numerical Questions. 213

Note, I have chose to use the same Numbers for the respective Value of each Quantity throughout all the Questions, because they will be more satisfactory in proving the Work than various Numbers would have been. Not but that any Numbers may be taken at Pleasure, provided that the Number represented by  $a$ , be greater than that by  $e$ , &c. I have omitted the Numerical Calculations purely for the Learner to practise on.

**Question 16.** *There are two Numbers, the Sum of their Squares is 2368; And the Greater of them is in Proportion to the Less, As 6 To 1. What are those Numbers?*

Let  $a$  = the Greater Number,  $e$  = the Lesser, and  $z$  = 2368.

Then	1	$aa + ee = z$	} By the Question.
And	2	$a : e :: 6 : 1$	
	3	$1a = 6e$	
2 ∴	4	$aa = 36ee$	
3 ⊖ 2	5	$ee = z - 36ee$	
1 — 4	6	$37 ee = z$	
5 + 36ee	7	$ee = \frac{z}{37} = 64$	
6 ÷ 37	8	$e = \sqrt{\frac{z}{37}} = 8$	
7 w 2	9	$6e = 6 \sqrt{\frac{z}{37}} = 48$	
8 × 6	10	$a = 48$	
3. 9			

Proof {

If  $a = 48$

And  $e = 8$

$aa = 2304$

$ee = 64$

---

$aa + ee = 2368$

And  $48 : 8 :: 6 : 1$

**Question 17.** *There are three Numbers in Continued Proportion, the Sum of the Extremes is 156, and the Mean is 72; What are the two Extremes?*

That is, Suppose  $a . m . e$  in ∴ and  $m = 72$ .

Then {	1	$a + e = s = 156$	} By the Question.
	2	$a : m :: m : e$	
	3	$ae = mm$	
2 ∴	4	$aa + 2ae + ee = ss$	
1 ⊖ 2	5	$4ae = 4mm$	
3 × 4	6	$aa + 2ae + ee = ss - 4mm$	
4 — 5	7	$a - e = \sqrt{ss - 4mm}$	
6 w 2	8	$2a = s + \sqrt{ss - 4mm}$	
1 + 7			

$8 \div 2$	9	$a = \frac{s + \sqrt{ss - 4mm}}{2} = 108$	} Or {	$a = 48$
$1 - 9$	10	$e = \frac{s - \sqrt{ss - 4mm}}{2} = 48$		$e = 108$

Question 18. *There are three Numbers in  $\div$  their Sum is 74, and the Sum of their Squares is 1924; What are those Numbers?*

That is,  $a, e, y$  are in  $\div$ .

Then {	1	$a + e + y = s = 74$	} Quere $a, e, y$ .
	2	$aa + ee + yy = z = 1925$	
	3	$a : e :: e : y$	
<hr/>			
3 $\div$	4	$ay = ee$	
1 $- e$	5	$a + y = s - e$	
2 $- ee$	6	$aa + yy = z - ee$	
4 $\times \frac{1}{2}$	7	$2ay = 2ee$	
6 $\div 7$	8	$aa + 2ay + yy = z + ee$	
5 $\odot 2$	9	$aa + 2ay + yy = ss - 2se + ee$	
8 and 9	10	$z + ee = ss - 2se + ee$	
10 $\div$	11	$2se = ss - z$	
11 $\div 25$	12	$e = \frac{ss - z}{25} = 24$	
5,	13	$a + y = s - e = 50$	
13 $\odot 2$	14	$aa + 2ay + yy = 2500$	
4 $\times 4$	15	$4ay = 4ee = 2304$	
14 $- 15$	16	$aa - 2ay + yy = 196$	
16 $\div 2$	17	$a - y = \sqrt{196} = 14$	
13 $\div 17$	18	$2a = 50 + 14 = 64$	
18 $\div 2$	19	$a = 32$	} Or {
13 $- 19$	20	$y = 50 - 32 = 18$	

Note, In all Questions about Continual Proportionals. (either Arithmetical or Geometrical) where three Terms are sought, the Mean is easiest found first (as above;) and if all the Terms be affirmative, then 'tis equal whether the first, or last Term be the greatest.

Question 19. *There are three Numbers in  $\div$  their Sum is 76; and if the Sum of the Extremes be multiplied into the Mean, that Product will be 1248; What are those Numbers?*

Viz.



<i>viz.</i> {	1	$a : e :: e : y$	} By the Question.
	2	$a + e + y = s = 76$	
	3	$ae + ye = p = 1248$	
	4	$ay = ee$	
1 ∴	5	$ae + ee + ye = se$	
2 × e	6	$ee = se - p$	
5 — 3	7	$ee - se = -p$	
6 — se	8	$ee - se + \frac{1}{4}ss = \frac{1}{4}ss - p$	
7 C □	9	$e - \frac{1}{2}s = \sqrt{\frac{1}{4}ss - p}$	
8 in 2	10	$e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = \begin{cases} 52. & \text{Per Theor. 3.} \\ 24. & \text{Chap. 8.} \end{cases}$	
9 + $\frac{1}{2}s$	11	$a + y = 52$	
2 — 10	12	$4ay = 4ee = 2304$	
4 × 4	13	$aa + 2ay + yy = 2704$	
11 ⊖ 2	14	$aa - 2ay + yy = 400$	
13 — 12	15	$a - y = \sqrt{400} = 20$	
14 in 2	16	$2a = 52 + 20 = 72$	
11 + 15	17	$a = 36$	} { Or $a = 16$ and $y = 36$
16 ÷ 2	18	$y = 52 - 36 = 16$	
11 — 17			

N. B. If you take  $e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = 52$  (at the 10th Step) Then it will be  $76 - 52 = 24 = a + y$ , which is impossible, *viz.* that the Mean shou'd be greater than the Sum of the two Extremes.

Therefore it must be  $e = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - p} = 24$ . (See Pag. 201.)

**Question 20.** *There are three Numbers in Arithmetical Progression, the First being added to twice the Second, and three times the Third, their Sum will be 62; and the Sum of all their Squares is 275: What are those Numbers?*

Suppose	1	$a, e, y$ in Arithmetical Progression.	
And {	2	$a + 2e + 3y = 62$	} by the Question.
	3	$aa + ee + yy = 275$	
Then	4	$a + y = 2e$ per Sect. I. Chap. 6.	
2 — 4	5	$2e + 2y = 62 - 2e$	
5 ÷ 2	6	$e + y = 31 - e$	
6 — e	7	$y = 31 - 2e$	
4 — 7	8	$a = 4e - 31$	
8 ⊖ 2	9	$aa = 16ee - 248e + 961$	
7 ⊖ 2	10	$yy = 961 - 124e + 4ee$	
8 + 10	11	$aa + yy = 20ee - 372e + 1922$	

3 — 11	12	$ee = 372e - 20ee - 1647$
12 + 20ee	13	$21ee = 372e - 1647$
13 — 37ee	14	$21ee - 372e = -1647$
14 ÷ 21	15	$ee = \frac{124}{7} e = -\frac{549}{7}$
15 C □	16	$ee = \frac{124}{7} e + \frac{3844}{9} = \frac{3844}{9} - \frac{549}{7} = \frac{1}{49}$
16 w 2	17	$e = \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$ The Mean.
17 + $\frac{62}{7}$	18	$e = \frac{62}{7} \pm \frac{1}{7} = 9$ Or $8\frac{5}{7}$
18 × 4	19	$4e = 36$ Or $8\frac{6}{7}$
8, 19	20	$a = 36 - 31 = 5$ Or $34\frac{6}{7} - 31 = 3\frac{6}{7}$
18 × 2	21	$2e = 18$ Or $17\frac{3}{7}$
7, 21	22	$y = 31 - 18 = 13$ Or $31 - 17\frac{3}{7} = 13\frac{4}{7}$

**Question 21.** *There are three Numbers in Arithmetical Progression; the Square of the first Term being added to the Product of the other two is 576: The Square of the Mean being added to the Product of the two Extremes, makes 612: And the Square of the last Term being added to the Product of the first into the second, is 792: What are those Numbers?*

Suppose	1	$a, e, y$ In Arith. Progres. As before.
Then {	2	$aa + ye = 576$
	3	$ee + ya = 612$
	4	$yy + ae = 792$
		By the Question.
1 ∴	5	$a + y = 2e$ Per Sect. 1. Chap. 6.
5 × e	6	$ae + ye = 2ee$
2 + 4	7	$aa + ye + yy + ae = 1368$
7 — 6	8	$aa + yy = 1368 - 2ee$
3 — ee	9	$ya = 612 - ee$
9 × 2	10	$2ya = 1224 - 2ee$
8 + 10	11	$aa + 2ya + yy = 2592 - 4ee$
5 C 2	12	$aa + 2ya + yy = 4ee$
11, 12	13	$4ee = 2592 - 4ee$
13 + 4ee	14	$8ee = 2592$
14 ÷ 8	15	$ee = 324$
15 w 2	16	$e = \sqrt{324} = 18$ The Mean.
8,	17	$aa + yy = 1368 - 2ee = 720$
10,	18	$2ya = 1224 - 2ee = 576$
17 — 18	19	$aa - 2ya + yy = 720 - 576 = 144$



# Chap. 9. Of Numerical Questions. 217

$$\begin{array}{lcl}
 1 \text{ w } 2 & \left\{ \begin{array}{l} 20 \\ 21 \\ 22 \\ 23 \end{array} \right. & \left\{ \begin{array}{l} a - y = \sqrt{144} = 12 \\ 2a = 2e + 12 = 48 \\ a = 24 \\ y = 2e - 24 = 12 \end{array} \right. \\
 5 + 20 & & \\
 21 \div 2 & & \\
 5 - 22 & &
 \end{array}
 \quad \text{Or} \quad \begin{cases} a = 12 \\ y = 24 \end{cases}$$

**Question 22.** 'Tis required to find two such Numbers, that the Sum of their Squares may be 8226½; and their Product being added to the Square of the Lesser, may be 6921½.

*Viz.*  $\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right.$

$$\begin{array}{lcl}
 1 & aa + ee = 8226\frac{1}{2} \\
 2 & ae + ee = 6921\frac{1}{2} \end{array} \left. \vphantom{\begin{array}{l} 1 \\ 2 \end{array}} \right\} \text{Quere } a \text{ and } e$$


---


$$\begin{array}{lcl}
 1 - 2 & 3 & aa - ae = 1305 \\
 3 \pm & 4 & ae = aa - 1305 \\
 4 \div a & 5 & e = \frac{aa - 1305}{a} \\
 5 \odot 2 & 6 & ee = \frac{a^4 - 2610aa + 1703025}{aa} \\
 1 - aa & 7 & ee = 8226,5 - aa \\
 6, 7 & 8 & \frac{a^4 - 2610aa + 1703025}{aa} = 8226,5 - aa \\
 8 \times aa & 9 & a^4 - 2610aa + 1703025 = 8226,5aa - a^4 \\
 9 + a^4 & 10 & 2a^4 - 2610aa + 1703025 = 8226,5aa \\
 10 \pm & 11 & 2a^4 - 10836,5aa = -1703025 \\
 11 - 2 & 12 & a^4 - 5418,25aa = -851512,5 \\
 12 \square & 13 & a^4 - 5418,25aa + 733958,265625 = 6487845,765625 \\
 13 \text{ w } 2 & 14 & aa - 2709,125 = \sqrt{6487845,765625} = 2547,125 \\
 14 + 27 \&c. & 15 & aa = 2709,125 + 2547,125 \\
 Suppose & 16 & aa = 2709,125 + 2547,125 = 5256,25 \\
 Then & 17 & a = \sqrt{5256,25} = 72,5 \\
 And 5, & 18 & e = \frac{aa + 1305}{a} = \frac{5256,25 - 1305}{72,5} = 54,5 \\
 Or let & 19 & aa = 2709,125 - 2547,125 = 162 \\
 10 \text{ w } 2 & 20 & a = \sqrt{162} = 12,72 \&c. \\
 Then & 21 & e = \frac{162 - 1305}{12,72} \text{ Which is impossible.} \\
 Therefore & & a = 72,5 \\
 And & & e = 54,5 \end{array} \left. \vphantom{\begin{array}{l} a = 72,5 \\ e = 54,5 \end{array}} \right\} \text{As at the 17th and 18th Steps.}$$

This Question may be perform'd with less Trouble, by substituting Letters for the known Numbers.

*Viz.*  $\left\{ \begin{array}{l} aa + ee = z \\ ae + ee = p \end{array} \right.$  Then let  $z - p = d = aa - ae, \&c.$

F f

Question

**Question 23.** It is required to find three such Numbers, that the Sum of the First and Second, being multiplied with the Third, may be 37824; and the Sum of the Second and Third, multiplied with the First, may be 59944, also, that the Sum of the First and Third, being multiplied with the Second, may be 52456.

Let  $a, e, y$  represent the three Numbers.

Then  $\left\{ \begin{array}{l} 1 \quad ay + ey = 37824 = b \\ 2 \quad ea + ya = 59944 = c \\ 3 \quad ae + ye = 52456 = d \end{array} \right\} \text{Quere } a, e, y.$

$1 + 2 + 3$	4	$2ae + 2ay + 2ye = b + c + d$
Let	5	$z = b + c + d$
$4 \div 2$	6	$ae + ay + ye = \frac{1}{2}z = \frac{b+c+d}{2}$
$6 - 3$	7	$ay = \frac{1}{2}z - d$
$7 \div a$	8	$y = \frac{z - 2d}{2a}$
$6 - 2$	9	$ye = \frac{1}{2}z - c$
$6 - 1$	10	$ae = \frac{1}{2}z - b = \frac{z - 2b}{2}$
$10 \div a$	11	$e = \frac{z - 2b}{2a}$
$8 \times 11$	12	$ye = \frac{z - 2d}{2a} \times \frac{z - 2b}{2a} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$
$9, 12$	13	$\frac{z - 2c}{2} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$
$13 \times 4aa$	14	$2zaa - 4caa = zz - 2dz - 2bz + 4bd$
$14 \div$	15	$az = \frac{zz - 2dz - 2bz + 4bd}{2z - 4c} = 55696$
$15 \div 2$	16	$a = \sqrt{55696} = 236$
II	17	$e = \frac{z - 2b}{2a} = 158$
8,	18	$y = \frac{z - 2d}{2a} = 96$

**Question 24.** 'Tis required to find two such Numbers, that their Sum, being subtracted from the Sum of their Squares, may leave 14. And if their Product be added to their Sum, it may make 14.

Let



# Chap. 9. Of Numerical Questions. 219

Let  $a$  and  $e$  be put for the Numbers, and let  $y = a + e$

Then  $\left\{ \begin{array}{l} 1 \quad aa + ee - y = 14 \\ 2 \quad ae + y = 14 \end{array} \right\}$  By the Question.

$$1 + y \quad 3 \quad aa + ee = 14 + y$$

$$2 - y \quad 4 \quad ae = 14 - y$$

$$4 \times \frac{1}{2} \quad 5 \quad 2ae = 28 - 2y$$

$$3 + 5 \quad 6 \quad aa + 2ae + ee = 42 - y$$

$$6 \div 2 \quad 7 \quad a + e = \sqrt{42 - y}$$

But  $8 \quad a + e = y$  By Substitution above.

$$7, 8, \quad 9 \quad y = \sqrt{42 - y}$$

$$9 \text{ } \odot \quad 2 \quad 10 \quad yy = 42 - y$$

$$10 + y \quad 11 \quad yy + y = 42$$

$$11 \text{ } \odot \quad \square \quad 12 \quad yy + y + \frac{1}{4} = 42 + \frac{1}{4} = 42,25$$

$$12 \div 2 \quad 13 \quad yy + \frac{1}{2} = \sqrt{42,25} = 6,5$$

$$13 - \frac{1}{2} \quad 14 \quad y = 6,5 - \frac{1}{2} = 6$$

Consequent  $15 \quad a + e = 6$  By Restitution from above.

$$3, 14 \quad 16 \quad aa + ee = 14 + 6 = 20$$

$$5, 15 \quad 17 \quad 2ae = 28 - 12 = 16$$

$$16 - 17 \quad 18 \quad aa - 2ae + ee = 4$$

$$18 \div 2 \quad 19 \quad a - e = \sqrt{4} = 2$$

$$15 + 19 \quad 20 \quad 2a = 8 \quad \left\{ \begin{array}{l} \text{If } a = 4 \quad \text{And } e = 2 \\ \text{Then } aa + ee - a - e = 14 \\ \text{And } ae + a + e = 14 \\ \text{According to the Question.} \end{array} \right.$$

$$23 \div 2 \quad 21 \quad a = 4 \quad \text{Proof}$$

$$15 - 21 \quad 22 \quad e = 6 - 4 = 2$$

Question 25. Three Men discoursing of their Money, saith the First, if 100 l. were added to my Money, it would be as much as both your Money put together; said the second Man, if 100 l. were added to my Money, I should have twice as much as both you have; saith the third Man, if 100 l. were added to my Money, I should have then three times as much Money as both you have: How much Money had each Man?

Let  $a$  represent the first Man's Money,  $e$  the Second, and  $y$  the Third?

Then  $\left\{ \begin{array}{l} 1 \quad a + 100 = e + y \\ 2 \quad e + 100 = 2a + 2y \\ 3 \quad y + 100 = 3a + 3e \end{array} \right\}$  By the Question.

$$1 - a \quad 4 \quad e + y - a = 100 = s$$

$$2 - e \quad 5 \quad 2a + 2y - e = 100 = s$$

$$3 - y \quad 6 \quad 3a + 3e - y = 100 = s$$

$$4, 6 \quad 7 \quad e + y - a = 3a + 3e - y$$

$$7 \div 2 \quad 8 \quad 2y = 4a + 2e$$

$$5 - 8 \quad 9 \quad 2a - e = s - 4a - 2e$$

$$9 + 4a \quad 10 \quad 6a + e = s = 100$$

E f 2

4 + 6

$$\begin{array}{ll}
 4 + 6 & \text{II} \quad 2a + 4e = 2s = 200 \\
 10 \times 4 & \text{I2} \quad 24a + 4e = 4s = 400 \\
 12 - 11 & \text{I3} \quad 22a = 2s = 200 \\
 13 \div 22 & \text{I4} \quad a = \frac{s}{11} = \frac{100}{11} = 9\frac{1}{11} l. \\
 10 - 6a & \text{I5} \quad e = s - 6a = 100 - \frac{600}{11} = \frac{500}{11} = 45\frac{5}{11} l. \\
 8 \div 2 & \text{I6} \quad y = 2a + e = \frac{200}{11} + \frac{500}{11} = \frac{700}{11} = 63\frac{7}{11} l.
 \end{array}$$

Answer. The  $\left\{ \begin{array}{l} \text{First} \\ \text{Second} \\ \text{Third} \end{array} \right\}$  Man had  $\left\{ \begin{array}{l} 9 l. \quad 1 s. \quad 9\frac{1}{11} d. \\ 45 l. \quad 9 s. \quad 1\frac{5}{11} d. \\ 63 l. \quad 12 s. \quad 8\frac{7}{11} d. \end{array} \right.$

Question 26. Three Men have each such a Sum of Money, that if the first and second Mens Money be added to half of what the third Man hath; that Sum will be 92 l. And if the second and third Mens Money be added to one third Part of the first Man's Money, that Sum will be 92 l. Lastly, if one fourth Part of the second Man's Money be added to the first and third Mens Money, that Sum will also be 92 l. How much was each Man's Money?

Put  $a$  for the first Man's Money,  $e$  for the Second, and  $y$  for the Third.

$$\begin{array}{ll}
 \text{Then } \left\{ \begin{array}{l} 1 \quad a + e + \frac{1}{2}y = s \\ 2 \quad \frac{1}{3}a + e + y = s \\ 3 \quad \frac{1}{4}e + a + y = s \end{array} \right\} & \text{By the Question} \quad \text{And } s = 92
 \end{array}$$

$$\begin{array}{ll}
 1. \quad 2 & 4 \quad a + e + \frac{1}{2}y = \frac{1}{3}a + e + y \\
 4 - e & 5 \quad a + \frac{1}{2}y = \frac{1}{3}a + y \\
 5 \times \frac{2}{2} \times \frac{3}{3} & 6 \quad 6a + 3y = 2a + 6y \\
 6 + & 7 \quad 4a = 3y \\
 2 \times 3 & 8 \quad a + 3e + 3y = 3s \\
 8 - 7 & 9 \quad a + 3e = 3s - 4a \\
 9 - a & 10 \quad 3e = 3s - 5a \\
 10 \div 3 & 11 \quad e = \frac{3s - 5a}{3} \\
 3 \times 4 & 12 \quad e + 4a + 4y = 4s = 368 \\
 12 - 2 & 13 \quad 3\frac{2}{3}a + 3y = 3s = 276 \\
 13. \quad 7 & 14 \quad 3\frac{2}{3}a + 4a = 3s = 276 \\
 14 \times 3 & 15 \quad 11a + 12a = 9s = 828 \\
 15 \div 23 & 16 \quad a = \frac{9s}{23} = \frac{828}{23} = 36 l. \quad \text{The 1st Man's Money.} \\
 & 17 \quad e = \frac{3s - 5a}{3} = \frac{276 - 180}{3} = 32 l. \quad \text{The 2d Man's Money.} \\
 & 18 \quad y = \frac{4a}{3} = \frac{144}{3} = 48 l. \quad \text{The 3d Man's Money.}
 \end{array}$$



Chap. 9. Of Numerical Questions. 221

Question 27. Four Men walking abroad found a Purse of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found; that if the First took 25 Shillings from the Second, it would make his Number equal with what the Second had then left. If the Second took 30 Shillings from the Third, his Money would then be triple to what the Third had left. And if the Third took 40 Shillings from the Fourth, his Money would then be double to what the Fourth had left. Lastly, the Fourth taking 50 Shillings from the First, he would then have three times as much as the First had left, and 5 Shillings more.

'Tis required to tell how many Shillings each Man had? put *a* for the first Sum, *e* the Second, *y* the Third, and *u* the Fourth.

Then {

1

$a + 25 = e - 25$

2

$e + 30 = 3y - 90$

3

$y + 40 = 2u - 80$

4

$u + 50 = 3a - 145$

By the Question.

1 + 25

5

$a + 50 = e$

2 - 30

6

$3y - 120 = e$

5. 6

7

$a + 50 = 3y - 120$

7 + 120

8

$a + 170 = 3y$

8 ÷ 3

9

$y = \frac{a + 170}{3}$

3 - 40

10

$u = 2u - 120$

9, 10

11

$2u - 120 = \frac{a + 170}{3}$

11 + 120

12

$2u = \frac{a + 170}{3} + 120 = \frac{a + 530}{3}$

12 ÷ 2

13

$u = \frac{a + 530}{6}$

4 - 50

14

$u = 3a - 195$

13, 14

15

$3a - 195 = \frac{a + 530}{6}$

15 × 3

16

$18a - 1170 = a + 530$

16 +

17

$17a = 1700$

17 ÷ 17

18

$a = 100$

by the 5

19

$e = 150$

by the 9

20

$y = 90$

by the 14

21

$u = 105$

The 1st

Second

Third

Fourth

Man's N. of Shillings.

Question

Question 28. Four Men have each a Sum of Money which being put all together makes 250 Pounds. And if to the first Man's Money be added 8 Pounds it will be just as much as the second Man's Money decreased by 8 Pounds, and as much as 8 times the third Man's Money, and but as much as one eighth Part of the fourth Man's Money; How much had each Man?

Let  $a, e, y, u$  represent the Four Mens Money.

Then  $\left\{ \begin{array}{l} 1 \quad a + e + y + u = s \\ 2 \quad a + b = e - b \\ 3 \quad yb = \frac{u}{b} = a + b \end{array} \right\}$  by the Question. Let  $s = 250$  and  $b = 8$ . Or any other Number at Pleasure.

$$2 + b \quad 4 \quad a + 2b = e$$

$$3 \div b \quad 5 \quad y = \frac{a + b}{b} \quad \text{Because } yb = a + b$$

$$3 \times b \quad 6 \quad u = ba + bb. \quad \text{For } \frac{u}{b} = a + b$$

$$4 + 5 + 6 \quad 7 \quad e + y + u = a + 2b + \frac{a + b}{b} + ba + bb$$

$$1 - a \quad 8 \quad e + y + u = s - a$$

$$7, \quad 8 \quad 9 \quad a + 2b + \frac{a + b}{b} + ba + bb = s - a$$

$$9 \times b \quad 10 \quad ba + 2bb + a + b + bba + bbb = bs - ba$$

$$10 \div \quad 11 \quad 2ba + bba + a = bs - bbb - 2bb - b$$

$$11 \div \quad 12 \quad a = \frac{bs - bbb - 2bb - b}{bb + 2b + 1} = 16,691358, \text{ \&c.}$$

$$\text{by the } 4 \quad 13 \quad e = a + 2b = 32,691358, \text{ \&c.}$$

$$\text{by the } 5 \quad 14 \quad y = \frac{a + b}{b} = 3,086419, \text{ \&c.}$$

$$\text{by the } 6 \quad 15 \quad u = ba + bb = 197,530864, \text{ \&c.}$$

That is  $\left\{ \begin{array}{l} a = 16 \text{ . } 13 \text{ . } 9,92592 \\ e = 32 \text{ . } 13 \text{ . } 9,92592 \\ y = 3 \text{ . } 1 \text{ . } 8,74056 \\ u = 197 \text{ . } 10 \text{ . } 7,40736 \end{array} \right.$

Consequently  $a + e + y + u = 249 \text{ . } 19 \text{ . } 11,99976$  which should be just 250 *l.* the Sum proposed in the Question. Now what it wants of that Sum proceeds from the Imperfection of the Decimal Parts being not continued on to more Places, which would have brought it nearer the Truth, tho' not perhaps exactly so. *Vide Sect. 5. Chap. 5. Part I.*

Question



# Chap. 9. Of Numerical Questions. 223

Question 29. Several Merchants enter into Partnership, every one put into the Stock 65 times as many Pounds as there were Partners; with that Stock they traded, and gain'd as many Pounds per 100 l. as there were Partners. Now if 10 l. 10 s. be added to, and subtracted from their Gain, the Product of that Sum, and Difference will be 6491 l. 6 s. 3 d.

Quere, How many Merchants there were, &c.

Let	1	$a =$	The Number of Merchants.
$1 \times 65$	2	$65a =$	Every one's Sum he put into Stock.
$2 \times a$	3	$65aa =$	The whole Stock.
And	4	$100 : a :: 65aa : \frac{65aaa}{100}$	by the Question.
Viz.	5	$\frac{65aaa}{100} =$	The whole Gain.
$5 + 10,5$	6	$\frac{65aaa}{100} + 10,5$	
$5 - 10,5$	7	$\frac{65aaa}{100} - 10,5$	
$6 \times 7$	8	$\frac{4225aaaaaa}{10000} - 110,25 = 6491,3125$	by the Question.
$8 \times 10000$	9	$4225a^6 - 1102500 = 64913125$	
$9 +$	10	$4225a^6 = 66015625$	
$10 \div 4225$	11	$a^6 = \frac{66015625}{4225} = 15625$	
$11 \text{ w } 6$	12	$a = \sqrt[6]{15625} = 5$	The Number of Merchants.
$12 \times 65$	13	$65a = 325.$	The Number of Pounds each put in

Question 30. Three Merchants join Stocks together; the first Man's Stock was less than the second Man's by 13 l. The second and third Man's Stock was 175 l. in trading they gain 48 l. more than their whole Stock was; the first Man's proportional Part of the Gain was 78 l. What was each Man's Stock and Part of the Gain?

Let  $a, e, y$  represent each Man's Stock.

Then	1	$a + e + y = s$	The whole Stock.
	2	$s + 48 =$	The whole Gain.
And	3	$a + 13 = e$	By the Question.
	4	$e + y = 175$	
$4 + a$	5	$a + e + y = 175 + a$	
$1, 5$	6	$s = 175 + a$	

6, 2	7	$5 + 48 = 223 + a$	
But	8	$175 + a : 223 + a :: a : 78$	Per Quest.
8	9	$aa + 223a = 78a + 13650$	
9 — 78a	10	$aa + 145a = 13650$	
10 C□	11	$aa + 145a + 5256,25 = 18906,25$	
11 w 2	12	$a + 72,5 = \sqrt{18906,25} = 137,5$	
12 — 72,5	13	$a = 137,5 - 72,5 = 65$	
3,	14	$e = a + 13 = 78$	
4 — 14	15	$y = 97$	
Then	16	$65 : 78 :: 78 : 93 \text{ l. } 12 \text{ s.} = e\text{'s}$	— — — Gain.
Again	17	$65 : 78 :: 97 : 116 \text{ l. } 8 \text{ s.} = y\text{'s}$	— — — Gain.
Proof {	18	$116 \text{ l. } 8 \text{ s.} + 93 \text{ l. } 12 \text{ s.} + 78 \text{ l.} = 288 \text{ l.}$	The Gain.
	19	$65 + 78 + 97 = 240.$	The whole Stock.
18 — 19	20	$288 - 240 = 48.$	The Gain more than the Stock.

Question 31. A Father at his Death left his three Sons all his Money in this Manner; to the Eldest he gave half of it, wanting 44 Pounds; to the Second he gave one third of it, and 14 Pounds more; to the Youngest he gave the Remainder, which was less than the Share of the second Son by 82 Pounds; What was each Son's Share?

Let  $a, e, y$  be the three Shares; and  $z =$  the whole Sum.

$$\text{Then } \left\{ \begin{array}{l} 1 \quad a + e + y = z \\ 2 \quad a = \frac{1}{2}z - 44 \\ 3 \quad e = \frac{1}{3}z + 14 \\ 4 \quad y = \frac{1}{3}z + 14 - 82 \end{array} \right. \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \text{By the Question.}$$

$$\begin{array}{rcl} 2 + 3 + 4 & 5 & a + e + y = \frac{2z}{3} + \frac{z}{2} - 98 \\ & 6 & z = \frac{2z}{3} + \frac{z}{2} - 98 \\ & 7 & 3z = 2z + \frac{3z}{2} - 194 \\ & 8 & 6z = 4z + 3z - 588 \\ & 9 & z = 588. \text{ The whole Sum that was left.} \\ 2, & 9 & 10 \quad a = \frac{588}{2} - 44 = 250. \text{ The Eldest Son's Share.} \\ 3, & 9 & 11 \quad e = \frac{588}{3} + 14 = 210. \text{ The second Son's, \&c.} \\ 4, & 9 & 12 \quad y = \frac{588}{3} + 14 - 82 = 128. \text{ The Youngest \&c.} \end{array}$$

Question 32. A Man playing at Hazard or Dice. won the first Throw just so much Money as he had in his Pocket; the second Throw he won the Square Root of what he then had and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was 112 l. 16 s. What Money had he when he began to play?

Suppose.



# Chap. 9. Of Numerical Questions. 225

Suppose	1	$a =$ His first Sum.
$1 \times 2$	2	$2a =$ His Sum after the first Throw.
And	3	$5 + \sqrt{2a} =$ The Winnings at the 2d Throw.
$2 + 3$	4	$2a + 5 + \sqrt{2a} =$ The Sum after the 2d Throw.
$4 \odot 2$	5	$4aa + 22a + 25 + 4a\sqrt{2a} + 10\sqrt{2a} =$ The Winnings at the 3d Throw; and therefore
$4 + 5$	6	$4aa + 24a + 30 + 4a\sqrt{2a} + 11\sqrt{2a} = 2256 \text{ Sh.}$

But to avoid these Surd Quantities. let us instead of supposing  $a =$  the first Sum, make a second Trial.

Viz. let	1	$2aa =$ The first Sum.
$1 \times 2$	2	$4aa =$ The Sum after the first Throw.
Then	3	$2a + 5 =$ The Sum won at the second Throw.
$2 + 3$	4	$4aa + 2a + 5 =$ His Sum after the 2d Throw.
$4 \odot 2$	5	$16a^4 + 16a^3 + 44aa + 20a + 25 =$ The Winnings at the 3d Throw; And therefore
$4 + 5$	6	$16a^4 + 16a^3 + 48aa + 22a + 30 = 2256 \text{ Shillings.}$

Yet again, to avoid these high Æquations, let us make a third Supposition; Thus,

The	1	$\frac{aa}{2} =$ The first Sum.
$1 \times 2$	2	$aa =$ The Sum after the first Throw.
Then	3	$a + 5 =$ The Winnings at the 2d Throw.
$2 + 3$	4	$aa + a + 5 =$ The Sum after the 2d Throw.
Substi.	5	$e = aa + a + 5$
$5 \odot 2$	6	$ee =$ The Winnings at the 3d Throw. Then
$5 + 6$	7	$ee + e = 2256 \text{ Shillings by the Question.}$
$7 \text{ C}\square$	8	$ee + e + 0,25 = 2256,25$
$8 \text{ w } 2$	9	$e + 0,5 = \sqrt{2256,25} = 47,5$
$9 - 0,5$	10	$e = 47$
$5, 10$	11	$aa + a + 5 = 47$
$11 - 5$	12	$aa + a = 42$
$12 \text{ C}\square$	13	$aa + a + 0,25 = 42,25$
$1 \text{ w } 2$	14	$a + 0,5 = \sqrt{42,25} = 6,5$
$14 - 0,5$	15	$a = 6$
$15 \odot 2$	16	$aa = 36$
$16 \div 2$	17	$\frac{aa}{2} = \frac{36}{2} = 18$

} The Shill. he had in his Pocket  
when he began to play.

Note, In resolving of this Question, I have made three different Suppositions for the Thing sought; purely as an Instance to shew the young Learner, how well he ought to consider the Nature of the Question, when he first States it, and make choice of

representing the Thing sought, so as to avoid running it into Surds if possible, viz. as in the first Supposition of  $a =$  the first Sum, &c. Not but that such *Æquations* may be solved, as shall be shew<sup>d</sup> in the next Chapter. However, it is most like an Artist to perform Things of this Nature the nearest and easiest Way they can be done.

**Question 33.** Suppose there were two equal Circles, whose Peripheries (viz. Circumferences) are divided into 44310 equal Parts; and that those Circles were so placed upon one Axis, as to move the contrary Way to each other; and suppose one of them to move, but one of those equal Parts the first Day, two Parts the second Day, three Parts the third Day, and so on in Arithmetical Progression, viz. 1, 2, 3, 4, 5, &c. And the other to move every Day the Cubes of those Parts, viz. 1. 8. 27. 64. 125, &c. of the same Parts. How many Parts, and how many Days must each Circle move, before the same two Points meet that were together when they began to move?

In order to give a ready Solution to this Question (or any other in this Kind) it will be convenient to premise this Lemma.

### Lemma.

The Sum of any Series of Cubes whose Roots are in Arithmetical Progression (the first Term, and common Difference being Unity or 1) is equal to the Square of the Sum of all those Roots.

As in these,

Terms in Arith. &c. Their Cubes.

1	1
2	8
3	27
4	64
5	125
6	216, &c.

$21 \times 21 = 441$  Sum of their Cubes.

Let	1	$a =$ The Sum of all the Parts the 1 <sup>st</sup> Circle moves.
Then	2	$aa =$ The Sum of all the Parts the 2 <sup>d</sup> moves,
Consequen.	3	$aa + a = 44310$ By the Quest. (per Lem.
2 $\square$	4	$aa + a + 0,25 = 44310,25.$



# Chap. 9. Of Numerical Questions. 227

$4 \text{ w } 2$	$5$	$a + 0,5 = \sqrt{44310,25} = 210,5$	$\left\{ \begin{array}{l} \text{the Number of Parts the first} \\ \text{Circle must move.} \end{array} \right.$
$5 - 0,5$	$6$	$a = 210$	
$6 \text{ @ } 2$	$7$	$aa = 44100$	$\left\{ \begin{array}{l} \text{The Number of Parts the 2d} \\ \text{Circle moves.} \end{array} \right.$

Next to find the Number of Days they moved, there is given, the first Term  $= 1$ , the common Difference  $= 1$ . And the Sum of all the Terms  $= 210$ , thence to find the last Term, which in this Case is the same with the Number of all the Terms.

Let  $a = 1$  the first Terms.  $e = 1$  the common Difference, and  $s = 210$  the Sum of all the Terms, to find  $y =$  the last Term. As per Sect. 1. Chap. 6.

Then  $yy + ey = 2s + aa - ae$  by the 16th Step, Page 186.

That is,  $yy + y = 210 \times 2 = 420$ , &c.

Hence  $y = 20$  the Number of Days required.

I shall now proceed to give an Example or two of the Method used in arguing about unlimited Questions, viz. such Questions which admit of various Answers, such as those in *Alligation Alternate*, promised in Page 117.

In order to shorten that Work, it will be convenient for the Learner to know the two Signs of Comparison  $>$  And  $<$ . The Sign  $>$  is of Greater than, As  $b > a$  signifies that  $b$  is Greater than  $a$ . The Sign  $<$  is of Lesser than. As  $b < d$  signifies that  $b$  is Lesser than  $d$ , &c.

## Example 1.

Question 34. A Tobacconist hath three Sorts of Tobacco, viz. one of 2s. 8d. the Pound, another of 20d. the Pound, and a third Sort of 16d. the Pound; of these he would make a Mixture to contain 56 Pound that may be sold for 22d. the Pound: How much of each Sort may he take?

Let  $a =$  the Quantity of that worth 32 Pence the Pound.  $e =$  that of 20 Pence the Pound, And  $y =$  that of 16 Pence the Pound;

Then  $a + e + y = 56$

And  $32a + 20e + 16y = 1232$

$\left\{ \begin{array}{l} \text{viz. each Quantity multi-} \\ \text{plied into its own Price, e-} \\ \text{quals their Sum multiplied} \\ \text{into the mean Price.} \end{array} \right.$

G g 2

This

This Question being thus stated, it appears by *Rule I. Page 176.* that it is capable of innumerable Answers; because for any one of these three Letters  $a . e . y$ . there may be taken any Number at Pleasure, provided it be less than 56. But although that may be truly done, yet there are several Ways of arguing about these Sorts of Questions, which will limit or bound them to all their proper or possible Answers in whole Numbers. Thus,

Let	1	$a + e + y = 56$	} As above.
And	2	$32a + 20e + 16y = 1232$	
$1 - a$	3	$e + y = 56 - a$	
$2 - 32a$	4	$20e + 16y = 1232 - 32a$	
$3 \times 16$	5	$16e + 16y = 896 - 16a$	
$4 - 5$	6	$4e = 336 - 16a$	
$6 \div 4$	7	$e = 84 - 4a$ Hence $a < \frac{3 \cdot 4}{4} = 21$	
$3 - 7$	8	$y = 3a - 28$ Hence $a > \frac{2 \cdot 8}{3} = 9\frac{1}{3}$	

From the two last Steps it appears, that the Quantity signified by  $a$ , ought to be less than 21, and Greater than  $9\frac{1}{3}$ ; That is, any Number betwixt  $9\frac{1}{3}$  and 21, may be taken for the Value of  $a$ . Consequently there may be Eleven Answers to this Question in whole Numbers.

Suppose  $a = 10$  Then  $e = 84 - 40 = 44$  Per 7th Step.

And  $y = 30 - 28 = 2$  Per 8th Step.

Again, if  $a = 11$  Then  $e = 84 - 44 = 40$  Per 7th Step.

And  $y = 33 - 28 = 5$  Per 8th Step. And so on for the Rest, which will be as in the following Table.

$a$	$e$	$y$	$a$	$e$	$y$	$a$	$e$	$y$
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be easy to find out and collect all the limited Answers to any Question (of this kind) wherein there are only three Quantities propos'd to be mix'd: But when there are more than Three, then the Work requires a little more Trouble; because the single Limits of all the Quantities above Two must be found. That is, if there are four Quantities concern'd in the Question, the Limits of two of them must be found; If five Quantities are concern'd, then the Limits of three of them must be found, &c. As in the following Question.

Question



# Chap. 9. Of Numerical Questions. 229

**Question 35.** Suppose it were required to mix four Sorts of Wines together; viz. one Sort worth 7 s. 4 d. the Gallon; another Sort worth 4 s. 7 d. the Gallon; a third Sort worth 3 s. 8 d. the Gallon; and a fourth Sort worth 2 s. 9 d. the Gallon; How much of each Sort may be taken to make a Mixture of 63 Gallons, so as that the whole Quantity may be sold for 5 s. 6 d. the Gallon; without Loss, &c.

First let all these several Rates, and the mean Rate, be reduced to one Denomination, viz. into Pence.

$$\text{viz. } \left. \begin{array}{l} 7 \text{ s. } 4 \text{ d.} = 88 \text{ d.} \\ 3 \text{ s. } 8 \text{ d.} = 44 \text{ d.} \end{array} \right\} \begin{array}{l} 4 \text{ s. } 7 \text{ d.} = 55 \text{ d.} \\ 2 \text{ s. } 9 \text{ d.} = 33 \text{ d.} \end{array} \text{ And } 5 \text{ s. } 6 \text{ d.} = 66 \text{ d.}$$

Then put  $a$  = the Quantity of that worth 88 d. the Gallon;  $e$  = that of 55 d. the Gallon;  $y$  = that of 44 d. the Gallon; and  $u$  = that of 33 d. the Gallon.

Then	1	$a + e + y + u = 63$	By the Question.
And	2	$88a + 55e + 44y + 33u = 4158$	$= 63 \times 66$
$1 - a$	3	$e + y + u = 63 - a$	
$2 - 88a$	4	$55e + 44y + 33u = 4158 - 88a$	
$3 \times 33$	5	$33e + 33y + 33u = 2079 - 33a$	
$4 - 5$	6	$22e + 11y = 2079 - 55a$	
$6 \div 11$	7	$2e + y = 189 - 5a$	Hence $a < \frac{189}{5} = 37\frac{4}{5}$
$3 \times 55$	8	$55e + 55y + 55u = 3465 - 55a$	
$8 - 4$	9	$11y + 22u = 33a - 693$	
$9 \div 11$	10	$y + 2u = 3a - 63$	Hence $a > \frac{63}{3} = 21$

From the 7th and 10th Steps it appears, that the Quantity of that Sort of Wine denoted by  $a$ . must be less than  $37\frac{4}{5}$  Gallons, and greater than 21 Gallons: That is, it may be  $a$  = any Number of Gallons betwixt 21 and  $37\frac{4}{5}$ .

Whence it follows, that there may be collected 16 Answers to this Question from the Limits of  $a$  only.

Next to find the Limits of  $e$ ,  $y$ , and  $u$ .

Suppose	11	$a = 22$	Then will $5a = 110$ . And $3a = 66$
But	12	$2e + y = 189 - 5a = 79$	Per 7th Step.
$12 - 2e$	13	$y = 79 - 2e$	Hence $e < \frac{79}{2} = 39\frac{1}{2}$
Again	14	$e + y + u = 63 - a = 41$	Per third Step.
$14 - e$	15	$y + u = 41 - e$	
$15 - 13$	16	$u = e - 38$	Hence $e < 38$

From

From the 13th and 16th Steps it appears, that if  $a = 22$  Then  $e = 39$ .  $y = 79 - 2e - 1$ . And  $u = e - 28 = 1$ .

Again.

Suppose	17	$a = 23$	Then $5a = 115$ .	And $3a = 69$
But	18	$2e + y = 189 - 5a = 74$ .	Per 7th Step.	
$18 - 2e$	19	$y = 74 - 2e$	Hence $e < \frac{74}{2} = 37$	
Again	20	$e + y + u = 63 - a = 40$ .	Per 3d Step.	
$20 - e$	21	$y + u = 40 - e$		
$21 - 19$	22	$u = e - 34$ .	Hence $e > 34$ .	

From the 19th and 22d Steps it appears, that if  $a = 23$ . Then  $e$  may be either 35 or 36.

Once more for a further Illustration.

Let	23	$a = 24$	Then $5a = 120$ .	And $3a = 72$
But	24	$2e + y = 189 - 5a = 69$ .	Per 7th Step.	
$24 - 2e$	25	$y = 69 - 2e$ .	Hence $e < \frac{69}{2} = 34\frac{1}{2}$	
Again	26	$e + y + u = 63 - a = 39$	Per 3d Step.	
$26 - e$	27	$y + u = 39 - e$		
$27 - 25$	28	$u = e - 30$	Hence $e > 30$ .	

From hence it appears, that if  $a = 24$ , Then  $e$  may be either 31. 32. 33. Or 34. viz. it may be any Number betwixt 30 and  $34\frac{1}{2}$  by the 25th, and 28th Steps, from whence the Values of  $y$  and  $u$  may be easily found.

That is, if	{	$e = 31$ :	Then $y = 7$ .	And $u = 1$
		$e = 32$ .	$y = 5$ .	$u = 2$
		$e = 33$ .	$y = 3$ .	$u = 3$
		$e = 34$ .	$y = 1$ .	$u = 4$

Proceeding on in this manner with all the other single Values of  $a$ , there may be found above 120 Answers to this Question in whole Numbers: And if you please to put  $a =$  Fractions, there may be found an indefinite Number of Answers; whereas the Rule of Alligation in Vulgar Arithmetick affords but only one Answer in Fractions; to wit, that of  $a = 31\frac{1}{2}$ .  $e = 10\frac{1}{2}$ .  $y = 10\frac{1}{2}$ .  $u = 10\frac{1}{2}$ . As may be easily try'd, per Rule Page 115, &c.

These two Examples being well understood (especially if the last be thorowly pursu'd) may suffice to shew the Method of limiting the Answers to all Sorts of Questions of this kind, I shall therefore conclude this Chapter of Questions, with giving a Solution to the Ænigma (or Riddle) proposed (but not answered) by Mr. John Kersey, in the Close of the Appendix to his *Arithmetick*, which affords several pretty Questions, the Solution whereof will discover a certain Sentence consisting of three Words, which must be found by the



# Chap. 9. Of Numerical Questions. 231

the Help of Figures placed (or supposed to be placed) over the Twenty Four Letters of the *Alphabet*.

Thus  $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \&c. \text{ Called Indices.} \\ a \cdot b \cdot c \cdot d \cdot e \cdot f \cdot g \cdot \&c. \text{ to the last Letter.} \end{array} \right.$

So that if the *Index* of any Letter be once found, the Letter to which it belongs, is consequently known.

## The Enigma.

1. If the Difference between the Indices of the second Letter of the second Word; and the third Letter of the first Word, be multiplied into the Difference of their Squares, the Product will be 576. And if their Sum be multiplied into the Sum of their Squares, that Product will be 2336, the *Index* of the said third Letter being the Greatest.

Let	1	$a =$ The Greater <i>Index</i> , or that of the 3d Letter,
And	2	$e =$ The Lesser, or that of the 2d Letter.
Then $\left\{ \begin{array}{l} \\ \end{array} \right.$	3	$a - e \times aa - ee = 576$
	4	$a + e \times aa + ee = 2336$ } By the Question.
$3 \times$	5	$aaa - aae - aee + eee = 576$
$4 \times$	6	$aaa + aae + aee + eee = 2336$
$6 - 5$	7	$2aae + 2aee = 1760$
$6 + 7$	8	$aaa + 3aae + 3aee + eee = 4096$
$8 \div 3$	9	$a + e = \sqrt[3]{4096} = 16$
$4 \div a + e$	10	$aa + ee = \frac{2336}{a + e} = \frac{2336}{16} = 146$
$9 \ominus 2$	11	$aa + 2ae + ee = 256$
$11 - 10$	12	$2ae = 110$
$10 - 12$	13	$aa - 2ae + ee = 36$
$13 \div 2$	14	$a - e = \sqrt{36} = 6$
$9 + 14$	15	$2a = 22$ } From hence it appears, that the 3d Letter of
$15 \div 2$	16	$a = 11$ } the 1st Word is <i>t</i> , and the 2d Letter of the
$9 - 16$	17	$e = 5$ } 2d Word is <i>e</i> .

Note, In order to set down the Letters (as they become found) in their proper Places, it may be convenient to supply the vacant Places with Stars.

Thus  $\left\{ \begin{array}{l} \text{First Word.} \\ \star \star t \star \star \end{array} \right. \quad \left\{ \begin{array}{l} \text{Second Word.} \\ \star \star e \star \star \star \end{array} \right. \quad \left\{ \begin{array}{l} \text{Third Word.} \\ \star \star \star \star \star \end{array} \right.$

2. The

2. The Indices last found, are the two Extremes of Four Numbers in Arithmetical Progression, the lesser Mean being the *Index* of the first Letter of the third Word; and the greater Mean is the *Index* of the fourth and last Letter of the first Word.

*Viz.* 5 . 7 . 9 . 11 are the four Terms in *Arith. Progression*.

Whence it appears, that *G* (whose *Index* is 7) is the first Letter of the third Word; and that *i* (whose *Index* is 9) is the fourth or last Letter of the first Word; which being placed down, will stand thus,

*\*\*li    \*e\*\*\*.    G\*\*\*.*

3. The second Letter of the third Word, is the same with the third Letter of the first Word; and the fifth Letter of the third Word is the same with the last Letter of the first Word.

Whence the Letters will stand thus, *\*\*li.    \*e\*\*\*.    Gl\*\*i\*.*

4. The Sum of the Squares of the Indices of the first and second Letters of the first Word is 520. And the Product of the same Indices is seven Ninths of the Square of the greater *Index*, which is the *Index* of the said first Letter.

Let *a* = the Greater, and *e* = the lesser *Index*.

Then	1	$aa + ee = 520$	} According to the <i>Data</i> .
And	2	$ae = \frac{7}{9}aa$	
<hr/>			
$2 - a$	3	$e = \frac{7}{9}a$	
$3 \odot 2$	4	$ee = \frac{49}{81}aa$	
$1 - 4$	5	$aa = 520 - \frac{49}{81}aa$	
$5 \times 81$	6	$81aa = 42120 - 49aa$	
$6 + 49aa$	7	$130aa = 42120$	
$7 \div 130$	8	$aa = \frac{42120}{130} = 324$	
$8 \text{ w } 2$	9	$a = \sqrt{324} = 18$	It's Letter is <i>s</i> :
$3, 9$	10	$e = \frac{7}{9}a = 14.$	It's Letter is <i>o</i> .

Hence the Letter will stand thus, *Soli.    \*e\*\*\*.    Gl.\*\*i\*.*

5. The Difference between the two last *Indices*, is the *Index* of the first Letter of the 2d Word; *viz.*  $18 - 14 = 4$  being the *Index* of the Letter *D*.

Then the Letters will stand thus, *Soli.    De\*\*\*.    Gl.\*\*i\*.*

6. The third and last Letter of the second Word, also the third Letter of the third Word, are the same with the second Letter of the first Word.

Hence



# Chap. 9. Of Numerical Questions. 233

Hence the Letters will stand thus, *Soli Deo Glo \* i \**

7. The *Sum* of the *Indices* of the fourth Letter of the third Word, and the sixth or last Letter of the same Word, being added to their *Product* is 35. And the Difference of their Squares is 288. The *Index* of the last Letter being the least.

Put  $a$  = the Greater, and  $e$  = the Lesser *Index*, as before.

Then	1	$ae + a + e = 35$	} By the <i>Data</i> .
And	2	$aa - ee = 288$	
$1 - a$	3	$ae + e = 35 - a$	
$3 \div a + 1$	4	$e = \frac{35 - a}{a + 1}$ For $e \times a + 1 = ae + e$	
$4 \odot 2$	5	$ee = \frac{1225 - 70a + aa}{aa + 2a + 1}$	
$2 + 5$	6	$aa = 288 + \frac{1225 - 70a + aa}{aa + 2a + 1}$	
$6 \times aa \&c.$	7	$\{ a^4 + 2a^3 + aa = 288aa + 576a + 288 + 1225$	
$7 \pm$	8	$\{ - 70a + aa$ $a^4 + 2a^3 - 288aa - 506a = 1513$	

This last *Æquation* being resolved according to the *Method* which shall be shewed in the next Chapter, it will be  $a = 17$ . It's Letter  $r$ ; And from the 4th Step  $= \frac{35 - a}{a + 1} = 1$  the *Index* of the Letter  $a$ .

Then these two Letters being placed according to the *Data* above, are all that are required by the *Ænigma* to complete these Words,

*Soli Deo Gloria.*

## C H A P. X.

## The Solution of Affected Equations in Numbers.

Before we proceed to the Solution of *Affected Equations*, it may not be amiss to shew the Investigation (or Invention) of those *Theorems* or *Rules* for extracting the Roots of simple Powers, made use of in Chapter II. Part I.

I shall here make Choice of the same Letters, to represent the Numbers both given and sought; as in my Compendium of *Algebra*.

*G*, always denote the given *Resolvend*.  
 Viz. Let  $\left\{ \begin{array}{l} r = \text{any Number taken as near the true Root} \\ \quad \text{as may be, whether it be greater or less.} \\ e = \text{the unknown Part of the Root sought, by} \\ \quad \text{which } r \text{ is to be either increased or de-} \\ \quad \text{creased.} \end{array} \right.$

Then if  $r$  be any Number less than the true Root it will be  $r + e =$  the Root sought.

But if  $r$  be taken Greater than the true Root it will then be  $r - e =$  the Root sought.

And put  $D$  for the *Dividend* that is produced from  $G$ , after it is lessened and Divided by  $r$  &c. (into the Co-efficients of *Affected Equations*) according as the Nature of the Root requires.

These Things being premised, we may proceed to raising the *Theorems*.

## Section I.

I. For the Square Root, viz.  $aa = G$ . Quere  $a$ .

Let	1	$r + e = a$	
1 $\ominus$ 2	2	$rr + 2re + ee = aa = G$	
2 $- rr$	3	$2re + ee = G - rr$ Call it $D$ , viz. $D = G - rr$	
Then	4	$\left\{ \frac{D}{2r + e} = e \right\}$	This shews the 1st Method of extracting the Square Root, Sect. 5. Ch. II. Part I.
3 $\div$ 2	5	$re + \frac{1}{2}ee = \frac{G - rr}{2} = D$	

Which gives this Theorem  $\left\{ \frac{D}{r + \frac{1}{2}e} = e \right.$

The



The Arithmetical Operations of both these *Theorems*. you have in the *Examples* of *Section 2. Page 126*; To which I refer the Learner, supposing him by this time to understand them without any more Words than what is there exprest.

## II. To extract the Cube Root; viz. $aaa = G$ . Quere $a$ .

Let	1	$r + e = a$ Supposing $r$ less than the true Root.
1 $\ominus$ 3	2	$rrr + 3rre + 3ree + eee = aaa = G$
2 $-$ rrr	3	$3rre + 3ree + eee = G - rrr$
3 $\div$ 3r	4	$re + ee + \frac{eee}{3r} = \frac{G - rrr}{3r} = D$

Let  $\frac{eee}{3r}$  be rejected or cast off, as being of small Value:

Then it will be,  $re + ee = D$ , which gives this following

$$\text{Theorem } \sqrt[3]{\frac{D}{r + e}} = e$$

By this *Theorem* or *Rule*. the 1st and 2d *Examples* in *Case 1. Page 132*. are perform'd; the which being compared with this *Theorem* may be very easily understood.

Again, Suppose  $aaa = G$ , (as before) And let  $r$  be taken greater than the true Root.

Then	1	$r - e = a$	$\left. \begin{array}{l} \text{\textit{See}} \text{ being rejected} \\ \text{as above.} \end{array} \right\}$
1 $\ominus$ 3	2	$rrr - 3rre + 3ree = a^3 = G$	
2 $\pm$	3	$3rre - 3ree = rrr - G$	
3 $\div$ 3r	4	$re - ee = \frac{rrr - G}{3r} = D$	

Which gives this *Theorem*  $\sqrt[3]{\frac{D}{r - e}} = e$

By this *Theorem* the third *Example* in *Case 2. Page 133*. is perform'd.

## III. To extract the Biquadrate Root; viz. $a^4 = G$ . Quere $a$ .

Let	1	$r + e = a$ Supposing $r$ Less than <i>Just</i> .	$\left. \begin{array}{l} \text{Rejecting all the} \\ \text{Powers of } e \text{ as a-} \\ \text{bove} \end{array} \right\}$
1 $\ominus$ 4	2	$r^4 + 4rrre + 6rree = a^4 = G$	
2 $-$ r <sup>4</sup>	3	$4rrre + 6rree = G - r^4$	
3 $\div$ 2rr	4	$2re + 3ee = \frac{G - r^4}{2rr} = D$	

Which gives this *Theorem*  $\sqrt[4]{\frac{D}{2r + 3e}} = e$

By this *Theorem* the *Biquadrate Root* of any *Number* may be extracted. But as I have already said, Page 134. those Extractions may be very well perform'd, by two Extractions of the *Square Root*. *Vide Example Page 135.*

IV. To *extract* the *Surfsolid Root*, viz.  $a^5 = G$ . Quere  $a$ .

If  $r$  be taken less than just, then  $r + e = a$ . As before.

And  $\left\{ \frac{G - r^5}{5r^3} = D \right.$  Which gives this *Theorem*  $\left\{ \frac{D}{r + \frac{1}{2}e} = e \right.$

By this *Theor.* the *Surfsolid Root Examp. 1. Page 136*, is extracted. But if  $r$  be taken greater than just; Then  $r - e = a$ .

And  $\left\{ \frac{r^5 - G}{5r^3} = D \right.$  Which gives this *Theorem*  $\left\{ \frac{D}{r - \frac{1}{2}e} = e \right.$

By this last *Theorem* the *Example* in *Page 137* is perform'd.

I presume it needless to pursue the raising of these *Theorems*, for extracting the *Roots* of simple *Powers* any further; because the *Method* of doing it is general, how highsoever they are; and therefore it may be easily understood by what is already done.

Notwithstanding I have already shewed the *Solution* of *Quadratic Equations* two several *Ways*, viz. by casting off the lowest *Term*: And by completing the *Square*, *Vide Section 2. Page 195 &c.* Yet it may not be amiss to shew, how those *Equations* may be resolved into *Numbers* by this *Universal Method* of continued *Series*, wherein if the first  $r$  be taken equal to the first true *Root* or single *Side* of the *Resolvend*; And every single *Value* of  $e$  (as it becomes found) be still added to it, for a new  $r$ . Then those *Roots* may be extracted without repeating a second *Operation*, As before in the single *Powers*.

*Case I.* Let  $aa + 2ba = G$ . 'Tis required to find the *Value* of  $a$ .

Put	1	$r + e = a$
1 $\odot$ 2	2	$rr + 2re + ee = aa$
1 $\times$ 2b	3	$2br + 2be = 2ba$
2 $+$ 3	4	$rr + 2br + 2re + 2be + ee = aa + 2ba = G$
4 $-$ rr &c.	5	$2re + 2be + ee = G - rr - 2br$
5 $\div$ 2	6	$re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D$

Which gives this *Theorem*  $\left\{ \frac{D}{r + b + \frac{1}{2}e} = e \right.$

Suppose



Suppose  $b = 364$  And  $G = 38692865$   
 If  $r = 6000$  Then  $rr = 36000000$  And  $2br = 4368000$   
 But  $36000000 + 4368000 = 40368000 > 38692865 = G$   
 Therefore the first  $r < 6000$ . Let  $r = 5000$  Then

$$\begin{array}{r} \text{1st } r = 5000 \\ b = 364 \\ \hline \end{array} \quad \begin{array}{r} 19346432,5 = \frac{1}{2}G \\ - 1432000, = \frac{1}{2}rr + br \\ \hline \end{array}$$

$$\begin{array}{r} \text{1st } r + b = 5364 \\ + \frac{1}{2}e = 400 \\ \hline \end{array} \quad \begin{array}{r} 5026432,5 = D \text{ (800 = } e \text{)} \\ 46112 \\ \hline \end{array}$$

$$\begin{array}{r} \text{1 Divisor } 5764) \\ \text{2d } r + b = 6164 \\ + \frac{1}{2}e = 30 \\ \hline \end{array} \quad \begin{array}{r} 41523 \\ 37164 \\ \hline \end{array} \quad (60 = e$$

$$\begin{array}{r} \text{2 Divisor } 6194) \\ \text{3d } r + b = 6224 \\ + \frac{1}{2}e = 35 \\ \hline \end{array} \quad \begin{array}{r} 4359 \\ 43592,5 \\ \hline \end{array} \quad \begin{array}{r} (7 = e \\ 867 = e \\ \hline \end{array}$$

$$3 \text{ Divisor } 6227,5$$

$$\begin{array}{l} \text{First } r = 5000 \\ + e = 867 \end{array} \} = 5867 = a \text{ As was required.}$$

Case 2. If  $aa - 2ba = G$  Then proceeding as above, there will arise this Theorem  $\left\{ \frac{D}{r - b + \frac{1}{2}e} = e. \&c. \right.$

And in Case 3. viz.  $2ba - aa = G$  you will have this Theorem  $\left\{ \frac{D}{b - r - \frac{1}{2}e} \&c. \right.$  As above.

I think it needless to trouble the Reader with the Work of these two Theorems in Numbers; because if the last Example of Case 1. be understood, the other will be easy. Not but that the Method of completing the Square is very ready and easy, as you may observe by the Work in several Questions of this Chapter.

## Section 3.

In the Solution of all Affected Equations, that are above (or higher than) Quadratics. it will be the best way to take  $r =$  the next nearest Root of the Equation: And then it will be  $r + e = a$  if  $r$  be less than just; Or  $r - e = a$  if  $r$  be greater than just (as at the Beginning of this Chap.)

And all the Powers of the unknown Part of the Root (viz.  $e$ ) above its Square ( $ee$ ) are to be rejected or cast off; As before in

in raising the *Theorems* for the simple Powers. And therefore it is, that to supply the Want of those Powers (above *ee* in the *Theorem*) the Operation must be repeated: As in the *Example* of extracting the *Cube Root*. Page 133. *viz.* when the Figures in the *Root* consist of more than three Places, (*Vide* Page 140, and 141.)

Suppose  $aaa + ba = G$ . Quere  $a$ .

Let	1	$r + e = a$ <i>viz.</i> Let $r$ be supposed <i>Less</i> than <i>just</i> .
1 $\odot$ 3	2	$rrr + 3rre + 3ree = aaa$
1 $\times$ $b$	3	$br + be = ba$
2 $+$ 3	4	$rrr + br + 3rre + be + 3ree = a^3 + ba = G$
4 $\div$ $3r$	5	$\frac{1}{3}rr + \frac{1}{3}b + re + \frac{be}{3r} + ee = \frac{G}{3r}$
5 $-$ &c.	6	$re + \frac{be}{3r} + ee = \frac{G}{3r} - \frac{1}{3}rr - \frac{1}{3}b = D$

Which gives this *Theorem*  $\left\{ \begin{array}{l} \frac{D}{r + \frac{b}{3r} + e} = e \end{array} \right.$

But if  $r$  be taken greater than *just*, Then it will

be  $re + \frac{be}{3r} - ee = \frac{1}{3}rr + \frac{1}{3}b - \frac{G}{3r} = D$  Which produces this

*Theorem*  $\left\{ \begin{array}{l} \frac{D}{r + \frac{b}{3r} - e} = e \end{array} \right.$

By either of these two *Theorems* the Value of  $a$  may be easily found. Or rather otherwise as in the following Example.

Let  $aaa + 24a = 587914$

Here  $b = 24$ .

Suppose the First  $r = 90$  Then  $r^3 = 729000 > 587914$  without the  $24 \times 90$  being added to it: Therefore  $r < 90$  Again, Suppose  $r = 80$  Then  $r^3 = 512000$  And  $24r = 1920$  But  $512000 + 1920 = 513920 < 587914$  Hence  $> 80$ , but nearer to it than 90. Therefore

it must be	1	$r + e = a$ <i>Less</i> than <i>just</i>
1 $\odot$ 3	2	$rrr + 3rre + 3ree = aaa$
1 $\times$ 24	3	$24r + 24e = 24a$
2, in Numb.	4	$512000 + 192000 + 240ee = aaa$
3, in Numb.	5	$1920 + 24e = 24a$
4 $+$ 5	6	$513920 + 19224e + 240ee = 587914$
6 $-$ 513920	7	$19224e + 240ee = 73994$
7 $\div$ 240	8	$80,1e + ee = 308,31 = D$
8 $\div$	9	$e = \frac{D}{80,1 + e}$

Operation



# Chap. 10. Of Affected Equations. 239

$$\text{Operation } 80,1) \quad 308,31 = D \quad (3,7, = e \\ + e = ,3 \quad 2493$$

$$\text{1. Divisor } 83,1) \quad 59,01 \quad \text{First } r = 80, \\ + e = ,7 \quad 58.66 \quad + e = 3,7$$

$$\text{2. Divisor } 83,8) \quad ,35 \quad r + e = 83,7$$

Or rather new  $r = 83,7$  for a second Operation, which being involved and tryed (as above) will be found Greater than just; Therefore

it must be	1	$r - e = a$
1 $\oplus$ 2	2	$rrr - 3rre + 3ree = aaa$
1 $\times$ 24	3	$24r - 24e = 24a$
2, in Numb.	4	$586376,253 - 21017,07e + 251,1ee = aaa$
3, in Numb.	5	$2008,8 - 24e = 24a$
4 + 5	6	$588385,053 - 21041,07e + 251,1ee = 587914$
6 +	7	$21041,07e - 251,1ee = 471,053$
7 $\div$ 251,1	8	$83,7955e - ee = 1,87595778 = D$
8 $\div$	9	$e = \frac{D}{83,7955 - e}$

$$\text{2. Operation } 83,7955) \quad 1,87595778 \quad (,0223 = e \\ - e = ,02 \quad 1,675510$$

$$\text{1. Divisor } 83,7755) \quad ,2004477 \\ - e = ,002 \quad ,1675470$$

$$\text{2. Divisor } 83,7735) \quad ,03290078 \\ - e = ,0003 \quad ,02513196$$

$$\text{3. Divisor } 83,7732) \quad ,00776882$$

Having once found half the Places of Figures for the Value of  $e$ , it will be needless to form New Divisors (as above); for the rest of the Figures may be as truly found by plain Division only. Thus

$$\text{The last Divisor is } 83,7732) \quad ,007768820 \quad (,0223 = e \} \text{Add} \\ \quad \quad \quad 7539588 \quad (,0000927$$

$$\text{Last } r = 83,7 \quad 2292320 \quad ,0223927 = e \\ - e = ,0223927 \quad 1675464$$

$$r - e = 83,6776073 = a \quad 6168560 \\ \quad \quad \quad 5864124 \text{ \&c.}$$

But if more Exactness be required, you may make the New  $r = 83,6776073$  And proceed with it to a third Operation; which

which will afford Twenty Seven Places of Figures for the Value of  $a$ . That is, every Operation will produce triple the Places of Figures to those of the Precedent  $r$ . And this tripling the Places of Figures in the Root, at every Operation, holds good, and is to be observed in the Solution of all Adfected  $\mathcal{A}$ equations (how high soever they are) according to this Method of resolving them. See Page 141.

*Example 2.* Suppose  $aaa - ba = G$ . Quere  $a$ .

If  $r + e = a$  Then  $re - \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$

which gives this Theorem  $\left\{ \frac{D}{r - \frac{\frac{1}{3}b}{r} + e} = e \right.$

But if  $r - e = a$  Then  $re + \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$

which gives this Theorem  $\left\{ \frac{D}{r + \frac{\frac{1}{3}b}{r} + e} = e \right.$

Or you may proceed otherwise, as in the last *Example*.

Let  $aaa - 6438a = 104785688$  Here  $b = 6438$   
 Suppose the First  $r = 500$ .  $rrr = 125000000$  and  $br = 3219000$   
 Then  $125000000 - 3219000 = 121781000$   
 But  $121781000 > 104785688$  Therefore  $r < 500$   
 Again, Suppose  $r = 400$   $rrr = 64000000$ . and  $br = 2575200$   
 Then will  $64000000 - 2575200 = 6142800$   
 But  $6142800 < 104785688$  Hence  $r > 400$

Consequently  $r$  is betwixt 400 and 500. But 500 is the next nearest; Therefore, Let  $r = 500$  being Greater than just.

Then	1	$r - e = a$
1 $\odot$ 3	2	$rrr - 3rre + 3ree = aaa$
1 $\times$ b	3	$br - be = ba$
2, in Numb.	4	$125000000 - 750000e + 1500ee = aaa$
3, in Numb.	5	$3219000 - 6438e = 6438a$
4, — 5	6	$121781000 - 743562e + 1500ee = 104785688$
6 $+$	7	$743562e - 1500ee = 16995312$
7 $\div$ 1500	8	$495e - ee = 11330 = D$
8 $\div$	9	$e = \frac{D}{495 - e}$

Operation



$$\begin{array}{r} \text{Operation } 495) \quad 11330 \quad (23,8 = e \\ -e = \quad 20 \quad 950 \end{array}$$

$$\begin{array}{r} \text{1. Divisor } 475) \quad 1830 \\ -e = \quad 3 \quad 1416 \end{array}$$

$$\begin{array}{r} \text{First } r = 500 \\ -e = \quad 23,8 \end{array}$$

$$\begin{array}{r} \text{2. Divisor } 472) \quad 414,0 \\ \quad 377,6 \end{array}$$

$$r - e = 476,2 = a$$

Let New  $r = 476$  for a 2d Operation. Then  $r^3 = 107850176$  and  $br = 3064488$  But  $107850176 - 3064488 = 104785688$  the same with the Resolvend. Consequently  $a = 476$  just.

*Example 3.* Let  $ba - aaa = G$ . Quere  $a$ .

$$\text{If } r + e = a \text{ Then } \frac{\frac{1}{3}be}{r} - re - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$$

$$\text{which gives this Theorem } \left\{ \frac{D}{\frac{\frac{1}{3}b}{r} - r - e} = e \right.$$

$$\text{But if } r - e = a \text{ Then } re - \frac{\frac{1}{3}be}{r} - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$$

$$\text{which gives this Theorem } \left\{ \frac{D}{r - \frac{\frac{1}{3}b}{r} - e} = e \right.$$

Or otherwise as before in the two last Examples. Thus

Let  $123456a - aaa = 12272861$ . Here  $b = 123456$ . Suppose the First  $r = 200$  Then  $rrr = 8000000$ . and  $br = 24691200$ . then  $24691200 - 8000000 = 16691200$  but  $16691200 > 12272861$ : therefore  $r$  is here less than just, because the highest Power is —, or Negative.

Again, Suppose  $r = 300$  then  $r^3 = 27000000$  and  $br = 37036800$  Then  $37036800 - 27000000 = 10036800 < 12272861$  Consequently  $r < 300$  and  $r > 200$

Let  $r = 300$ . being the next nearest, but more than just.

Then	1	$r - e = a$
1 $\odot$ 3	2	$rrr - 3rre + 3ree = aaa$
1 $\times$ b	3	$br - be = ba$
2, in Numb.	4	$27000000 - 2700000e + 900ee$
3, in Numb.	5	$37036800 - 123456e$
5 — 4	6	$10036800 + 146544e - 900ee = 12272861$
6 —	7	$146544e - 900ee = 2236961$
7 $\div$ 900	8	$162e - ee = 2484 = D$
1 $\div$ &c.	9	$e = \frac{D}{162 - e}$

I i

Operation

$$\begin{array}{r} \text{Operation. } 162) \quad 2484 \quad (16,6 = e \\ - e = \quad 10 \quad 152 \end{array}$$

$$\begin{array}{r} \text{1. Divisor } 152) \quad 964 \\ - e = \quad 6 \quad 876 \end{array}$$

$$\begin{array}{r} \text{First } r = 300 \\ - e = \quad 16,6 \end{array}$$

$$\begin{array}{r} \text{2. Divisor } 146) \quad 88,0 \\ \quad 87,6 \end{array}$$

$$r - e = 283,4 = a$$

Or New  $r = 283$  which being involved, &c. will appear to be the true Root. That is,  $a = 283$  just.

Note, These are usually called the three Forms of Cubic Æquations; and in the Solution of the third or last Form, viz.  $ba - aaa = G$ , you may meet with some seeming Difficulties; especially in making Choice of the First  $r$  because this Æquation is an ambiguous Æquation, and hath two Affirmative Roots, viz. a Greater and Lesser Root. But having once found either of them, the other may be easily obtained by Division only; As in the Quadratic Æquations *Vide Chapter 8.*

As for Instance, in the last Example,  $a = 283$

And  $123456a - aaa = 12272861$ .

Make these two

Æquations  $= 0$ . To wit, Let  $a - 283 = 0$ .

And  $-aaa + 123456a - 12272861 = 0$ .

Then,  $a - 283) -aaa + 123456a - 12272861$  ( $-aa$   
 $-aaa + 283aa$

$$\begin{array}{r} -283aa + 123456a \quad (\quad -283a \\ -283aa + 80089a \end{array}$$

$$\begin{array}{r} +43367a - 12272861 \quad (+43367 \\ +43367a - 12272861 \\ (0) \quad (0) \end{array}$$

Hence it appears that  $-aa - 283a + 43367 = 0$   
 Consequently  $aa + 283a = 43367$  this Æquation being Solved,  $a = 110,2722$  &c. which is the lesser Root of the aforesaid Æquation  $ba - aaa = G$  &c.

After this manner all the possible, and impossible Roots of any Æquation may be easily discovered, any one of its Roots being once found. I shall therefore omit inserting more Examples of that Kind.

Suppose  $aaa + baa + ca = G$ . Quere  $a$ .

Let  $b = 74$ ,  $c = 8729$ . and  $G = 560783$

By Trial (as before) it will be found that the next nearest  $r = 40$  being something less than just.

Therefore



Therefore	1	$r + e = a$
1 $\times c$	2	$cr + ce = ca$
1 $\odot 2 : \times b$	3	$brr + 2bre + bee = baa$
1 $\odot 3$	4	$rrr + 3rre + 3ree = aaa$
2, in Numb.	5	$349160 + 8729e$
3, in Numb.	6	$118400 + 5920e + 74ee$
4, in Numb.	7	$64000 + 4800e + 120ee$
5 + 6 + 7	8	$531560 + 19449e + 194ee = 560783$
8 - 531560	9	$19449e + 194ee = 29223$
9 $\div 194$	10	$100,2e + ee = 153,06 = D$
10 $\div$	11	$e = \frac{D}{100,2 + e}$

Operation.	100,2)	153,06	(1,5 = e
	+ e = 1	101,2	First r = 40
			+ e = 1,5
1. Divisor	101,2)	51,86	
	+ e = ,5	50,85	r + e = 41,5 = 1

2. Divisor 101,7      1,01

Or New  $r = 41,5$  for a second Operation, which being duly involved, &c. will be found more than just.

Therefore	1	$r - e = a$
Then	2	$cr - ce = ca$
	3	$brr - 2bre + bee = baa$
	4	$rrr - 3rre + 3ree = aaa$

These being turn'd into Numbers, &c. As above, they will be  $2003775e - 198,5ee = 390,375$  — — — which being Divided by 198,5 the Co-efficient of  $ee$  — — — will become

$100,946e - ee = 1,966624$  &c. = D.

Operation.	100,946	1,966624	(,019 = e
- e =	,01	1,00936	

1. Divisor	100,936)	957264
- e =	,009	908343

2. Divisor 100,927)      \* 489210      (,0004847

\* Here I proceed by plain Division, without forming New Divisors.

403708	,0194847 = e
855020	
807416	

Last r = 41,5	476040
- e = ,0194847	403708

$r - e = 41,4805153 = a$       723320 &c.

Let the last *Æquations* in the *Ænigma*, Chap. 9. be here proposed for a Solution.

$$\text{Viz. } aaaa + baaa - caa - da = G$$

$$b = 2, c = 288. d = 506. \text{ and } G = 1513 \text{ Quere } a.$$

By Tryals it will be found, that the next nearest  $r = 20$  being something more than just.

Therefore	1	$r - e = a$
I $\times d$	2	$dr - de = da$
I $\oplus 2 : \times c$	3	$crr - 2cre + cee = caa$
I $\oplus 3 : \times b$	4	$brrr - 3brre + 3bree = baaa$
I $\oplus 4$	5	$r^4 - 4rrre + 6rree = aaaa$

These being turned into Numbers, and those duly collected, according as the Signs of the *Æquation* direct, they will become

$$50680 - 22374e + 2232ee = 1513. \text{ which being all divided by } 2232. \text{ the Co-efficient of } ee, \text{ will become } 10e - ee = 22 = D.$$

$$\text{Then } \left\{ \frac{D}{10 - e} = e \right.$$

$$\begin{array}{r} \text{Operation, } 10) \quad 22 \quad (3 = e \\ \quad - e = 3 \quad 21 \end{array}$$

$$\begin{array}{r} \text{Divisor } 7) \quad 1 \\ \text{First } r = 20 \\ \quad - e = 3 \end{array}$$

$$r - e = 17 = a \text{ just.}$$

See the End of Chap. 9.

By what hath been already done, about the Solution of these few *Æquations* (being carefully observed,) I presume the Learner will easily conceive how to proceed in the Solution of all Kinds of *Æquations*, be they never so high, or adfected; therefore I shall not here propose many various Examples, but only take them as they fall in Course when I come to the next Part, wherein you will (perhaps) find such *Æquations* with their Solutions as are not common.

CH A P.



C H A P. XI.

Of Simple Interest, Annuities or Pensions, &c.

*Interest* or the Use paid for the *Loan of Money*, is either *Simple* ; Or *Compound*.

Section I. Of Simple Interest.

*Simple Interest*, is that which is paid for the Loan of any Principal or Sum of Money, lent out for some Time. at any Rate *per Cent.* agreed on between the Borrower and the Lender ; which according to the Laws of *England*. ought to be six Pounds for the Use of 100 *l.* for one Year, and twelve Pounds for the Use of 100*l.* for two Years. And so on for a greater, or lesser Sum, proportional to the Time proposed.

There are several Ways of computing (or answering Questions about) *Simple Interest*; as by the Single. and Double Rule of Three (See Page 96. &c.) others make use of Tables composed at several Rates *per Cent.* As Sir Samuel Moreland in his Doctrine of *Interest*. both Simple and Compound, which is all perform'd by Tables; where in he hath detected several Material Errors committed by Doctor *Newton*, Mr. *Kersey* upon *Wingate* and Mr. *Clavil* &c. in the Business of Computing Interest, &c. by their Tables, too tedious to be here repeated.

But I shall in this Tract take other Methods, and shew that all Computations relating to Simple Interest, are grounded upon Arithmetical Progression; and from thence raise such General Theorems, as will suit with all Cases. In order to that,

Let  $\begin{cases} P = \text{Any Principal or Sum put to Interest.} \\ R = \text{The Ratio of the Rate, per Cent. per Annum.} \\ t = \text{The Time of the Principal's Continuance at Interest.} \\ A = \text{The Amount of the Principal, and its Interest.} \end{cases}$

Note; the Ratio of the Rate, is only the Simple Interest of 1*l.* for one Year. at any given Rate; and it is thus found.

Viz.  $100 : 6 :: 1 : 0,06 = \text{the Ratio at 6 per Cent. per Ann.}$

Or  $100 : 7 :: 1 : 0,07 = \text{the Ratio at 7 per Cent. \&c.}$

Again  $100 : 7,5 :: 1 : 0,075 = \text{the Ratio at 7 and } \frac{1}{2} \text{ per Cent.}$

And if the given Time be whole Years; then  $t = \text{the Number of those Years}$ : But if the Time given, be either pure Parts of a Year, or Parts of a Year mix'd with Years; those Parts must be turn'd into Decimals; and then  $t = \text{those Decimals, \&c.}$  Now the Common  
Parts

Parts of a Year may be easily turn'd or converted into Decimal Parts, if it be considered

That One  $\left\{ \begin{array}{l} \text{Day is the } \frac{1}{365} \text{ Parts of a Year} = 0,00274 \text{ fere} \\ \text{Month is the } \frac{1}{12} \text{ Part of a Year} = 0,0833333 \text{ \&c.} \\ \text{Quarter is the } \frac{1}{4} \text{ Part of a Year} = 0,25 \end{array} \right.$

Half a Year = 0,5

And three Quarters = 0,75

These Things being premised, we may proceed to raising the Theorems.

Let  $R$  = the Interest of 1  $\text{\textit{l.}}$  for one Year. As before.

Then  $2R$  = the Interest of 1  $\text{\textit{l.}}$  for two Years.

And  $3R$  = the Interest of 1  $\text{\textit{l.}}$  for three Years.

$4R$  = the Interest of 1  $\text{\textit{l.}}$  for four Years. And so on for any Number of Years proposed.

Hence it is plain, that the Simple Interest of one Pound is a Series of Terms in Arithmetical Progression increasing; whose first Term and common Difference is  $R$ . And the Number of all the Terms is  $t$ . Therefore the last Term will always be  $tR$  = the Interest of 1  $\text{\textit{l.}}$  for any given Term signified by  $t$ .

Then  $\left\{ \begin{array}{l} \text{As one Pound : Is to the Interest of 1  $\text{\textit{l.}}$  :: So is any} \\ \text{Principal or given Sum : To its Interest.} \end{array} \right.$

That is,  $1 \text{\textit{l.}} : tR :: P : tRP$  = the Interest of  $P$ . Then the Principal being added to its Interest, their Sum will be =  $A$  the Amount required : Which gives this general Theorem.

$$\text{Theorem } tRP + P = A$$

From whence the three following Theorems are easily deduced,

$$\text{Theorem 2. } \frac{A}{tR + 1} = P. \quad \text{Theorem 3. } \frac{A - P}{tP} = R$$

$$\text{Theorem 4. } \frac{A - P}{RP} = t.$$

These four Theorems resolve all Questions about Simple Interest.

**Question 1.** What will 256  $\text{\textit{l.}}$  10s. Amount to in 3 Years, 1 Quarter, 2 Months, and 18 Days, at 6 per Cent. per Ann.

Here is given  $P = 256 \text{ \textit{s.}}$   $R = 0,06$ . And  $t = 3,46599$

For 3 Years = 3,

Quere  $A$ . Per Theorem 1.

one Quarter = 0,25

2 Months =  $0,16667 = 0,08333 \times 2$

18 Days =  $0,04932 = 0,00274 \times 18$

Hence  $t = 3,46599 \times 0,06 = 0,2079594 = tR$

Then  $0,2079594 \times 256 \text{\textit{s.}} = 53 \text{ } 34 \text{ } 15 \text{ } 86 = tRP$

And  $53,341586 + 256 \text{\textit{s.}} = 309,841586 = tRP + P = A$

That is, 309  $\text{\textit{l.}}$  16s. 10d. being the Answer required.

**Question**

$$\begin{array}{r} 15.61.0 \\ \times 2.2 \\ \hline 312 \\ 3120 \\ \hline 34420 \end{array} \quad \begin{array}{r} 15.0 - 2 \frac{2}{5} \\ \hline 46.4 - 7 \frac{5}{5} \\ \hline 5 \end{array} \quad \times \sqrt{e} \quad 309.16.10$$



**Question 2.** *What Principal or Sum being put to Interest, will raise a Stock of 309l. 16s. 10d. in three Years, one Quarter, two Months and 18 Days ; at 6 per Cent. per Annum ?*

Or the same Question otherwise stated thus.

*What is 309l. 16s. 10d. due 3 Years, one Quarter, 2 Months and 18 Days hence, worth in ready Money ; Abating or Discounting 6 per Cent. &c.*

Here is given  $A = 309,841586$   $R = 0,06$   $t = 3,46599$  (found as before) Thence to find  $P$ . Per Theorem 2.

$$\text{First } 3,46599 \times 0,06 = 0,2079594 = tR$$

$$\text{Then } tR + 1 = 1,2079594 \quad 309,841586 = A \quad (256,5 = P)$$

That is,  $256,5 = 256 \text{ l. } 10 \text{ s.}$  the Answer required.

**Question 3.** *At what Rate of Interest, per Cent, &c. will 256l. 10s. Amount to 309l. 16s. 10d. In 3 Years, one Quarter, two Months and 18 Days.*

Here is given,  $P = 256,5$ ,  $A = 309,841586$  and  $t = 3,46599$

To find  $R$ . Per Theorem 3.

$$\text{First } 309,841586 - 256,5 = 53,341586 = A - P$$

$$\text{Next } 3,46599 \times 256,5 = 889,026435 = tR$$

$$\text{And } tR = 889,026435 \quad 53,341586 \quad (00,06 = \text{Ratio.})$$

$$\text{Then } 11 : 0,06 :: 100 : 6 = \text{the Rate required.}$$

**Question 4.** *In what Time will 256 l. 10 s. raise a Stock of (or Amount to) 309l. 16s. 10 d. at 6 per Cent. &c.*

Here is given,  $P = 256,5$   $A = 309,841586$  and  $R = 0,06$

To find  $t$  Per Theorem 4.

$$\text{First } 309,841586 - 256,5 = 53,341586 = A - P$$

$$\text{And } 256,5 \times 0,06 = 15,39 = PR$$

$$\text{Then } 15,39 \quad 53,341586 \quad (3,46599 = t)$$

That is  $t = 3$  Years and  $.46599$  Decimal Parts of a Year ; which may be brought into Common Parts of a Year, thus

$$\begin{array}{rcl} 0,46599 & \text{And } 0,08333 & 0,21599 \text{ (2 Months.} \\ -0,25 = \text{one Quarter} & & ,16666 \end{array}$$

$$\begin{array}{rcl} 0,21599 & & 0,02074 \quad ,04933 \text{ (18 Days.} \\ \text{Hence } t = 3 \text{ Years, one Quarter, 2 Months, and 18 Days; the Answer} & & \end{array}$$

required.

It must needs be easy to conceive. that what is here done at 6 per Cent. may be done at any other Rate of Interest, by forming the Ratio, viz.  $R$  accordingly.

Scholium

## Scholium.

Altho' it be according to the Laws and Custom of *England*. to compute Interest at the Proportion of 6 *per Cent.* (as above) yet he that takes up Money at Interest for any Time less than even or complete Years, pays more Interest than seems reasonably due, according to the true Rules of Art.

As for Instance ; If 100*l.* be forborn at Interest one whole Year. it amounts to 106*l.* But (I say) if it be paid at the half Year's End, it should not amount to 103*l.* As appears from this following Proportion.

Let  $a$  = the Amount due at the half Year's End ; Then it will be  $100 : a :: a : 106$  the Amount at the Year's End. *Ergo*  $aa = 10600$  And  $a = \sqrt{10600} = 102,9563 = 102*l.* 19*s.* 1½*d.* which is less than 103*l.* by 10½*d.* And if it be paid in less than half a Year's Time, the Error must needs be the Greater.$

## Section 2. Of Annuities or Pensions in Arrears ; Computed at Simple Interest.

*Annuities* or *Pensions*, &c. are said to be in Arrears, when they are payable or due, either Yearly or Half-yearly, &c. and are unpaid for any Number of Payments. Therefore the Business is, to compute what all those Payments will amount unto, allowing any Rate of Simple Interest for their Forbearance, from the Time each particular Payment became due : Now in order to that,

Put  $\left\{ \begin{array}{l} u = \text{the Annuity, Pension, or yearly Rent, \&c.} \\ t = \text{the Time of its Continuance, or being unpaid.} \\ R = \text{the Ratio, or Interest of 1 *l.* for 1 Year,} \\ A = \text{the Amount of the Annuity and its Interest.} \end{array} \right. \left. \begin{array}{l} \text{As be-} \\ \text{fore.} \end{array} \right\}$

Then if  $u$  = the first Year's Rent, due without Interest.

$Ru$  = the Interest } Due at the End of the second Year.  
And  $2u$  = the Rent.

$2Ru$  = the Interest } Due at the End of the third Year.  
And  $3u$  = the Rent.

$3Ru$  = the Interest } Due at the End of the fourth Year.  
 $4u$  = the Rent.

$4Ru$  = the Interest } Due at the End of the fifth Year.  
 $5u$  = the Rent.

And so on for any Number of Years. Hence it is evident, that  $R + 2Ru + 3Ru + 4Ru + 5u = A$  the Sum of all the Rents and their Interest, being forborn 5 Years.

From



From whence it follows, that  $Ru + 2Ru + 3Ru + 4Ru = A - tu$ .

Here  $t = 5$ . Divide by  $u$ , Then  $R + 2R + 3R + 4R = \frac{A - tu}{u}$

Next to find the Sum of this Progression (See Page 185.) Thus

Let  $R + 2R + 3R + 4R \&c. = z$ . Then  $1 + 2 + 3 + 4 \&c. = \frac{z}{R}$

Here the Sum of the first and last Terms are  $4 + 1 = 5 = t$   
And the Numbers of all the Terms is  $4 = t - 1$ . Therefore

$\frac{t-1}{2} \times t =$  the Sum of all the Terms. That is,  $\frac{tt-t}{2} = \frac{z}{R}$

Hence  $\frac{ttR-tR}{2} = z$ . Consequently  $\frac{ttR-tR}{2} = \frac{A-tu}{u}$

Now from this Equation it will be easy to deduce the following Theorems.

Theorem 1.  $\left\{ \frac{ttRu-tRu+2tu}{2} = A \right.$  Or  $\left\{ \frac{ttu-tu}{2} \times R : + tu = A \right.$

Theorem 2.  $\left\{ \frac{2A}{ttR-tR+2t} = u \right.$  Theorem 3.  $\left\{ \frac{2A-2tu}{ttu-tu} = R : \right.$

Let  $\frac{2}{R} - 1 = x$ . Then  $t = \sqrt{\frac{2A}{Ru} + \frac{xx}{4}} : - \frac{1}{2}x \}$  Theorem 4.

Question 1. If 250l. Yearly Rent (or Pension, &c.) be forborn or unpaid Seven Years; What will it amount to in that Time, at 6 per Cent. for each Payment as it becomes due?

Here is given  $u = 250$ .  $t = 7$ . And  $R = 0,06$  To find  $A$ . Per Theor. 1. First  $250 \times 7 = 1750 = tu$ .  $1750 \times 7 = 12250 = ttu$  Again  $12250 - 1750 = 10500 = ttu - tu$ . And  $\frac{10500}{2} \times 0,06 = 315$  Lastly  $315 + 1750 = 2065 = A$ . Viz. 2065l. the Answer required.

But if the Annuity, Rent or Pension, is to be paid by Quarterly or Half-yearly Payments, &c.

Then  $\frac{0,06}{2} = 0,03 = R$  for Half-yearly Payments.

And  $\frac{0,06}{4} = 0,015 = R$  for Quarterly; Or  $0,045 = R$  for Three-quarterly Payments. Example of Half-yearly Payments.

Suppose 250l. per Annum, to be paid by Half-yearly Payments, were in Arrears or unpaid for seven Years; What would it amount to allowing 6 per Cent. per Annum for each Payment as it became due.

In this Example there is given  $u = 125 = \frac{250}{2}$   $t = 14$  the Number of Payments; And  $R = 0,03 = \frac{0,06}{2}$ . Thence to find  $A$ .

First  $125 \times 14 = 1750 = tu$ .  $1750 \times 14 = 24500 = ttu$ .

Again  $24500 - 1750 = 22750 = ttu - tu$ . Then  $\frac{22750}{2} = 11375$   
K k And

And  $11375 \times 0,03 = 341,25$  Lastly  $341,25 + 1750 = 2091,25$ , that is,  $A = 2091\text{ l. } 5\text{ s.}$  the Answer required.

N. B. Hence it may be observed, that Half-yearly Payments are more advantageous than Yearly:

For  $2091\text{ l. } 5\text{ s.} > 2065\text{ l.}$  by  $26\text{ l. } 5\text{ s.}$  Consequently, Quarterly Payments are more advantageous than Half-yearly Payments.

**Question 2.** *What yearly Rent, Pension, &c. being forborn or unpaid seven Years, will raise a Stock of 2065 l. allowing 6 per Cent. per Annum for each Payment as it becomes due?*

Here is given  $A = 2065$  .  $t = 7$  And  $R = 0,06$ . To find  $u$ . Per Theorem 2.

First  $7 \times 0,06 = 0,42 = tR$ , and  $0,42 \times 7 = 2,94 = ttR$ .

Then  $ttR - tR = 2,52$

Lastly  $ttR - tR + 2t = 16,52$   $4130 = 2A$  ( $250 = u$ ).

That is,  $250\text{ l. per Annum}$ , &c. will raise  $2065\text{ l.}$  the Stock required.

**Question 3.** *In what Time will 250 l. yearly Rent, raise a Stock of 2065 l. allowing 6 per Cent. &c. for the Forbearance of the Payments as they become due?*

Here is given  $u = 250$ .  $A = 2065$  And  $R = 0,06$  To find  $t$ . Per Theorem 4. First

$$\frac{2}{R} = 33,3333 \text{ And } 33,3333 - 1 = 32,3333 = x = \frac{2}{R} - 1$$

Then  $16,16666$  &c.  $= \frac{1}{2} x$ ,  $261,3605$  &c.  $= \frac{1}{4} xx$ .

$$\text{Again } 4\frac{1}{3} = 275,333 = 2A \div Ru \text{ And } 275,3333 + 261,3605 = 536,6938 = \frac{2A}{Ru} + \frac{1}{4} xx. \text{ Then } \sqrt{536,6938} = 23,1666.$$

Lastly  $23,1666 - 16,1666 = 7 = t$  the Time required.

**Question 4.** *If 250 l. yearly Rent, being forborn seven Years, will amount to 2065 l. allowing Simple Interest for every Payment as it becomes due, what must the Rate of the Interest be per Cent. &c.*

Here is given  $u = 250$ .  $A = 2065$ . And  $t = 7$ . To find  $R$ : Per Theorem 3.

$$\text{Thus } \begin{cases} ttu = 12250 . 4130 = 2A \\ tu = 1750 . 3500 = 2tu \end{cases}$$

$$ttu - tu = 10500 \quad 630 = 2A - 2tu \quad (0,06 = R)$$

Then  $1 : 0,06 :: 100 : 6$  the Rate required.

Section



Section 3. The Present Worth of Annuities or Pensions, &c. computed at Simple Interest.

The Business of Purchasing Annuities, or taking of Leases, &c. for any assigned Time, depends upon the true Equating of the Principal or Money laid out on the Purchase, with the Annuity or yearly Rent, by allowing (or discompting) the same Rate of Interest to both Parties. Which may be easily perform'd by duly applying the respective Theorems of the two last Sections together. As will fully appear by the following Question.

Question 1. What is 75*l.* Yearly Rent, to continue nine Years, worth in ready Money, at 6 per Cent. per Annum Simple Interest?

1. Per Theorem 1. of the last Section, find what the proposed Yearly Rent would amount to, if it were forborn 9 Years at 6 per Cent.

$$\begin{array}{ll} \text{Thus } u = 75. \quad t = 9. & \text{And } R = 0.06 \quad \text{Quere } A. \\ tu = 6075 & \text{Then 2) } \left. \begin{array}{l} 5400 \quad (2700 \\ R = 0.06 \end{array} \right\} \text{Multiply} \\ \underline{tu = 675} & \end{array}$$

$$tu - tu = 5400$$

$$\begin{array}{r} 162, \\ + tu = 675. \end{array} \Bigg\} = 837 = A$$

2. Then by Theorem 2. Section 1. find what Principal, being put to Interest for the same Time, and at the same Rate, will amount to 837 *l.* = *A.*

Thus  $tR = 0.54 = 9 \times 0.06$ .  $tR + 1 = 1.54$   $837 (543.5064 = P$   
That is,  $P = 543 \text{ } l. \text{ } 10 \text{ } s. \text{ } 1 \frac{1}{2} \text{ } d.$  Which is the Worth of 75 *l.* a Year, as was required.

From the Work of these two Operations, (duly consider'd) it must needs be easy to conceive, how the two Theorems by which they were perform'd, may be combined into one.

$$\text{For 1. } \frac{tuRu - tRu + 2tu}{2} = A. \quad \text{And 2. } PtR + P = A.$$

Consequently  $PtR + P = \frac{tuRu - tRu + 2tu}{2}$  And from this Equation may be deduced the following Theorems.

$$\text{Theorem 1. } \left\{ \frac{tuRu - tRu + 2tu}{2tR + 2} = P. \text{ Or } \frac{tR - 1 + 2t}{2tR + 2} : u = P \right.$$

By this Theorem all Questions of the same kind with the last (*viz.* that above may be easily and readily answered at one Operation.



Theorem 2.  $\left\{ \frac{2PtR + 2P}{ttR - tR + 2t} = u. \text{ Or } \frac{tR + 1}{ttR - tR + 2t} : \times 2P = u. \right.$

Theorem 3.  $\left\{ \frac{2P - 2tu}{ttu - tu - 2Pt} = R \right.$

Let  $\frac{2}{R} - \frac{2P}{u} - 1 = x$ . Then will  $tt \pm xt = \frac{2P}{Ru}$

Which gives this Theorem 4.  $\left\{ \sqrt{\frac{2P}{Ru} + \frac{xx}{4}} : \pm \frac{x}{2} = t \right.$

By the second and third Theorems, two very useful Questions may be easily answer'd.

1. *As for Instance : If it be required to find what Annuity, or yearly Rent, &c. may be purchased, for any proposed Sum, to continue any assigned Time, allowing any Rate of Interest.*

This Question may be answered by Theorem 2.

2. *Again : If it be required to find how long any yearly Rent, Pension, or Annuity, &c. may be purchased (or enjoy'd) for any proposed Sum, at any given Rate of Interest.*

All Questions of this Kind are easily answered per Theorem 4.

In these Questions it is supposed, that the Purchaser or yearly Rent. is to commence or be immediately enter'd upon. But if it be required to find the Value or Purchase of any Annuity or yearly Rent, &c. in Reversion ; That is, when it is not to be enter'd upon until after some Time, or Number of Years are past ; then you must first find what the Sum propos'd to be laid out in the Purchase, would amount to, if it were put to Interest. during the Time the Annuity, &c. is not to be in present Possession ; and make that Amount the Sum for the Purchase proceeding with it as in either of the two last Questions, &c.

Note, From the first Question of this Section it will be easy to conceive how to perform the Equation of Payments, between Debtor and Creditor, at any Rate of Interest, without doing any Damage to either Party.

That is when several Sums of Money are to be paid, at several different Times, to find the Time when all the Payments may be truly discharged at once : As if one Sum were to be paid at the End of two Months, another at six Months, and perhaps a third Sum at eight Months End. &c. And it were requir'd to find the Time when all those Sums may be truly discharged at one Payment without Loss, &c.

CHAP.



C H A P. XII.

Of Compound Interest, and Annuities, &c.

*Compound Interest* is that which arises from any Principal and its Interest put together, as the Interest still becomes due; so that at every Payment, or at the Time when the Payments became due, there is created a new Principal; And for that Reason it is called Interest upon Interest, or Compound Interest.

As for Instance; Suppose 100*l.* were lent out for two Years; at 6 per Cent. per Annum, Compound Interest: Then, at the End of the first Year, it will only amount to 106*l.* As in Simple Interest. But for the second Year this 106*l.* becomes Principal, which will amount to 112*l.* 7*s.* 2½*d.* at the second Year's End, whereas by Simple Interest it would have amounted to but 112*l.*

And altho' it be not lawful to let out Money at Compound Interest; yet in purchasing of Annuities or Pensions, &c. And taking Leases in Reversion, it is very usual to allow Compound Interest to the Purchaser for his ready Money; and therefore it is very requisite to understand it.

Section 1. Of Compound Interest.

Let  $\left\{ \begin{array}{l} P = \text{the Principal put to Interest.} \\ t = \text{the Time of its Continuance.} \\ A = \text{the Amount of the Principal and Interest.} \\ R = \left\{ \begin{array}{l} \text{the Amount of 1*l.* and its Interest for 1 Year, at any} \\ \text{given Rate, which may be thus found.} \end{array} \right. \end{array} \right. \right\} \text{As before.}$

Viz. 100 : 106 :: 1 : 1,06 = the Amount of 1*l.* at 6 per Cent.  
Or 100 : 107 :: 1 : 1,07 = the Amount of 1*l.* at 7 per Cent.  
and so on for any other assigned Rate of Interest.

Then if  $R =$  the Amount of 1*l.* for one Year, at any Rate,

$RR =$  the Amount of 1*l.* for two Years.

$RRR =$  the Amount of 1*l.* for three Years.

$R^4 =$  the Amount of 1*l.* for four Years.

$R^5 =$  the Amount of 1*l.* for five Years. Here  $t = 5$

For  $1 : R :: R : RR :: RR : RRR :: RRR : R^4 :: R^4 : R^5 : \&c.$  in  $\div$ .

That is  $\left\{ \begin{array}{l} \text{As one Pound : is to the Amount of one Pound at one} \\ \text{Year's End : : So is that Amount : To the Amount of} \\ \text{one Pound at two Year's End, \&c.} \end{array} \right.$

Whence

Whence it is plain, that Compound Interest is grounded upon a Series of Terms increasing in Geometrical Proportion continued; wherein  $t$  (*viz.* the Number of Years) does assign the Index of the last and highest Term.

*Viz.* the Power of  $R$ , which is  $Rt$ .

Again, As  $1 : Rt :: P : PRt = A$  the Amount of  $P$  for the Time, that  $Rt =$  the Amount of 1 l.

That is  $\left\{ \begin{array}{l} \text{As one Pound : is to the Amount of one Pound} \\ \text{for any given Time} :: \text{So is any proposed Prin-} \\ \text{cipal (or Sum) : To its Amount for the same Time.} \end{array} \right.$

From the Premises, (I presume) the Reason of the following Theorems, may be very easily understood.

Theorem 1.  $PRt = A$  As above.

Frome hence the two following Theorems are easily deduced.

Theorem 2.  $\left\{ \frac{A}{Rt} = P \right.$  Theorem 3.  $\left\{ \frac{A}{P} = Rt \right.$

By these three Theorems, all Questions about Compound Interest, may be truly resolved by the Pen only, *viz.* without Tables; Tho' not so readily as by the Help of Tables calculated on Purpose, As will appear farther on.

Question 1. What will 256 l. 10 s. Amount to in seven Years, at 6 per Cent. per Annum. Compound Interest?

Here is given  $P = 256.5$  .  $t = 7$  . and  $R = 1.06$  which being involved until its Index  $= t$  (*viz.* 7) will become  $R^7 = 1.50363$   
Then  $1.50363 \times 256.5 = 385.6811 = A = 385$  l. 13 s. 7½d, which is the Answer required.

Question 2. What Principal or Sum of Money must be put (or let) out to raise a Stock of 385 l. 13 s. 7½d. in seven Years, at 6 per Cent. per Annum Compound Interest?

Here is given  $A = 385.6811$   $R = 1.06$  and  $t = 7$  To find  $P$ . by Theorem 2.

Thus  $Rt = 1.50363$   $385.6811 = A$  ( $256.5 = P$ ).

That is,  $P = 256$  l. 10 s. which is the Principal or Sum as was required.

Question



# Chap. 12. Of Compound Interest. 255

**Question 3.** *In what Time will 256 l. 10 s. raise a Stock of (or amount to) 385 l. 13 s. 7½ d. allowing 6 per Cent. per Annum Compound Interest?*

Here is given  $P = 256,5$   $A = 385,6811$   $R = 1,06$  To find  $t$  by the third Theorem  $\left\{ Rt = \frac{A}{P} \frac{385,6811}{256,5} = 1,50363, \right.$

which being continually divided by  $R = 1,06$  until nothing remain, the Number of those Divisions will be  $= 7 = t$ .

Thus  $1,06) 1,50363$  ( $1,41852$  And  $1,06) 1,41852$  ( $1,338225$  Again  $1,06) 1,338225$  ( $1,262477$ . And so on until it become  $1,06) 1,06$  ( $1$ . which will be at the seventh Division. Therefore it will be  $t = 7$  the Number of Years required by the Question.

**Question 4.** *If 256 l. 10 s. will amount to (or raise a Stock of) 385 l. 13 s. 7½ d. in seven Years Time; What must the Rate of Interest be, per Cent. per Annum.*

Here is given  $P = 256,5$ ,  $A = 385,6811$  and  $t = 7$  Quere  $R$ .

By Theorem 3.  $\left\{ \frac{A}{P} = Rt = 1,50363. \right.$  As before in the last Question

And if  $Rt = R^7 = 1,50363$  Then  $R = \sqrt[7]{1,50363}$  which may be thus extracted.

Put	1	$r + e = R$	Then
1 $\odot$ 7	2.	$r^7 + 7r^6e + 21r^5ee = R^7 = 1,50363 = G$	
2 $- r^7$	3	$7r^6e + 21r^5ee = G - r^7$	
3 $\div 7r^5$	4	$re + 3ee = \frac{G - r^7}{7r^5} = D$	
4 $\div$	5	$e = \frac{D}{r + 3e}$	Let $r = 1$ Then $D = 0,0719$

Operation  $r = 1,00$   $0,0719$  ( $0,06 = e$   
 $+ 3e = ,18$   $708$

Divisor  $1,18$  (11) to be rejected.

First  $r = 1,00$   
 $+ e = 0,06$  }  $= 1,06 = R$

Then  $1 : 0,06 :: 100 : 6$  The Rate per Cent. required.

The first three Questions may be much more easily perform'd by the following Table, which is only the Amounts of one Pound for Thirty Nine Years.

That

That is, of  $R$ .  $RR$ .  $RRR$ .  $R^4$ .  $R^5$ . and so on to  $R^3$ .

Years    1	The Amounts of 1 l. at 6 per Cent. &c. Com- pound Interest.	Years    1	The Amounts of 1 l. at 6 per Cent. &c. Com- pound Interest.	Years    1	The Amounts of 1 l. at 6 per Cent. &c. Com- pound Interest.
1	1,06 = $R$	14	2,2609039557	27	4,8223459407
2	1,1236 = $RR$	15	2,3965581931	28	5,1116866971
3	1,191016 = $R^3$	16	2,5403516847	29	5,4183878990
4	1,26247696	17	2,6927727857	30	5,7434911729
5	1,3382255776	18	2,8543391529	31	6,0881006432
6	1,4185191122	19	3,0255995021	32	6,4533866818
7	1,5036302590	20	3,2071354722	33	6,8405898828
8	1,5938480745	21	3,3995636005	34	7,2510252757
9	1,6894789590	22	3,6035374166	35	7,6860867923
10	1,7908476965	23	3,8197496616	36	8,1472519998
11	1,8982985583	24	4,0489346413	37	8,6360871198
12	2,0121964718	25	4,2918707197	38	9,1542523470
13	2,1329282601	26	4,5493829629	39	9,7035074878

The Title of this Table shews its Construction, and its Use will easily appear by an Example or two.

*Example 1. What will 375 l. 10s. amount to in nine Years at 6 per Cent. per Annum, &c.*

The Tabular Number against 9 Years is 1,689479 which being multiplied with the Principal 375,5 will produce 634,3993 &c. viz. 634 l. 8s. fere, being the Amount or Answer required.

*Example 2. What Principal (or Sum) must be put to Interest to raise a Stock of 634 l. 8s. in nine Years Time, at 6 per Cent. per Annum, &c.*

If the proposed Stock. (viz. 634.4) be divided by the Tabular Number that is against the given Number of Years, (viz. 9.) the Quotient will be the Principal (or Sum) requir'd. viz. against 9 is 1.689479)

Then 1.689479) 634,4 (375,5 = 375 l. 10s. the Principal (or Sum) as was required.

*Example 3. In what Time will 375 l. 10s. raise a Stock of (or amount to) 634 l. 8s. at 6 per Cent. &c.*

Divide



Divide the proposed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5) and the Quotient will shew the Tabular Number that stands over-against the Time sought.

Thus 375,5) 634,4 (1,689479 &c. this Number being sought in the Table, will be found to stand against 9 Years, which is the Time required.

But if the Quotient cannot be truly found in the Table of Amounts for Years, as above; Then take out of that Table the nearest Number that is less, and make it a Divisor, by which you must divide the first Quotient; And then seek the second Quotient in the Table of Amounts for Days, (which is inserted a little further on) and it will assign the Number of Days. As in this

### Example.

*In what Time will 563 l. Amount to 860 l. at 6 per Cent. per Annum, Compound Interest?*

*Answer.* In 7 Years and 99 Days.

Thus 563) 860 (1,52753 which shews the Time to be more (or above) seven Years; For over-against 7 Years is 1,50363 which being made the new Divisor:

*Viz.* 1,50363) 1,52753 (1,01589, &c. this Number is the nearest Amount to 99 Days.

*Note, If the Stock, Principal, and Time be given; the Rate of Interest will be best found by extracting the Root, &c. As before in the fourth Question.*

The next Thing that I shall here propose, is to make this Table (which is only calculated for the Rate of 6 per Cent.) universally useful for all Rates of Compound Interest. which I may presume to say, is a new Improvement of my own. being well satisfied it never was published before; and not only so. but I have heard several very good Artists affirm it was impossible to be done.

The Method of performing it is briefly thus, Let  $x =$  the Difference between  $1,06 = R$  the Amount of 1 l. for one Year (in the Table) and any other proposed Amount of 1 l. for one Year; which admits of two Cases.

*Case 1.* If the proposed Rate be Greater than  $1,06 = R$ , then will  $R + x =$  the true Amount of 1 l. for one Year at that Rate.

*Case 2.* But if the proposed Rate be Less than  $1,06 = R$ , then it will be  $R - x =$  the Amount of 1 *l.* &c.

Make  $\begin{cases} t-1=b. t-2=c. t-3=d. t-4=f \text{ \&c.} \\ \frac{1}{2}tb=g. \frac{1}{3}cg=m. \frac{1}{4}dm=n. \frac{1}{5}fn=s \text{ \&c.} \end{cases}$

Then will  $R^t + tR^bx + gR^cxx + mR^dxxx \text{ \&c.} =$  the Amount of 1 *l.* at the given Rate, for any Time denoted by  $t$ . In *Case 1.*

And  $R^t - tR^bx + gR^cxx - mR^dxxx \text{ \&c.} =$  the Amount of 1 *l.* in *Case 2.*

Which is no more but this, Let  $R + x$  Or  $R - x$  (which soever it is) be involved (as directed in *Seçt. 5. Chap. 2.*) to the same Power or Height as the Index  $t$  the given Time in the Question denotes: rejecting all the Powers of  $x$  above  $xxx$  Or  $xxxx$  at most, as useless.

Then multiply that Power of  $R + x$  Or  $R - x$  into the given Principal, and their Product will be the Amount required.

An Example or two in each Case will render all easy.

*Example 1.* Suppose it were required to find what 256 *l.* would amount to in fifteen Years, at 8 *l.* per Cent. per Annum, Compound Interest? Here  $t = 15$ .

First  $100 : 108 :: 1 : 1,08$  the Amount of 1 *l.* at 8 per Cent.

Next  $1,08 - 1,06 = 0,02 = x$ . And  $R + x = 1,08$  As in *Case 1.*

Then  $R^{15} + 15R^{14}x + 105R^{13}xx + 455R^{12}xxx \text{ \&c.} =$  the Amount of 1 *l.* for 15 Years, at 8 per Cent.

Here  $x = 0,02$ .  $xx = 0,0004$ . and  $xxx = 0,000008$

By the Table  $R^{15} = 2,396558$

And  $\begin{cases} 15R^{14}x = 2,260904 \times 15 \times 0,02 & = 0,678271 \\ 105R^{13}xx = 2,132928 \times 105 \times 0,0004 & = 0,089583 \\ 455R^{12}xxx = 2,012196 \times 455 \times 0,000008 & = 0,007324 \end{cases}$

Sum  $= 3,171744$

Then  $3,171744 \times 256 = 811,966464 = A$   
That is, 811 *l.* 19 *s.* 4 *d.* fere. Which is the Answer as was required.

*Example 2.* What will 365 *l.* amount to in seven Years, at Four and a half per Cent. &c.

First  $100 : 1,045 :: 1 : 1,045$  the Amount of 1 *l.* at  $4\frac{1}{2}$  *l.* per Cent.

Next



# Chap. 12. Of Compound Interest. 259

Next  $1,06 - 1,045 = 0,015 = x$ . Consequently  $R - x = 1,045$  as in Case 2.

Then  $R^7 - 7R^6x + 21R^5xx - 35R^4xxx \&c. =$  the Amount of 1*l.* for 7 Years, at  $4\frac{1}{2}$  per Cent.

Here  $x = ,015$ .  $xx = ,000225$ . and  $xxx = ,000003375$

By the Table  $R^7 = + 1,503630$

And  $\begin{cases} - 7R^6x = - 0,148944 \\ + 21R^5xx = + 0,006323 \\ - 35R^4xxx = - 0,000141 \end{cases}$

$R^7 - 7R^6x + 21R^5xx - 35R^4xxx = 1,360868$

Then  $1,360868 \times 365 = 496,71682 = A$ .

That is 496 *l.* 14*s.*  $3\frac{3}{4}$ *d.* is the Answer required.

If the Reason of these two Operations be but well understood, it will be very easy to conceive how to find *P*, the Principal, by having *A*, *t*, and *x* given (because *R* and its Powers are always given by the Table.)

For  $R^t + tR^bx + gR^cxx + mR^dxxx \times P = A$  (as above.)

Therefore  $\frac{A}{R^t + tR^bx + gR^cxx + mR^dxxx} = P$

Or if *A*, *P*, and *t* be given, *x* may be found.

For  $R^t + tR^bx + gR^cxx + mR^dxxx = \frac{A}{P}$  This Equation being solved, (as in Chap: 10.) the Value of *x* will be found; and then either  $R + x$ , Or  $R - x$  will shew the Rate of Interest, &c.

But I shall leave the Numerical Operations to the Learner's Practice, supposing enough done to shew how all Questions of this kind that are limited by whole Years, may be computed.

And if the Time given or sought be not terminated by whole Years, but by Weeks, Months, Quarters or Half-Years, &c. for resolving such Questions, the best Way will be to reduce those Parts of a Year into Days; that done, find an answer according to the Demand of the Question (and agreeing to 1*l.* as before) for those Number of Days; and in order to that, it will be requisite to find the Amount of 1*l.* for one Day, (as in my Compendium of Algebra, Page 110) which I shall here insert.

Put *a* = the Amount sought, then it will be

$1 : a :: a : aa :: aa : aaa :: aaa : aaaa :: aaaa : a^365$

That is  $\left\{ \begin{array}{l} \text{As one Pound : Is to its Amount for one Day} \\ \text{: : So is that Amount : To the Amount of two} \\ \text{Days : : And so is that of two Days : To that of} \\ \text{three Days. And so on in \dots to 365 Days.} \end{array} \right.$

Then the last of the Terms will be  $a^{365} = 1,06$

Put	1	$r + e = a$ . And let $r = 1$
1 $\odot$ 365	2	$r^{365} + 365 r^{364} e + 66430 r^{363} ee = a^{365} = 1,06$
2, in Numb.	3	$1 + 365e + 66430ee = 1,06$
3 — 1	4	$365e + 66430ee = 0,06$
4 $\div$ 66430	5	$,00549e + ee = 0,0000009032 = D$
5 $\div$	6	$e = \frac{D}{,00549 + e}$

Operation  $,00549) 0,0000009032 \quad (,00016 = e$   
 $+ e = ,00016) \quad \quad \quad 55$

1. Divisor .0055	3532	First $r = 1$
2. Divisor ,00565	3390	$+ e = 0,00016$
		$r + e = 1,00016$

New  $r = 1,00016$  for a second Operation. Then

2, in Numb.	7	$1,06013401407 + 386,887e + 70402,172ee = 1,06$
		Hence it appears that $r - e = a$
Therefore	8	$1,06013401407 - 386,887e + 70402,172ee =$
		$1,06$
8 $\pm$	9	$386,887e - 70402,172ee = 0,00013401407$
9 $\div$	10	$,0054953e - ee = ,0000000019035503$
10 $\div$	11	$e = \frac{,0000000019035503}{,0054953 - e}$

Operation  $,0054953) ,0000000019035503 \quad (,00000003 = e$   
 $- e = ,00000003 \quad \quad \quad 164850$

Divisor ,0054950	255050	$(,0000000464$
	219800	

Last $r = 1,00016$	352503
$- e = 0,0000003464$	329700

$r - e = a = 1,0001596536$	228030
	219800

Which being farther pursued to a third Cp ration; it will be  $a = 1,000159653587453$  &c.

This



# Chap. 12. Of Compound Interest. 261

This Value of  $a$  is the Amount of 1*l.* for one Day from which, if 1*l.* be subtracted, the Remainder = ,000159653587 &c. will be the Interest of 1*l.* for one Day. Consequently, if any proposed Principal be multiplied into either of these, the respective Product will be the Amount or Interest of that Principal for one Day, at 6 per Cent. &c.

And that the Amount (or Interest) of any Principal or Sum may be easily computed for any Number of Days less than a Year; I have here inserted the following Table, which with a great deal of Care (and I believe exactness) is calculated from the last found (1,000159653587453) Amount of 1*l.* for one Day. To which also is annexed a Table of the Amounts of 1*l.* for Months.

Days	Amounts of 1 <i>l.</i> &c.	Days	Amounts of 1 <i>l.</i> &c.	Days	Amounts of 1 <i>l.</i> &c.
1	1,0001596536	26	1,0041592879	51	1,0081749166
2	1,0003193326	27	1,0043196055	52	1,0083358753
3	1,0004790372	28	1,0044799487	53	1,0084968597
4	1,0006387673	29	1,0046403175	54	1,0086578699
5	1,0007985229	30	1,0048007120	55	1,0088189057
6	1,0009583039	31	1,0049611320	56	1,0089799673
7	1,0011181105	32	1,0051215776	57	1,0091410545
8	1,0012779426	33	1,0052820488	58	1,0093021675
9	1,0014378002	34	1,0054425457	59	1,0094633062
10	1,0015976834	35	1,0056030682	60	1,0096244707
11	1,0017575920	36	1,0057636164	61	1,0097856608
12	1,0019175262	37	1,0059241901	62	1,0099468767
13	1,0020774859	38	1,0060847895	63	1,0101081184
14	1,0022374712	39	1,0062454146	64	1,0102693858
15	1,0023974820	40	1,0064060653	65	1,0104306789
16	1,0025575184	41	1,0065667416	66	1,0105919978
17	1,0027175803	42	1,0067284436	67	1,0107533424
18	1,0028776677	43	1,0068881712	68	1,0109147128
19	1,0030377808	44	1,0070489245	69	1,0110761090
20	1,0031979193	45	1,0072097035	70	1,0112375309
21	1,0033580850	46	1,0073705082	71	1,0113989786
22	1,0035182732	47	1,0075313385	72	1,0115604521
23	1,0036784885	48	1,0076921945	73	1,0117219513
24	1,0038387294	49	1,0078530762	74	1,0118834764
25	1,0039989958	50	1,0080139835	75	1,0120450272

Days



Days	Amounts of I. &c.	Days	Amounts of I. &c.	Days	Amounts of I. &c.
76	I,0I22066038	116	I,0I86908655	156	I,0252166658
77	I,0I23682062	117	I,0I88535031	157	I,0253803453
78	I,0I25298344	118	I,0I90161667	158	I,0255440509
79	I,0I26914885	119	I,0I91788563	159	I,0257077827
80	I,0I28531683	120	I,0I93415719	160	I,0258715406
81	I,0I30148739	121	I,0I95043134	161	I,0260353247
82	I,0I31766054	122	I,0I96670809	162	I,0261991349
83	I,0I33383627	123	I,0I98298745	163	I,0263629713
84	I,0I35001458	124	I,0I99926934	164	I,0265268338
85	I,0I36619547	125	I,0201555389	165	I,0266907225
86	I,0I38237895	126	I,0203184110	166	I,0268546374
87	I,0I39856501	127	I,0204813084	167	I,0270185784
88	I,0I41475365	128	I,0206442319	168	I,0271825456
89	I,0I43094488	129	I,0208071814	169	I,0273465389
90	I,0I44713869	130	I,0209701569	170	I,0275105585
91	I,0I46333511	131	I,0211331585	171	I,0276746046
92	I,0I47953408	132	I,0212961861	172	I,0278386764
93	I,0I49573565	133	I,0214592397	173	I,0280027746
94	I,0I51193981	134	I,0216223193	174	I,0281668989
95	I,0I52814655	135	I,0217854250	175	I,0283310494
96	I,0I54435589	136	I,0219485567	176	I,0284952262
97	I,0I56056781	137	I,0221117144	177	I,0286594291
98	I,0I57678232	138	I,0222748982	178	I,0288236583
99	I,0I59299941	139	I,0224381081	179	I,0289879137
100	I,0I60921910	140	I,0226013440	180	I,0291521953
101	I,0I62544138	141	I,0227646060	181	I,0293165031
102	I,0I64166624	142	I,0229278940	182	I,0294808372
103	I,0I65789370	143	I,0230912081	183	I,0296451975
104	I,0I67412375	144	I,0232545483	184	I,0298095841
105	I,0I69035638	145	I,0234179146	185	I,0299739969
106	I,0I70659161	146	I,0235813069	186	I,0301384359
107	I,0I72282944	147	I,0237447253	187	I,0303029012
108	I,0I73906985	148	I,0239081699	188	I,0304673928
109	I,0I75513286	149	I,0240716405	189	I,0306319206
110	I,0I77155846	150	I,0242351372	190	I,0307964557
111	I,0I78780665	151	I,0243986600	191	I,0309610250
112	I,0I80405744	152	I,0245622089	192	I,0311256216
113	I,0I82031083	153	I,0247257839	193	I,0312902445
114	I,0I83656680	154	I,0248893851	194	I,0314548937
115	I,0I85282578	155	I,0250530124	195	I,0316195692

Days



Days	Amounts of l. &c.	Days	Amounts of l. &c.	Days	Amounts of l. &c.
196	1,0317842709	236	1,0383939484	276	1,0450459680
197	1,0319489990	237	1,0385597318	277	1,0452128133
198	1,0321137534	238	1,0387255415	278	1,0453796853
199	1,0322785341	239	1,0388913778	279	1,0455446584
200	1,0324433410	240	1,0390572405	280	1,0457135092
201	1,0326081742	241	1,0392231298	281	1,0458804611
202	1,0327730339	242	1,0393890454	282	1,0460474397
203	1,0329379198	243	1,0395549876	283	1,0462144449
204	1,0331028321	244	1,0397209563	284	1,0463814768
205	1,0332677706	245	1,0398869515	285	1,0465485353
206	1,0334327355	246	1,0400529732	286	1,0467156206
207	1,0335977268	247	1,0402190214	287	1,0468827325
208	1,0337627444	248	1,0403850961	288	1,0470498711
209	1,0339277883	249	1,0405511973	289	1,0472170363
210	1,0340928586	250	1,0407173250	290	1,0473842283
211	1,0342579552	251	1,0408834793	291	1,0475514469
212	1,0344230782	252	1,0410496601	292	1,0477186923
213	1,0345882275	253	1,0412158674	293	1,0478859643
214	1,0347534033	254	1,0413821012	294	1,0480532631
215	1,0349186054	255	1,0415483616	295	1,0482205885
216	1,0350838338	256	1,0417146485	296	1,0483879407
217	1,0352490887	257	1,0418809620	297	1,0485553196
218	1,0354143699	258	1,0420473021	298	1,0487227252
219	1,0355796775	259	1,0422136687	299	1,0488901576
220	1,0357450115	260	1,0423800618	300	1,0490576166
221	1,0359103719	261	1,0425464815	301	1,0492251025
222	1,0360757587	262	1,0427129278	302	1,0493926150
223	1,0362411719	263	1,0428794007	303	1,0495601543
224	1,0364066116	264	1,0430459001	304	1,0497277204
225	1,0365720776	265	1,0432124261	305	1,0498953132
226	1,0367375701	266	1,0433789787	306	1,0500629327
227	1,0369030889	267	1,0435455579	307	1,0502305790
228	1,0370686342	268	1,0437121637	308	1,0503982521
229	1,0372342059	269	1,0438787961	309	1,0505659519
230	1,0373998041	270	1,0440454551	310	1,0507336786
231	1,0375654287	271	1,0442121407	311	1,0509014320
232	1,0377310798	272	1,0443788529	312	1,0510692121
233	1,0378967573	273	1,0445455918	313	1,0512370191
234	1,0380624612	274	1,0447123572	314	1,0514048529
235	1,0382281916	275	1,0448791493	315	1,0515727134

Days



Days	Amounts of I l. &c.	Days	Amounts of I l. &c.	Days	Amounts of I l. &c.
316	I,0517406008	339	I,0556094165	362	I,0594924636
317	I,0519085150	340	I,0557779484	363	I,0596616154
318	I,0520764559	341	I,0559465071	364	I,0598307942
319	I,0522444237	342	I,0561150927	365	I,06
320	I,0524124183	343	I,0562837053		
321	I,0525804397	344	I,0564523448		
322	I,0527484880	345	I,0566210112		
323	I,0529165631	346	I,0567897045	Months	The Amounts of I l. at 6 per Cent. For Months.
324	I,0530846650	347	I,0569584248		
325	I,0532527937	348	I,0571271720		
326	I,0534209493	349	I,0572959594		
327	I,0535891317	350	I,0574647472		
328	I,0537573410	351	I,0576335753		
329	I,0539255771	352	I,0578024303		
330	I,0540938401	353	I,0579713122		
331	I,0542621300	354	I,0581402211		
332	I,0544304467	355	I,0583091570		
333	I,0545987903	356	I,0584781199		
334	I,0547671608	357	I,0586471097		
335	I,0549355582	358	I,0588161265		
336	I,0551039824	359	I,0589851703		
337	I,0552724336	360	I,0591542411		
338	I,0554409116	361	I,0593233389		
				1	I,0048675505
				2	I,0097587942
				3	I,0146738462
				4	I,0196128224
				5	I,0245758394
				6	I,0295630141
				7	I,0345744641
				8	I,0396103076
				9	I,0446706634
				10	I,0497556507
				11	I,0548653894
				12	I,06

The Use of this Table is in all Respects like that of whole Years, in finding the Amount of any given Sum for any proposed Number of Days less than a Year.

*Example 1. Suppose it were required to find the Amount of 375 l. for 210 Days at 6 per Cent.*

The Amount of 1 l. for 210 Days is 1,0340928 &c. per Table.

Then  $1,0340928 \times 375 = 387,7848 \text{ &c.} = 387 \text{ l. } 15 \text{ s. } 8\frac{1}{4} \text{ d.}$  which is the Amount required.

And the rest of the Variations may be perform'd just as in the Examples of whole Years.

But if the Time given consists of Years, and Parts of a Year, As Quarters, Months, &c. Then reduce the odd Time or Parts of the Year into Days; and the Answer may then be found at two Operations; As in the following Example.

*Example*



## Chap. 12. Of Compound Interest. 265

**Example 2.** Suppose it were required to find what 265 l. would amount to in five Years and 135 Days, at 6 per Cent. &c.

First the Amount of 1 l. for  $\begin{cases} 5 \text{ Years is } 1,338225 \text{ } \text{\textit{£}}c. \\ 135 \text{ Days is } 1,021785 \text{ } \text{\textit{£}}c. \end{cases}$

Then  $1,338225 \times 1,021785 \times 265 \text{ l.} = 362,355232 \text{ } \text{\textit{£}}c.$  being the Amount or Answer required.

Or, if the Amount and Time are given, to find the Principal; then multiply the Amount of 1 l. for the Years, and the Amount of 1 l. for the odd Days together; And by their Product divide the given Amount, the Quotient will be the Principal required.

**Example 3.** What Principal will raise a Stock of 362 l. 7 s. 1½ d. Or 362,355232 l. in 5 Years and 135 Days, at 6 per Cent. &c.

The Amount of 1 l. for  $\begin{cases} 5 \text{ Years is } 1,338225 \text{ } \text{\textit{£}}c. \\ 135 \text{ Days is } 1,021785 \text{ } \text{\textit{£}}c. \end{cases}$

Then  $1,338225 \times 1,021785 = 1,367378 \text{ } \text{\textit{£}}c.$  the Divisor.

Next  $1,367378 \mid 362,355232 = A$  (265 l. the Principal required,

Again, if the Principal and its Amount are given, to find the Time, at 6 per Cent. &c. you must divide the Amount by its Principal, and then proceed as in the third Example, Page 256, for the Answer required.

But if the Amount and its Principal, with the Time of its being at Interest are given, to find the Rate of Interest; then proceed as in the Fourth Question, Page 255 &c.

Now in order to make this Table of Amounts for Days, useful for all Rates of Interest (as before in that for Years) you must first find the Simple Interest of 1 l. for one Day, both at the given Rate, and also at 6 per Cent. And call their Difference  $x$ .

Thus, Suppose the given Rate were. 8 per Cent. per Annum. First  $100 : 8 :: 1 : 0,08$  And  $100 : 6 :: 1 : 0,06$  the two Simple Interests for one Year.

Then  $365 \mid 0,08$  (0,00021917 &c. the Simple Interest of 1 l. for one Day, at 8 per Cent.

And  $365 \mid 0,06$  (0,00016438 &c. the Simple Interest of 1 l. for one Day, at 6 per Cent.

Their Difference  $0,00005479 = x$  which may do indifferently well for ordinary small Questions; but where Exactness is required, it will be convenient to make Use of this Proportion.

M m

Viz:

*Viz.* { As the Simple Interest of 1 *l.* for one Day, at 6 per Cent.:  
is to the Tabular Interest of 1 *l.* for one Day :: so is the  
Simple Interest of 1 *l.* for one Day, at any given Rate : to  
a fourth Number.

That is, 0,00016438 : 0,00015965 :: 0,00021917 : 0,00021286

Then 0,00021286 — 0,00015965 = 0,00005321 =  $x$ .

This  $x$  being involved with the respective Amounts for Days, in the same Manner as was done with those for Years (*vide* Page 258) the Result will be the Answer to the Question.

## Sect. 2. Annuities or Pensions in Arrear Computed at Compound Interest.

When Annuities, &c. are said to be in Arrear, see Page 248. And I shall here make use of the same Letters to represent the same things as before in that Page, save only that  $R$  is here equal to the Amount of 1 *l.* as in Section I. of this Chap.

Suppose  $u$  = the first Year's Rent of any Annuity without Interest.

Then will  $Ru + u$  = { the Amount of the first Year's Rent, and its Interest; More the 2d Year's Rent.

And  $RRu + Ru + u$  = { the Amount of the 1st and 2d Years Rents, with their Interest; More the 3d Year's Rent, &c.

Here  $RRu + Ru + u = A$  the Amount of any yearly Rent or Annuity, being forborn three Years. And from hence may be deduced these Proportions.

*Viz.*  $u : Ru :: Ru : RRu :: RRu : RRRu$  and so on in  $\div$  for any Number of Terms or Years denoted by  $t$ , wherein the last Term will always be  $uR^t - 1$

Consequently  $A - uR^t - 1$  = the Sum of all the Antecedents.

And  $A - u$  = the Sum of all the Consequents in the Series.

And therefore it would be  $u : uR :: A - uR^t - 1 : A - u$  *Vide* Page 188.

*Ergo*  $Au - uu = RuA - uuR^t$  which being divided all by  $u$ , will become  $A - u = RA - uR^t$ .

From this last Equation it will be easy to raise the following Theorems.

Theorem 1.  $\frac{uR^t - u}{R - 1} = A$ . Theorem 2.  $\frac{RA - A}{R^t - 1} = u$ .

Theorem



**Theorem 3.**  $\left\{ \frac{RA + u - A}{u} = R^t \right.$  If this Equation be continually divided by  $R$ , until nothing remain, the Number of those Divisions will be  $t$ . See Page 255.

4. Again  $\left\{ \frac{A}{u} R - R^t = \frac{A - u}{u} \right.$  If this Equation be resolved into Numbers, according to the Method proposed in Sect. 3. Chap. 10. the Root will shew the Value of  $R$ .

**Question 1.** If 30 l. yearly Rent, or Annuity, &c. be forborn (viz. remain unpaid) nine Years; what will it amount to, at 6 per Cent. per Annum, Compound Interest?

Here is given  $u = 30$ .  $t = 9$  And  $R = 1.06$  To find  $A$ . Per Theorem 1.  
 $R^9 = 1.689479$  By the Table of Amounts for Years.

$$30 = u$$

---


$$R^9 u = 50.684370$$

$$- u = 30,$$


---

$R - 1 = 0.06$   $20.684370$  ( $344.7395 = 344$  l.  $14$  s.  $9\frac{1}{2}$  d.  $= A$  the Amount required.

**Question 2.** What yearly Rent, or Annuity, &c. being forborn or unpaid nine Years, will raise a Stock of 344 l.  $14$  s.  $9\frac{1}{2}$  d.  $= 344,7395$  at 6 per Cent. &c.

Here is given  $A = 344,7395$   $t = 9$ . And  $R = 1.06$  To find  $u$ . Per Theorem 2.

$$AR = 344,7395 \times 1.06 = 365,42387$$

$$- A = 344,7395$$


---

$$R^t - 1 = 1.689479 - 1 = 0.689479$$

$$20.68437 (30 = u$$

**Question 3.** In what Time will 30 l. yearly Rent raise a Stock or Amount to 344 l.  $14$  s.  $9\frac{1}{2}$  d. allowing 6 per Cent. for the Forbearance of the Payments.

Here is given  $u = 30$   $A = 344.7395$  And  $R = 1.06$  to find  $t$ . Per Theorem 3.

$$\text{First } AR + u - A = 365,2387 + 30 - 344,7395 = 50.68437$$

$$\text{And } u = 30) 50.68437 (1.689479 = R^t \text{ Then}$$

$R = 1.06$   $1.689479$  ( $1.593848$  And  $1.06$ )  $1.593848$  ( $1.50363$  and so on until it become  $1.06$ )  $1.06$  (1. which will be at the ninth Division; therefore  $t = 9$ .

Or  $R^t = 1.689479$  being sought in the Table of Amounts for Years, will be found to stand over-against 9 Years, which is the Time required.

**Question 4.** *If 30l. per Annum, being unpaid nine Years, will amount to 344l. 14s. 9 $\frac{1}{2}$ d. allowing Compound Interest for every Payment as it becomes due, what must the Rate of Interest be per Cent. &c.*

Here is given  $u = 30$   $A = 344,7395$  And  $t = 9$  To find  $R$  by the last of the four  $\text{\AA}equations$ , *Viz.*  $\left\{ \frac{A}{u} R - R^t = \frac{A - u}{u} \right.$

$$\text{First } \frac{A}{u} = \frac{344,7395}{30} = 11,491317 \text{ And } \frac{A - u}{u} = 10,491317$$

Hence there is the  $\text{\AA}equation$   $11,491317 R - R^9 = 10,491317$

Let	1	$r + e = R$ And suppose $r = 1$
1 $\odot$ 9	2	$r^9 + 9r^8e + 36r^7ee = R^9$
1, in Numb.	3	$11,491317 + 11,491317e = 11,491317R$
2, in Numb.	4	$1,000000 + 9,000000e + 36ee = R^9$
3 — 4	5	$10,491317 + 2,491317e - 36ee = 10,491317$
6 $\pm$	6	$36ee = 2,491317e$
6 $\div$ 36e	7	$e = 0,06 \text{ \&c.}$

$$\text{First } r = 1 \left. \begin{array}{l} + e = 0,06 \end{array} \right\} = 1,06 = R \left\{ \begin{array}{l} \text{As may be easily try'd, by involving} \\ \text{it and ordering it, as the } \text{\AA}equation \\ \text{above directs.} \end{array} \right.$$

**Sect. 3.** *To find the present Worth of Annuities, Pensions, or Leases, &c. at Compound Interest.*

Let  $P$  = the present Worth of any Annuity, or Lease, &c. and the Rest of the Letters as before.

Then, from what has been said in *Section 3. Chap. II.* about purchasing of Annuities, &c. at Simple Interest, it will be easy to form the like *Theorems* here at Compound Interest, *viz.* by combining *Theorem I. Page 266.* And *Theorem I. Page 254.* into one *Theorem*.

$$\text{For } \left\{ \frac{u R^t - u}{R - 1} = A \right\} \left\{ \begin{array}{l} \text{The Amount of any yearly Rent being un-} \\ \text{paid any Number of Years. Per Theorem I. of} \\ \text{the last Section.} \end{array} \right.$$

$$\text{And } PR^t = A \left\{ \begin{array}{l} \text{The Amount of any Principal or Sum being put} \\ \text{to Interest, for the same Number of Years. Per} \\ \text{Theorem I. Page 254.} \end{array} \right.$$

Hence



## Chap. 12. Of Compound Interest. 269

Hence it follows, That  $PR^t = \frac{uR^t - u}{R - 1}$ .

Viz.  $PR^t + 1 - PR^t = uR^t - u$  being the very same Equation with that in my Compendium of *Algebra*, Page 112, which is there raised from the Consideration of purchasing Annuities, or taking of Leases, &c. to be grounded upon a Rank or Series of Geometrical Proportionals continually decreasing. Thus  $\frac{u}{R}$  is the first and greatest Term;  $R$  the common Ratio of all the Terms; and  $P$  is the Sum of all the Series.

That is,  $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$  &c. in  $\infty$

until the last Term  $= \frac{u}{R^t}$  Then will  $P - \frac{u}{R^t}$  be the Sum of all the

Antecedents. And  $P - \frac{u}{R}$  will be the Sum of all the Consequents. Therefore it will be

$\frac{u}{R} : \frac{u}{RR}$ . Or (in the same Ratio)  $u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R}$

which produces  $PR^t + 1 - uR^t = PR^t - u$ . As above.

From this Equation may be deduced these following Theorems.

Theorem 1.  $\left\{ \frac{u - \frac{u}{R^t}}{R - 1} = P \right.$  Theorem 2.  $\left\{ \frac{PR^t \times R - PR^t}{R^t - 1} = u \right.$

Theorem 3.  $\left\{ \frac{u}{P + u - PR} = R^t \right.$  Which being continually divided by  $R$ , will give  $t$ .

Theorem 4.  $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^t + 1 \right.$  The resolving of this Equation, will discover the Value of  $R$ .

Question 1. What is 30 l. yearly Rent, to continue seven Years, worth in ready Money, allowing 6 per Cent. Compound Interest to the Purchaser?

Here is given  $u = 30$ .  $t = 7$ . And  $R = 1.06$  To find  $P$ .

Per Theorem 1. Viz.  $\frac{u}{R^t} = \frac{30}{1.50368} = 19.9517$ .

And  $30 - 19.9517 = 10.0483 = u - \frac{u}{R^t}$

Then

Then  $R - 1 = 0,06$   $10,0483$  ( $167,4716 = P = 167 \text{ l. } 9 \text{ s. } 5 \text{ d.}$  being the Answer required.

**Question 2.** *What Annuity or yearly Rent, to continue seven Years, may be purchased for 167 l. 9 s. 5 d. allowing 6 per Cent. Compound Interest to the Purchaser?*

In this Question there is given  $P = 167,4716$ .  $t = 7$   
And  $R = 1,06$  To find  $u$ . By the second Theorem.

$$\text{First } PR^t \times R = 251,8153 \times 1,06 = 266,9242$$

$$\text{And } -PR^t = 167,4716 \times 1,50363 = 251,8153$$

Then  $R^t - 1 = 0,50363$   $15,1089$  ( $30 = u$ )  
That is  $u = 30 \text{ l.}$  the Answer required.

**Question 3.** *How long may one have a Lease of 30 l. yearly Rent, for 167 l. 9 s. 5 d. allowing 6 per Cent Compound Interest to the Purchaser?*

Here is given  $P = 167,4716$ .  $u = 30$ . And  $R = 1,06$  To find  $t$ .  
By the third Theorem.

$$\text{First } P + u = 167,4716 + 30 = 197,4716$$

$$\text{And } -PR = 177,5199$$

$$\text{Then } 19,9515 \text{ } 30 = u \text{ } (1,50363 = R^t)$$

If this  $1,50363 = R^t$  be either continually divided by  $1,06 = R$  until nothing remain (as before in Page 255.) Or if it be sought in the Table of Amounts for Years, &c. it will discover  $t = 7$  which is the true Answer required.

**Question 4.** *Suppose one should give 167 l. 9 s. 5 d. for the Purchase of a Pension, or Annuity of 30 l. per Annum, to continue seven Years; at what Rate of Interest, per Cent. would that Purchase be made, allowing Compound Interest to the Purchaser?*

In this Question there is given,  $P = 167,4716$ .  $u = 30$  And  $t = 7$ .  
To find  $R$ . Per Theorem 4.

The 4th Theorem is this Equation  $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^t + 1 \right.$

Which being brought into Numbers, and its Root extracted, as in the 4th Question of the last Section; the Value of  $R$  will be found  $1,06$ . viz.  $R = 1,06$ .

And then it will be,  $1 : 0,06 :: 100 : 6$  the Rate per Cent. as was required.

These



## Chap. 12. Of Compound Interest. 271

These four Questions include all the Varieties that can be proposed about purchasing Annuities or Leases, &c. which are to be either immediately enter'd upon, or in Possession at the Time when the Purchase is made.

But such Questions as relate to Annuities, or taking of Leases, &c. in Reversion, must be parted or divided into two distinct Questions, each to be separately consider'd by it self (See Page 252.) As in the following Examples.

*Example 1. Suppose it were required to compute the present Worth of 75 l. yearly Rent, which is not to commence or be enter'd upon, until ten Years hence; and then to continue seven Years after that Time: at 6 per Cent. &c. Compound Interest.*

The first Work in this Question, is to find what 75 l. per Annum, to continue seven Years, is worth in ready Money; as if it were to be immediately enter'd upon: And to perform that, there is given  $u = 75$ .  $R = 1,06$  And  $t = 7$ . To find  $P$ . As in the first Question of this Section.

$$\text{Thus, } \frac{u}{R^t} = \frac{75}{1,06^7} = 49,8793 \quad \text{And } 75 - 49,8793 = 25,1207$$

$$= u - \frac{u}{R^t}$$

Then,  $R - 1 = 0,06$   $25,1207 = 418,6789 = 418 l. 13 s. 6 \frac{3}{4} d.$  the Answer to the first Part of the Question.

Then the next Work will be, to find what Principal or Sum being put out ten Years, at 6 per Cent. &c. will Amount to 418 l. 13 s. 6  $\frac{3}{4}$  d. Here is given  $A = 418,6783$   $R = 1,06$  And  $t = 10$ . To find  $P$ . Per Theorem 2. Page 254.

Thus  $R^{10} = 1,790837$   $418,6783 = A$   $(233,7884 = 233 l. 15 s. 9 d.$  the present Worth of 75 l. per Annum in Reversion, &c. As was required.

*Example 2. What Annuity or yearly Rent to be entered upon ten Years hence, and then to continue seven Years, may be purchased for 233 l. 15 s. 9 d. ready Money, at 6 per Cent. &c. Compound Interest?*

In the 1st Work of this Question there is given,  $P = 233,7884$ .  $R = 1,06$ . And  $t = 10$  (the Time which the Annuity is not to be enter'd upon) To find  $A$ . Per Theorem 1. Page 254.

Thus,  $PR^t = 233,7884 \times 1,790847 = 418,6783 = A$  the Amount of

of 233 *l.* 15 *s.* 9 *d.* put to Interest ten Years, at 6 per Cent. &c. Then for the second Work of the Question there is given  $P = 418,6783$ .  $R = 1,06$ . And  $t = 7$  (the Time that the Annuity is to be enjoy'd) To find  $u$ . Per Theorem 2. of this Section.

$$\text{Thus } PR^t \times R = 418,6783 \times 1,50363 \times 1,06 = 667,3095$$

$$- PR^t = 418,6783 \times 1,50363 = 629,5372$$

$$R^t - 1 = 0,50363 \quad 37,7723 \quad (75 = u)$$

That is,  $u = 75$  *l.* the yearly Rent required by the Question.

These two Examples of finding  $P$  and  $u$  do fully shew the Method that must be used in resolving the two general, and indeed, the most useful Questions about Annuities, or Leases in Reversion: And if there be Occasion, either the Rate, or the Time, viz.  $R$  or  $t$ , may be found by a due Application of their respective Theorems.

Note, That which hath been done in the two last Sections about Annuities or yearly Rents, &c. at 6 per Cent. may also be done for any Rate of Interest, by applying the Difference of the Rates (viz.  $x$ ) As directed in the first Section of this Chapter.

Now because that Rents and Annuities, &c. are usually paid either by quarterly, or half-yearly Payments, and the Method of computing them by the Pen, may be thought a little Troublesom; I have inserted the following Tables of the Amounts of 1 *l.* for each, at 6 per Cent.

Half Years $t$	Amounts of 1 <i>l.</i> at 6 per Cent. &c. Com- pound Inte- rest.	Half Years $t$	Amounts of 1 <i>l.</i> at 6 per Cent. &c. Com- pound Inte- rest.	Half Years $t$	Amounts of 1 <i>l.</i> at 6 per Cent. &c. Com- pound Inte- rest.
1	1,0295630141	11	1,3777875592	21	1,8437905523
2	1,06	12	1,4185191122	22	1,8982985583
3	1,0913367949	13	1,4604548127	23	1,9544179853
4	1,1236	14	1,5036302590	24	2,0121964718
5	1,1568170026	15	1,5480821017	25	2,0716830644
6	1,191016	16	1,5938480745	26	2,1329282601
7	1,2262260228	17	1,6409670276	27	2,1959840483
8	1,26247696	18	1,6894789589	28	2,2609039557
9	1,2997995842	19	1,7394250493	29	2,3277430912
10	1,3382255776	20	1,7908476965	30	2,3965581901

Quarters



Quarterly Amounts.

Quarters of a Year = 1.	Amounts of 1 l. at 6 per Cent. &c. Com- pound Inte- rest.	Quarters of Years = 1.	Amounts of 1 l. at 6 per Cent. &c. Com- pound Inte- rest.	Quarters of Years = 1.	Amounts of 1 l. at 6 per Cent. &c. Com- pound Inte- rest.
1	1,0146738461	21	1,3578024938	41	1,8171263199
2	1,0295630141	22	1,3777875592	42	1,8437905523
3	1,0446706634	23	1,3980050019	43	1,8708460509
4	1,06	24	1,4185191122	44	1,8982985583
5	1,0755542769	25	1,4393342435	45	1,9261538989
6	1,0913367949	26	1,4604548127	46	1,9544179853
7	1,1073509032	27	1,4818853020	47	1,9830968140
8	1,1236	28	1,5036302590	48	2,0121964718
9	1,1400875335	29	1,5256942978	49	2,0417231330
10	1,1568170026	30	1,5480821017	50	2,0716830644
11	1,1737919574	31	1,5707984203	51	2,1020826228
12	1,191016	32	1,5938480745	52	2,1329282601
13	1,2084927856	33	1,6172359557	53	2,1642265211
14	1,2262260228	34	1,6409670276	54	2,1959840483
15	1,2442194748	35	1,6650463253	55	2,2282075801
16	1,26247696	36	1,6894789589	56	2,2609039557
17	1,2810023527	37	1,7142701133	57	2,2940801123
18	1,2997995842	38	1,7394250493	58	2,3277430912
19	1,3188726433	39	1,7649491048	59	2,3619000349
20	1,3382255776	40	1,7908476965	60	2,3965581931

Either of these Tables may also be made useful for any proposed Rate of Interest; by making the  $\frac{1}{2}$  or  $\frac{1}{4}$  of the Difference of the Rate =  $x$ , &c.

As for Instance, Suppose any of the aforesaid Questions about *Annuities* or *Rents*, &c. were to be computed at 8 per Cent. per Annum.

Then  $1,08 - 1,06 = 0,02 = x$  for yearly Payments; as before.  
Consequently 2)  $0,02$  ( $0,01 = x$  for half Years Payments.  
Or 4)  $0,02$  ( $0,005 = x$  for Quarterly Payments.

Now these Values of  $x$ , although they are not really true, yet they may serve indifferently well for small Rents; as I have already said, Page 265. But if you would work exactly.

Then  $\sqrt{1,08} = 1,0392304845$  &c.  
—  $\sqrt{1,06} = 1,0295680141$  vide Table Page 272.

Difference =  $0,0096624704 = x$  for  $\frac{1}{2}$  yearly Payments.  
N n And



And  $\sqrt{\phantom{x}} : \sqrt{\phantom{x}} 1,08 = 1,0194263092 \text{ \&c.}$

—  $\sqrt{\phantom{x}} : \sqrt{\phantom{x}} 1,06 = 1,0146738461$  See the last Table.

Their Difference  $0,0047524631 = \infty$ . for Quarterly Payments.

These are the true Values of  $\infty$ . which being involved with their respective Amounts (as before for Years, &c.) according as the Question requires, the Result will be the Answer at 8 per Cent. &c. The like may be done for any other Rate, either greater or less than 6.

Now, altho' this Method (See Page 257, and 258, &c.) of making the Tables that are only calculated for the Rate of 6 per Cent. General and Useful for all Rates of Compound Interest, be really true; yet it was rather propos'd, to shew what may be perform'd by the Pen, without a great many Tables of several Rates, than intended for common Practice.

For it must needs be confess'd. that Tables calculated on Purpose for any designed Rate of Interest, are much more ready and useful in common Practice. And therefore since the Legislative Power, have thought fit to reduce the Rate of Interest, and have settled it by an Act of Parliament, at 5 per Cent. I've therefore been at the Trouble (which was not a little) to calculate the following Tables for that Rate; but don't think it convenient to take the Tables at 6 per Cent. out of the Book, because the Examples are all suited to them; and not only so, but they may be found Useful in the taking of Leases for Houses, &c. For in those Cases, the Purchaser is always allowed more Interest for his Purchase-Money, than the common Rate paid upon the Loan of Money.

January the 15<sup>th</sup> W. 1710  
Resolved  
6709.01.23.45  
645 02395  
389  
324  
6502  
4928  
157223  
149224  
4779 900245  
423723  
Here

Ball Koonce  
See the work  
Boys &c



# Chap. 12. Of Compound Interest. 275

Here follow new *Tables* of the *Amounts* of one Pound, at the *Rate* of 5 per Cent. per *Annum* Compound Interest. For *Years*, *Half Years*, *Quarters*, *Months*, and *Days*.

## I. The Table of the Yearly Amounts of 1 l. &c.

Years = $t$ .	The Amounts of 1 l. &c.	Years = $t$ .	The Amounts of 1 l. &c.	Years = $t$ .	The Amounts of 1 l. &c.
1	$1,05 = R$	14	1,97993160	27	3,73345632
2	$1,1025 = RR$	15	2,07892818	28	3,92012914
3	$1,157625 = R^3$	16	2,18287459	29	4,11613599
4	1,21550625	17	2,29201832	30	4,32194239
5	1,27628156	18	2,40661923	31	4,53803949
6	1,34009564	19	2,52695019	32	4,76494147
7	1,40710042	20	2,65329770	33	5,00318854
8	1,47745544	21	2,78596250	34	5,25334797
9	1,55132822	22	2,92526072	35	5,51601536
10	1,62889463	23	3,07152375	36	5,79181613
11	1,71033936	24	3,22509994	37	6,08140694
12	1,79585633	25	3,38635494	38	6,38547729
13	1,88564914	26	3,55567269	39	6,70475115

## II. The Table of the Half Yearly Amounts of 1 l. &c.

Half Yrs. = $t$ .	The Amounts of 1 l. &c.	Half Yrs. = $t$ .	The Amounts of 1 l. &c.	Half Yrs. = $t$ .	The Amounts of 1 l. &c.
1	1,02469507	11	1,30779943	21	1,66912030
2	1,05	12	1,34009564	22	1,71033936
3	1,07592983	13	1,37318940	23	1,75257632
4	1,1025	14	1,40710042	24	1,79585633
5	1,12972632	15	1,44184887	25	1,84020513
6	1,157625	16	1,47745544	26	1,88564914
7	1,18621264	17	1,51394132	27	1,93221539
8	1,22550625	18	1,55132822	28	1,97993160
9	1,24552327	19	1,58963838	29	2,02882616
10	1,27628156	20	1,62889463	30	2,07892818

## III. The Table of the Quarterly Amounts of 1 l. &amp;c.

Quarters t.	The Amounts of 1 l. &c.	Quarters t.	The Amounts of 1 l. &c.	Quarters t.	The Amounts of 1 l. &c.
1	1,01227223	21	1,29194439	41	1,64888480
2	1,02469507	22	1,30779943	42	1,66912031
3	1,03727037	23	1,32384905	43	1,68960414
4	1,05	24	1,34009564	44	1,71033936
5	1,06288585	25	1,35654161	45	1,73132904
6	1,07592983	26	1,37318940	46	1,75257632
7	1,08913389	27	1,39004151	47	1,77408435
8	1,1025	28	1,40710042	48	1,79585633
9	1,11603014	29	1,42436869	49	1,81789549
10	1,12972632	30	1,44184887	50	1,84020513
11	1,14359059	31	1,45954358	51	1,86278856
12	1,157625	32	1,47745544	52	1,88564914
13	1,17183164	33	1,49558712	53	1,90879027
14	1,18621264	34	1,51394132	54	1,93221539
15	1,20077012	35	1,53252076	55	1,95592799
16	1,21550625	36	1,55132822	56	1,97993160
17	1,23042323	37	1,57036648	57	2,00422978
18	1,24552327	38	1,58963838	58	2,02882616
19	1,26080862	39	1,60914680	59	2,05372439
20	1,27628156	40	1,62889463	60	2,07892818

## IV. The Table of the Monthly Amounts of 1 l. &amp;c.

Months t.	The Amounts of 1 l. &c.	Months t.	The Amounts of 1 l. &c.	Months t.	The Amounts of 1 l. &c.
1	1,00407412	5	1,02053728	9	1,03727037
2	1,00816485	6	1,02469507	10	1,04149634
3	1,01227223	7	1,02886981	11	1,04573953
4	1,01639636	8	1,03306155	12	1,05

NOTE : The Amount of one Pound for one Day, is 1,0001336807225 &c. (found as that in Page 260.) But in the following Table; I take only nine of those Figures, as being sufficient in Practice, for computing the Interest of any Sum, not exceeding one hundred Millions of Pounds.

V. The



## V. The Table of the Dayly Amounts of 1 l. &c.

Days = t	The Amounts of 1 l. &c.	Days = t	The Amounts of 1 l. &c.	Days = t	The Amounts of 1 l. &c.
1	1,00013368	36	1,00482378	71	1,00953587
2	1,00026738	37	1,00495810	72	1,00967082
3	1,00040109	38	1,00509245	73	1,00980579
4	1,00053483	39	1,00522681	74	1,00994079
5	1,00066858	40	1,00536119	75	1,01007579
6	1,00080235	41	1,00549558	76	1,01021083
7	1,00093614	42	1,00563000	77	1,01034587
8	1,00106994	43	1,00576443	78	1,01048093
9	1,00120377	44	1,00589888	79	1,01061602
10	1,00133761	45	1,00603335	80	1,01075112
11	1,00147147	46	1,00616784	81	1,01088623
12	1,00160535	47	1,00630234	82	1,01102137
13	1,00173924	48	1,00643687	83	1,01115652
14	1,00187315	49	1,00657141	84	1,01129169
15	1,00200708	50	1,00670597	85	1,01142688
16	1,00214103	51	1,00684055	86	1,01156209
17	1,00227500	52	1,00697514	87	1,01169732
18	1,00240899	53	1,00710975	88	1,01183256
19	1,00254299	54	1,00724438	89	1,01196783
20	1,00267701	55	1,00737903	90	1,01210311
21	1,00281105	56	1,00751370	91	1,01223841
22	1,00294510	57	1,00764839	92	1,01237372
23	1,00307918	58	1,00778309	93	1,01250906
24	1,00321327	59	1,00791781	94	1,01264441
25	1,00334738	60	1,00805255	95	1,01277978
26	1,00348151	61	1,00818731	96	1,01291517
27	1,00361565	62	1,00832208	97	1,01305058
28	1,00374982	63	1,00845687	98	1,01318600
29	1,00388400	64	1,00859168	99	1,01332145
30	1,00401820	65	1,00872651	100	1,01345691
31	1,00415242	66	1,00886136	101	1,01359239
32	1,00428665	67	1,00899623	102	1,01372788
33	1,00442091	68	1,00913111	103	1,01386340
34	1,00455518	69	1,00926601	104	1,01399893
35	1,00468947	70	1,00940093	105	1,01413448

Days

Days = 1	The Amounts of 1 l. &c.	Days = 1	The Amounts of 1 l. &c.	Days = 1	The Amounts of 1 l. &c.
106	1,01427005	146	1,01970775	186	1,02517459
107	1,01440564	147	1,01984406	187	1,02531164
108	1,01454125	148	1,01998039	188	1,02544870
109	1,01467687	149	1,02011675	189	1,02558578
110	1,01481252	150	1,02025312	190	1,02572288
111	1,01494818	151	1,02038950	191	1,02586000
112	1,01508386	152	1,02052591	192	1,02599714
113	1,01521955	153	1,02066234	193	1,02613430
114	1,01535527	154	1,02079878	194	1,02627147
115	1,01549100	155	1,02093524	195	1,02640866
116	1,01562675	156	1,02107172	196	1,02654588
117	1,01576252	157	1,02120822	197	1,02668310
118	1,01589831	158	1,02134473	198	1,02682015
119	1,01603412	159	1,02148127	199	1,02695762
120	1,01616994	160	1,02161782	200	1,02709490
121	1,01630578	161	1,02175439	201	1,02723221
122	1,01644164	162	1,02189098	202	1,02736953
123	1,01657752	163	1,02202758	203	1,02750686
124	1,01671349	164	1,02216421	204	1,02764422
125	1,01684933	165	1,02230085	205	1,02778160
126	1,01698527	166	1,02243751	206	1,02791899
127	1,01712122	167	1,02257419	207	1,02805640
128	1,01725719	168	1,02271089	208	1,02819384
129	1,01739317	169	1,02284761	209	1,02833129
130	1,01752918	170	1,02298434	210	1,02846875
131	1,01766521	171	1,02312109	211	1,02860624
132	1,01780125	172	1,02325787	212	1,02874375
133	1,01793731	173	1,02339466	213	1,02888127
134	1,01807338	174	1,02353147	214	1,02901881
135	1,01820948	175	1,02366829	215	1,02915637
136	1,01834559	176	1,02380514	216	1,02929395
137	1,01848173	177	1,02394200	217	1,02943154
138	1,01861788	178	1,02407888	218	1,02956916
139	1,01875405	179	1,02421578	219	1,02970679
140	1,01889024	180	1,02435270	220	1,02984445
141	1,01902644	181	1,02448964	221	1,02998212
142	1,01916267	182	1,02462659	222	1,03011980
143	1,01929891	183	1,02476356	223	1,03025751
144	1,01943517	184	1,02490055	224	1,03039524
145	1,01957145	185	1,02503756	225	1,03053298

Days



# Chap. 12. Of Compound Interest. 279

Days = t	The Amounts of 1l. &c.	Days = t	The Amounts of 1l. &c.	Days = t	The Amounts of 1l. &c.
226	1,03067074	266	1,03619636	306	1,04175160
227	1,03080852	267	1,03633488	307	1,04189086
228	1,03094632	268	1,03647342	308	1,04203015
229	1,03108414	269	1,03661197	309	1,04216944
230	1,03122197	270	1,03675055	310	1,04230876
231	1,03135983	271	1,03688914	311	1,04244810
232	1,03149770	272	1,03702775	312	1,04258245
233	1,03163559	273	1,03716638	313	1,04272683
234	1,03177350	274	1,03730503	314	1,04286622
235	1,03191143	275	1,03744370	315	1,04300563
236	1,03204938	276	1,03758239	316	1,04314506
237	1,03218734	277	1,03772109	317	1,04328451
238	1,03232533	278	1,03785982	318	1,04342397
239	1,03246333	279	1,03799856	319	1,04356346
240	1,03260135	280	1,03813732	320	1,04370297
241	1,03273939	281	1,03827609	321	1,04384249
242	1,03287744	282	1,03841489	322	1,04398203
243	1,03301552	283	1,03855371	323	1,04412159
244	1,03315361	284	1,03869254	324	1,04426117
245	1,03329173	285	1,03883139	325	1,04440077
246	1,03342986	286	1,03897027	326	1,04454038
247	1,03356801	287	1,03910916	327	1,04468002
248	1,03370617	288	1,03924817	328	1,04481967
249	1,03384436	289	1,03938699	329	1,04495934
250	1,03398157	290	1,03952594	330	1,04509903
251	1,03412079	291	1,03966491	331	1,04523874
252	1,03425903	292	1,03980389	332	1,04537847
253	1,03439729	293	1,03994289	333	1,04551822
254	1,03453557	294	1,04008191	334	1,04565798
255	1,03467387	295	1,04022095	335	1,04579777
256	1,03481218	296	1,04036001	336	1,04593757
257	1,03495052	297	1,04049908	337	1,04607739
258	1,03508887	298	1,04063818	338	1,04621723
259	1,03522724	299	1,04077729	339	1,04635709
260	1,03536563	300	1,04091642	340	1,04649697
261	1,03550404	301	1,04105557	341	1,04663686
262	1,03564247	302	1,04119474	342	1,04677678
263	1,03578091	303	1,04133393	343	1,04691671
264	1,03591938	304	1,04147314	344	1,04705667
265	1,03605786	305	1,04161236	345	1,04719664

Days



Days = 1	The Amounts of 1 l. &c.	Days = 1	The Amounts of 1 l. &c.	Days = 1	The Amounts of 1 l. &c.
346	1,04733663	353	1,04831708	360	1,04929845
347	1,04747664	354	1,04845722	361	1,04943872
348	1,04761666	355	1,04859738	362	1,04957901
349	1,04775671	356	1,04873756	363	1,04971932
350	1,04789677	357	1,04887775	364	1,04985965
351	1,04803686	358	1,04901797	365	1,04999999
352	1,04817696	359	1,04915820	viz.	1.05

I think it needless to say any thing of the Use of these Tables; because I take it for granted, that whoever understands the Work of the foregoing Examples, at 6 per Cent. cannot but know how to make Use of these Tables of 5 per Cent. As Occasion requires.

Thus far concerning such Annuities or Leases, &c. that are limited by any assigned Time; and 'tis only such that can be computed by Theorems or certain Rules. However it may not perhaps be unacceptable, to insert a brief Account of some Estimates that have been reasonably made, by two very ingenious Persons, about the Proportion, or Difference of Mens Lives, according to their several Ages; which may be of good Use in computing the Values of Annuities, or taking of Leases for Lives, &c.

Sir William Petty in his Discourse made before the Royal Society (Anno 1674.) concerning the Use of *Duplicate Proportion*, in the Life of Man and its Duration; saith, 'That it's found by Experience 'there are more Persons living of between 16 and 26 Years old, 'than of any other Age or Decade of Years in the whole Life of 'Man (viz. 70 or 80 Years.) His Reason for that Assertion I shall omit; but supposing it true, he thence infers, 'That the Roots of 'every Number of Mens Ages under 16 (whose Root is 4) compared with the said Number 4 doth shew the Proportion of the 'Likelyhood of such Men reaching the Age of 70 Years.

As for Example, 'Tis 4 Times more likely, that one of 16 Years old should live to 70, than a new-born Babe: 'Tis 3 Times more likely, that one of 9 Years old should attain the Age of 70, than the said Infant. &c.

On the other Hand, 'Tis 5 to 4, that one of 25 Years old will die before one of 16: And 6 to 5, that one of 36 will die before one of 25. And so on according to the Roots of any other declining Age, compared with (4.6) the Root of 21, which is the Year of Perfection according to the Sense of our Law, and the Age for whose Life a Lease is most Valuable.



2. The ingenious and great Mathematician, Captain Edmund Halley, (in *Philosoph. Transact. Num. 196*) doth with great Industry and Skill, draw an Estimate of the Proportion of Mens Lives, from the Monthly Tables of the Births and Funerals in *Breslaw*, the Capital City of the Province of *Silesia*; or, as the Germans call it, *Schlesia*. Whence he proves, that it is 80 to 1 a Person of 25 Years old will not die in a Year: That it is  $5\frac{1}{2}$  to 1, that a Person of 40 will live 7 Years: That a Man of 30 Years old may reasonably expect to live 27 or 28 Years, &c.

Now from these and the like Proportions (he justly infers that) the Price of Insurance upon Lives ought to be regulated, there being a great Difference between the Life of a Man of 20, and one of 50. For Example; 'tis 100 to 1, that a Man of 20 dies not in a Year, and but 38 to 1, for a Man of 50 Years of Age. And upon these also depends the Valuation of Annuities for Lives: For it is plain, that the Purchaser ought to pay only such a Part of the Value of any Annuity, as he hath Chances that he is living.

And for that Purpose he hath taken the Pains (which was not a little) to compute the following Table, (that shews the Value of Annuities) for every fifth Year of Age to the 70th.

Age	Year's Purchase.	Age	Year's Purchase.	Age	Year's Purchase.
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The same ingenious Gentleman proceeds on, and shews how to estimate or find the Value of two Lives, and then of three Lives, which being too long a Discourse to be recited here, I have, for Brevities Sake, omitted it; and shall only add this serious Observation.

*Viz.* How unjustly we repine at the Shortness of our Lives, and think our selves wrong'd, if we attain not to old Age; whereas it appears, that the one half of those that are born, die in Seventeen Years Time. For by the aforesaid Bills of Mortality at *Breslaw*, it was found, that 1238 were in that Time reduced to 616. So that instead of murmuring at what we call a short Life, we ought to account it as a great Blessing that we have survived, per-



haps by many Years, that Period of Life, whereat the one half of the whole Race of Mankind does not arrive.

*Sect. 4. Of Purchasing Free-hold, or Real Estates; at Compound Interest.*

All Free-hold or Real Estates, are supposed to be purchased or bought to continue for ever (viz. without any limited Time); therefore the Business of computing the true Value of such Estates is grounded upon a Rank or Series of Geometrical Proportionals, continually decreasing, *ad Infinitum*.

Thus, Let  $P$ ,  $u$ ,  $R$ , denote the same Data as in the last Section. Then the Series will be,  $\frac{u}{R} \cdot \frac{u}{RR} \cdot \frac{u}{R^3} \cdot \frac{u}{R^4} \cdot \frac{u}{R^5}$  and so on in  $\infty$  until the last Term  $= 0$ . Then will  $P = 0$  (viz.  $P$ ) be the Sum of all the Antecedents. And  $P = \frac{u}{R}$  will be the Sum of all the Consequents; therefore it will be,  $u : \frac{u}{R} :: P : P = \frac{u}{R}$  which produces  $PR = u = P$ .

This Equation affords these following Theorems,

Theorem 1.  $PR = P = u$ . Theorem 2.  $\sum \frac{u}{R - 1} = P$ .

Theorem 3.  $\sum \frac{P + u}{P} = R$ .

Example. Suppose a Free-hold Estate of 75*l.* yearly Rent were to be sold; what is it worth, allowing the Buyer 6 per Cent. &c. Compound Interest for his Money?

In this Question there is given  $u = 75$ .  $R = 1,06$  To find  $P$ . Per Theorem 2. Thus  $R - 1 = 0,06$   $75 = u$  ( $1250*l.* =  $P$$  the Answer required. And so for any of the rest as Occasion requires. But if the Rent is to be paid, either by Quarterly, or Half-yearly Payments;

Then  $R = \sqrt{1,06}$  for Half-yearly } Payments at 6 per Cent.  
And  $R = \sqrt[4]{1,06}$  for Quarterly }

Or  $\left\{ \begin{array}{l} R = 1,08 \text{ for Yearly} \\ R = \sqrt{1,08} \text{ for Half-yearly} \\ R = \sqrt[4]{1,08} \text{ for Quarterly} \end{array} \right\}$  Payments at 8 per Cent.

The like is to be understood for any other proposed Rate of Interest either Greater, or Less than 6 per Cent.

The Application of these Theorems to Practice, is so very easy, that it's needless to insert more Examples.



A N  
INTRODUCTION  
TO THE  
**Mathematicks.**

P A R T III.

C H A P. I.

Of Geometrical Definitions, &c.

Sect. I. Of Lines and Angles.

**A** Point hath no Parts: That is, a Geometrical Point is not any Quantity, but only an Assignable Place in any Quantity, denoted by a *Point*. } *A.* *B.*  
As at *A.* and *B.*

Such a Place may be conceived so infinitely small, as to be void of Length, Breadth and Thickness; And therefore a Point may be said to have no Parts.

2. A *Line* is called a Quantity of one Dimension, because it may have any supposed Length, but no Breadth nor Thickness, being made or represented to the Eye, by the Motion of a Point.

That is, If the Point at *A.* be moved (upon the same Plane) to the Point at *B.* it will describe a Line, either Right, or Circular, (*viz.* Crooked) according to its Motion.

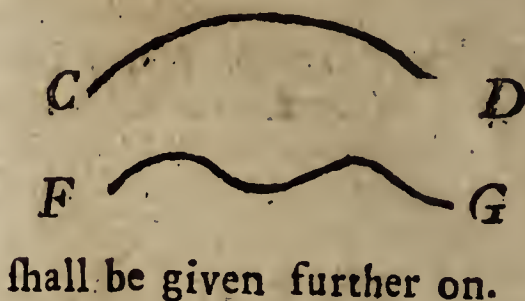
Therefore the Ends or Limits of a Line are Points.

3. A *Right Line*, is that Line which lieth even or strait betwixt those Points that limit its Length, being the shortest Line that can be drawn between any two } *A* ————— *B*  
Points. As the Line *A B.*

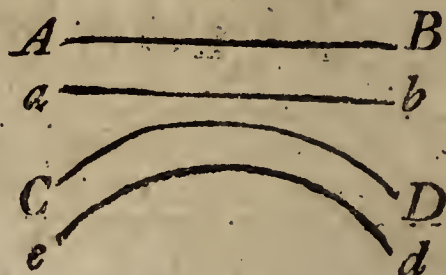
Therefore between any two Points, there can lie, or be drawn but one Right Line.

4. A *Circular, Crooked or Oblique Line*, is that which lies bending between those Points which limit its Length, as the Lines *CD* or *FG*, &c.

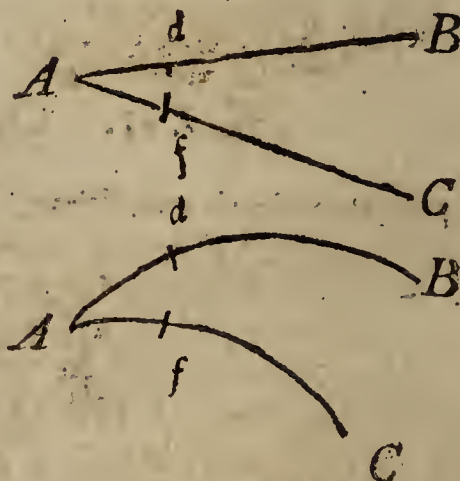
Of these Kinds of Lines there are various Sorts; but those of the Circle Parabola, Ellipsis, and Hyperbola are of most general Use in Geometry; of which a particular Account



5. *Parallel Lines*, as those that lie equally distant from one another in all their Parts, viz. such Lines as being infinitely extended (upon the same Plane) will never meet; as the Lines *AB* and *ab*, or *CD* and *cd*.



6. *Lines not Parallel, but Inclining* (viz. leaning) one towards another, whether they are Right Lines, or Circular Lines, will (if they are extended) meet, and make an Angle; the Point where they meet is called the Angular Point, as at *A*. And according as such Lines stand, nearer or further off each other, the Angle is said to be less, or greater, whether the Lines that include the Angle be long, or short. That is, the Lines *Ad*, and *Af* include the same Angle as *AB*, and *AC* doth; notwithstanding that *AB* is longer than *Ad*, &c.



7. All *Angles* included between the Right Lines, are called Right lin'd Angles; and those included between Circular Lines, are called Spherical Angles. But all Angles, whether Right-lin'd or Spherical, fall under one of these three Denominations.

*Viz.* { A Right Angle.  
An Obtuse Angle.  
An Acute Angle.

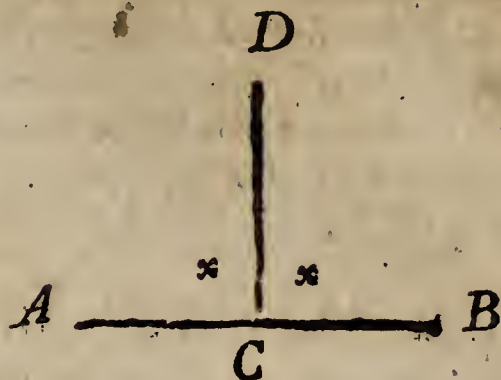
Angle is that which is included betwixt two Lines, other Perpendicular.

That

3. A Right Angle is that which is included betwixt two Lines, that meet one another Perpendicular.



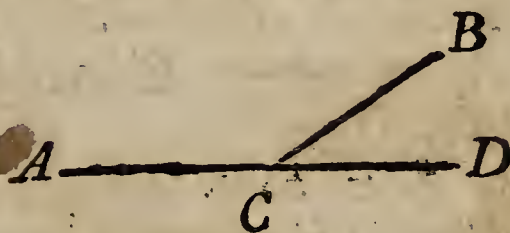
That is, when a Right-line as  $DC$ , meets with another Right-line as  $AB$ , so directly as that it neither inclines nor declines to one Side more than the other, but makes the Angles on both Sides of it equal, as at  $x, x$ ; Then are those Angles called Right Angles; and the Lines so meeting are said to be Perpendicular to each other.



That is,  $AC$ , and  $CD$ , are Perpendicular to  $DC$ , as well as  $DC$  is to either or both of them.

9. An *Obtuse Angle* is that which is greater than a Right Angle. Such is the Angle included between the Lines  $AC$  and  $CB$ .

10. An *Acute Angle* is that which is less than a Right Angle; as the Angle included between the Lines  $CB$  and  $CD$ .



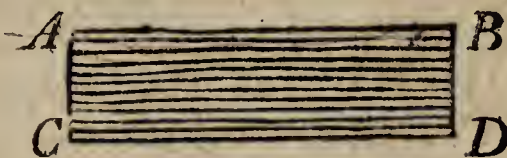
These two Angles are generally called *Oblique Angles*.

## Seet. 2. Of a Circle, &c.

*Before a Circle and its Parts are defin'd, it will be convenient to give a brief Account of Superficies in general.*

1. A *Superficies* or *Surface* is the upper, or very Out-side of any visible Thing. But by *Superficies* in *Geometry*, is meant only so much of the Out-side of any Thing as is inclosed within a Line, or Lines, according to the Form or Figure of the Thing designed; and it is produced or formed by the Motion of a Line, as a Line is described by the Motion of a Point; thus:

Suppose the Line  $AB$  were equally moved (upon the same Plane) to  $CD$ ; then will the Points at  $A$  and  $B$  describe the two Lines  $AC$  and  $BD$ ; and by so doing they will form



(and inclose the *Superficies* or Figures  $ABCD$ , being a Quantity of two Dimensions, viz. it hath Length and Breadth, but not Thickness. Consequently the Bounds or Limits of a *Superficies* are Lines.

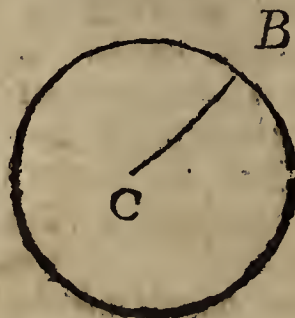
Note,



Note, *The Superficies of any Figure, is usually called its Area.*

2. A *Circle* is a Plain Regular Figure, whose Area is bounded or limited by one continued Line, called the *Circumference* or *Periphery* of the Circle, which may be thus described or drawn.

Suppose a Right Line, as  $CB$ , to have one of its extreme Points as  $C$ , so fixt upon any Plane, as that the other Point at  $B$  may move about it; then if the Point at  $B$  be moved round about (upon the same Plain) it will be the Circumference or Periphery of that Circle; the Point  $C$  will be its *Center*, and the contained Space will be it's *Area*, and the Right Line  $CB$ , by which the Circle is thus described, is called *Radius*.



### CONSECTARY.

From hence 'tis evident, that an infinite Number of Right Lines may be drawn from the Center of any Circle to touch its Periphery, which will be all equal to one another, because they are all *Radius's*.

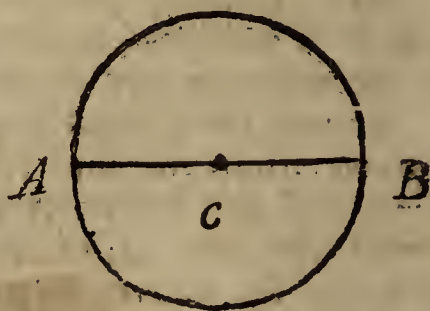
And with a little Consideration it will be easy to conceive that no more than two equal Right Lines can be drawn from any Point within a Circle to touch its Periphery, but from the Center only. (9. e. 3.)

3. *Equal Circles* are those which have equal *Radius's*; for it's plain by the last Definition that one and the same Radius (as  $CB$ ) must needs describe equal Circles, how many soever they are.

$D$

4. The *Diameter* of a Circle, is twice its Radius join'd into one Right Line, as  $AB$  drawn through the Center  $C$ , and ending at the Periphery on each Side.

That is, the Diameter divides the Circle into two equal Parts.

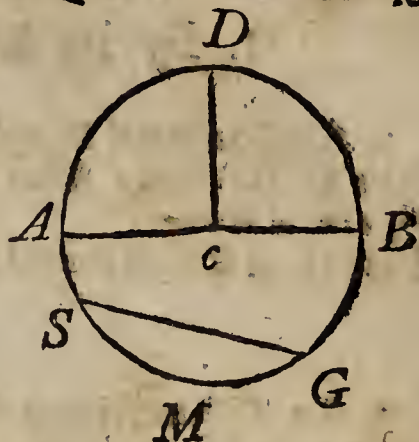


5. A *Semicircle* (viz. half a Circle) is a Figure included between the Diameter, and half the Periphery cut off by the Diameter; As  $ADB$ .

6. A



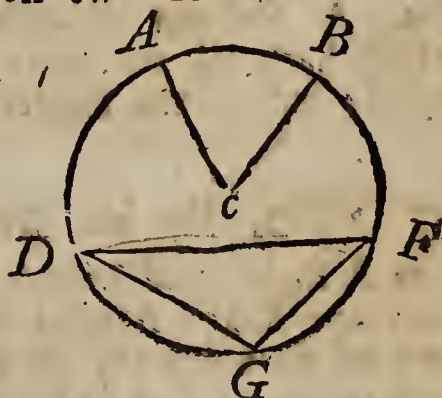
6. A *Quadrant* is half a Semicircle, viz. one Quarter of a Circle and it's made by the Radius (as  $DC$  standing Perpendicular upon the Diameter at the Center  $C$ , cutting the Periphery of the Semicircle in the Middle, as at  $D$ . Therefore a Quadrant, or half the Semicircle, is the Measure of a Right Angle.



7. A *Chord Line*, or the Subtense of an Arch, is any Right Line that cuts the Circle into two unequal Parts, as the Line  $SG$ ; and is always less than the Diameter.

8. A *Segment* of a Circle, is a Figure included betwixt the Chord and that Arch of the Periphery which is cut off by the Chord: And it may either be greater, or less than a Semicircle; as the Figure  $SMG$  or  $SDG$ .

9. A *Sector* is a Figure included between two Radius's of the Circle, and that Arch of its Periphery where they touch, as the Figure  $ACB$ , and the Arch  $AB$  is the Measure of the Angle at  $C$ , included betwixt the Radius's  $AC$ , and  $BC$ .



*Note*, All Angles of Sectors are called Angles at the Center of a Circle.

10. An *Angle* in the Segment of a Circle is that which is included between two Chords that flow from one and the same Point in the Periphery, as at  $D$ , and meet with the Ends of another Chord Line, as at  $F$  and  $G$ .

That is, the Angles at  $D$ , at  $F$ , and at  $G$ , are called Angles at the Periphery, or Angles standing on the Segment of a Circle.

### SECT. 3. Of Triangles.

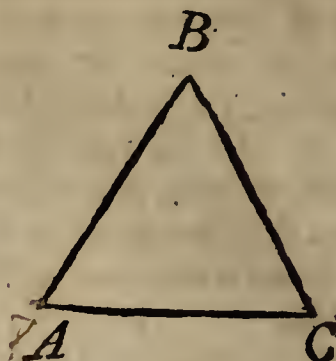
There are two Kinds of Triangles, viz. Plain and Spherical; but I shall not give any Definition of the Spherical, because they more immediately relate to Astronomy.

1. A *Plain Triangle* is a Figure whose Area is contained within the Limits of three Right Lines called Sides, including three Angles: And it may be divided, and takes its Name either according to its Sides or Angles.

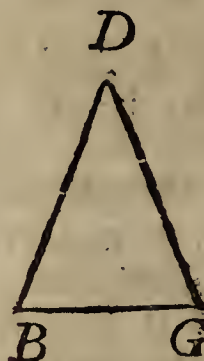
I. By

1. By its Sides.

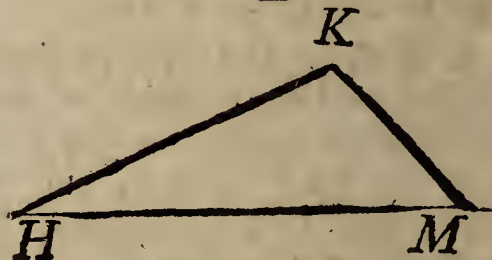
2. An *Equilateral Triangle* is that which hath all its three Sides equal, as the Figure  $ABC$ .  
That is,  $AB = BC = AC$ .



3. An *Isoceles Triangle* is that which hath only two of its Sides equal as the Figure  $BDG$ . That is,  $BD = DG$ ; but the third Side  $BG$  may be either greater or less, as Occasion requires.

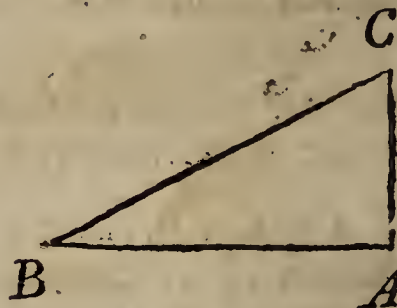


4. A *Scalenous Triangle* is that which hath all its three Sides unequal; such as the Figure  $HKM$ .



2. By its Angles.

5. A *Right-angled Triangle*, is that which hath one Right Angle; that is, when two of its Sides are Perpendicular to each other, as  $CA$  is supposed to be to  $BA$ . Therefore the Angle at  $A$ , is a Right Angle, Per *Def.* 8. *Señ.* 1.



*Note,* The longest Side of every Right-angled Triangle (as  $Bc$ ) is called Hypotenuse, and the longest of the other two Sides which include the Right Angle (as  $BA$ ) is called Base. The third Side (as  $cA$ ) is called Cathetus or Perpendicular.

6. An *Obtuse-angled Triangle* is that which hath one of its Angles Obtuse, and is called an *Amblygonium Triangle*. Such is the third Triangle  $HKM$ .

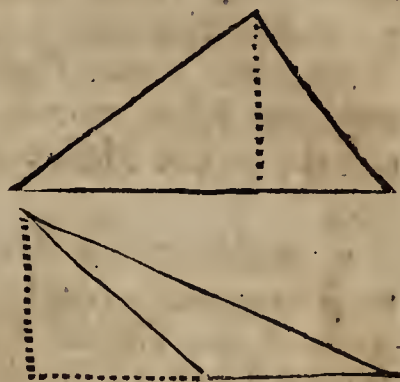
7. An *Acute-angled Triangle* is that which hath all its Angles Acute, and its called an *Oxygonium Triangle*; Such are the first and second Triangles  $ABC$ , and  $BDG$ .

*Note,* All Triangles that have not a Right Angle, whether they are Acute, or Obtuse, are in general Terms, called oblique Triangles with



without any other Distinction as before. And the longest Side of every oblique Triangle, is usually called the Base; the other two are only called Sides, or Legs.

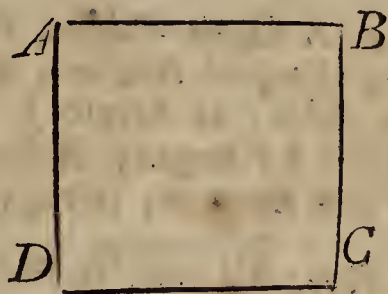
8. The *Altitude* or *Height* of any plain Triangle, is the Length of a Right Line let fall perpendicular from any of its Angles, upon the Side opposite to that Angle from whence it falls; and may be either within, or without the Triangle, as Occasion requires, being denoted by the two prick'd Lines in the annexed Triangles.



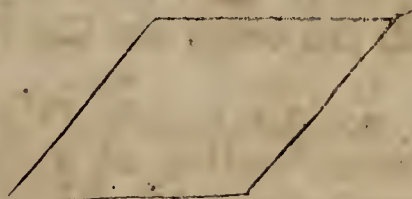
## Sect. 4. Of Four-Sided Figures, &c.

1. A *Square*, is a plain regular Figure, whose Area is limited by four equal Sides all perpendicular one to another.

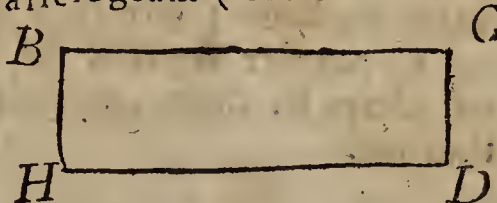
That is, when  $AB = BC = CD = DA$ , and the Angles  $A, B, C, D$  are all equal, then 'tis usually called a Geometrical Square.



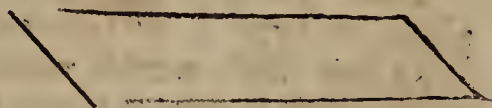
2. A *Rhombus*, or Diamond-like Figure, is that which hath four equal Sides, but no right Angle. That is, a *Rhombus* is a Square moved out of its Right Position, as the annexed Figure.



3. A *Rectangle*, or a right-angled Parallelogram (often called an Oblong, or long Square) is a Figure that hath four right Angles, and its two opposite Sides equal, viz.  $BC = HD$  and  $BH = CD$



4. A *Rhomboides*, is an oblique-angled Parallelogram, that is, it is a Parallelogram moved out of its right Position, like the annexed Figure.



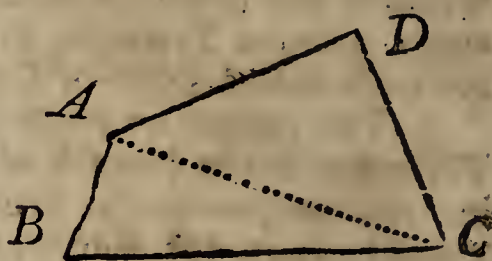
5. The *Altitude* or *Height* of any oblique-angled Parallelogram, viz. either of Rhombus, or Rhomboides, is a Right-line let fall perpendicular from any Angle upon the Side opposite to that Angle; and it may either be within or without the Figure. As in the prick'd Lines in the annexed Figure.





6. All four-sided Figures, which differ from those before-mention'd, are called *Trapezia*.

That is, when they have neither opposite Sides, nor opposite Angles, equal; as the Figure *ABCD*.



7. A Right-line drawn from any Angle in a four-sided Figure, to its opposite Angle, is called a *Diagonal Line*, and will divide the Area of the Figure into two Triangles, being denoted by the prick'd Line *AC* in the last Figure.

8. All Right-lined Figures that have more than four Sides, are called *Polygons*, whether they be regular, or irregular.

9. A *Regular Polygon*, is that which hath all its Sides equal, standing at equal Angles; and is nam'd according to the Number of its Sides (or Angles). That is, if it have five equal Sides, it is called a *Pentagon*; if six equal Sides, it is called a *Hexagon*; if seven, it is a *Heptagon*; if eight, it is an *Octagon*, &c.

*Note*, All regular Polygons may be inscribed in a Circle. That is, their angular Points, how many soever they have, will all just touch the Circle's Periphery.

10. An *Irregular Polygon* is that Figure which hath many unequal Sides standing at unequal Angles (like unto the annexed Figure, or otherwise); and of such kind of Polygons there are infinite Varieties; but they may all be reduced to regular Figures, by drawing Diagonal Lines in them, as shall be shewed farther on.



These are the most general and useful Definitions that concern plain or superficial Geometry.

As for those which relate to Solids, I thought it convenient to omit giving any Account of them in this Place, because they would rather puzzle and amuze the Learner than improve him, until he has gained a competent Knowledge in the most useful Theorems concerning Superficies; for then those Definitions may be more easily understood, and will help to form a clearer Idea of their respective Solids, than 'tis possible to conceive of them before; and therefore I have reserved those Definitions until we come to the fifth Part.



## Sect. 5. Of such Terms as are generally used in Geometry.

Whatsoever is proposed in Geometry, will either be a *Problem* or a *Theorem*.

Both which *Euclid* includes in the general Term of Proposition.

A *Problem* is that which proposes something to be done, and relates more immediately to Practical, than Speculative Geometry; that is, 'tis generally of such a Nature as to be perform'd by some known or common received Rules, without any Regard had to their Inventions, or Demonstrations.

A *Theorem* is when any common received Rule, or any new Proposition is required to be demonstrated, that so it may from thenceforward become a certain Rule to be relied upon in Practice, when Occasion requires it. And therefore several Rules are often called Theorems, by which Operations in Arithmetick, and Conclusions in Geometry are perform'd.

*Note*, By *Demonstration* is understood the highest Degree of Proof that human Reason is capable of attaining to, by a Train of Arguments, deduced or drawn from such plain Axioms, and other self-evident Truths, as cannot be denied by any one that considers them.

A *Corollary*, or *Confectary*, is some consequent Truth drawn, or gained from any Demonstration.

A *Lemma* is the Demonstration of some Premises laid down, or proposed as preparative to obviate and shorten the Proof of the Theorem under Consideration.

A *Scholium* is a brief Commentary or Observation made upon some preceding Discourse.

N. B. I advise the young Geometer to be very perfect in the Definitions, viz. not to rest satisfied with a bare Remembrance of them, but that he endeavour to gain a clear Idea, or Understanding of the Things defined; and for that Reason, I have been fuller in every Definition than is usual.

And that he may know from whence most of the following Problems, and Theorems contain'd in the two next Chapters are collected; I have all along cited the Proposition, and Book of *Euclid's Elements*. where they may be found.

As for Instance, at Problem 1. there is (3. e. 1.) which shews, that it is the third Proposition in *Euclid's* first Book. The like must be understood in the Theorems.

## C H A P. II.

The First Rudiments, or *Leading and Preparatory Problems in Plain Geometry.*

In order to perform the following Problems. the young Geometer ought to be provided with a thin strait Ruler, made either of Brass or Boxwood; and two Pair of very good Compasses, *viz.* one Pair called three-pointed Compasses, being very useful for drawing of Figures or Schemes, either with black Lead or Ink, and one Pair of plain Compasses with very fine Points, to measure and set off Distances; also he should have a very good Steel Drawing-Pen: And then he may proceed to the Work with this Caution; that he ought to make himself Master of one Problem before he undertake the next, that is, he ought to understand the Design, and, as far as he can, the Reason of every Problem as well as how to do it, and then a little Practice will render them very easy, they being all grounded upon these following Postulates.

## Postulates or Petition.

I. That a *Right-line* may be drawn from any one given *Point* to another.

II. That a *Right-line* may be produced, encreased, or made longer from either of its Ends

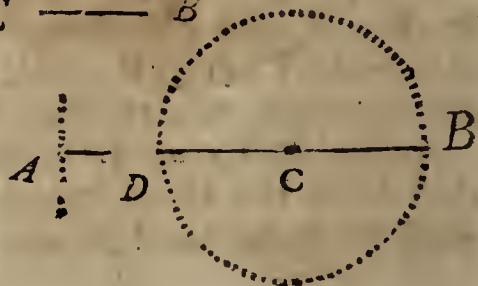
III. That upon any given *Point* (or *Center*) and with any given *Distance* (*viz.* with any *Radius*) a *Circle* may be described.

## P R O B L E M I.

*Two Right-lines being given; To find their Sum and Difference. ( 3. e. 1. )*

Let the given Lines be  $\left\{ \begin{array}{l} A \text{ ————— } C \\ C \text{ ————— } B \end{array} \right.$

Make the shortest Line as *CB* Radius, and with it describe a Circle; From its Center *C* set off the other Line *AC*, and joyn *ACB* with a *Right-line*. Then will  $AB = AC + CB$ ; and  $AD = AC - CB$ ; as was required.



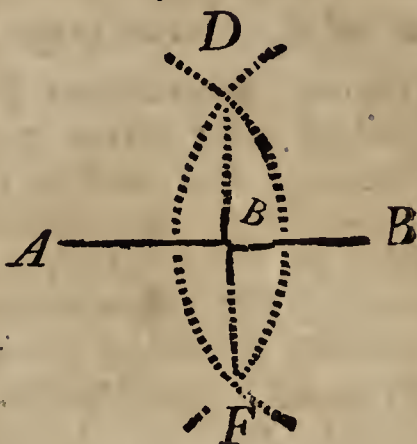
P R O



PROBLEM II.

To Bisect, or Divide a Right-line given ( as  $AB$  ) into two equal Parts. ( 10. e. 1. )

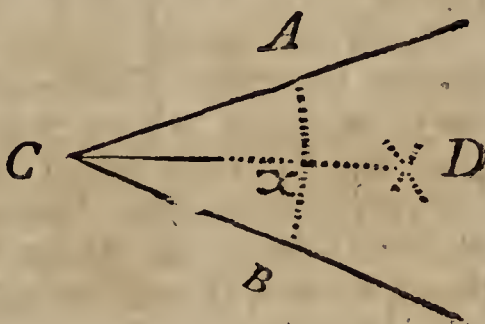
From both Ends of the given Line ( viz.  $A$  and  $B$  ) with any Radius greater than Half its Length, describe two Arches, that may cross each other in two Points, as at  $D$ , and  $F$ ; then joyn those Points  $DF$ , with a Right-line; and it will Bisect the Line  $AB$  in the Middle at  $C$ ; viz. it will make  $AC = CB$ ; as was required.



PROBLEM III.

To Bisect a Right-lin'd Angle given, into two equal Angles ( 9. e. 1. )

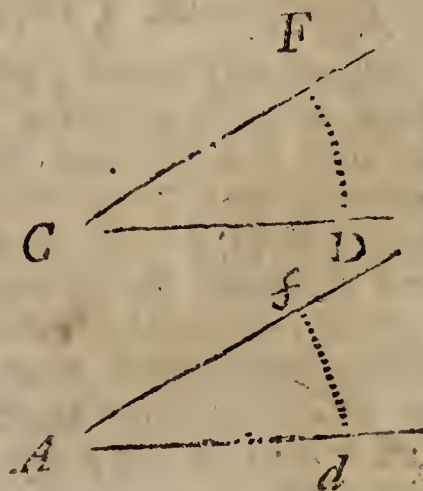
Upon the Angular Point, as at  $C$ , describe an Arch as  $AB$ ; and from those Points  $A$  and  $B$ , describe two equal Arches crossing each other, as at  $D$ . Then joyn the Points  $C$ , and  $D$ , with a Right line. and it will Bisect the Arch  $AB$ , and consequently the Angle; as was required.



PROBLEM IV.

At a Point  $A$ , in a Right-line given  $AB$ . To make a Right-lin'd Angle, equal to a Right-lin'd Angle given  $C$ . ( 23 e. 1. )

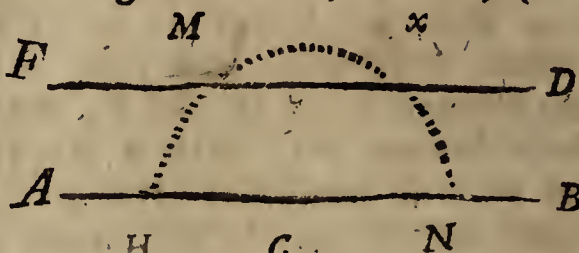
Upon the given Angle Point  $C$ , describe an Arch, as  $FD$ , (making  $CD$  any Radius at Pleasure) and with the same Radius, describe the like Arch upon the given Point  $A$ . as  $fd$ . That is, make the Arch  $fd$  equal to the Arch  $FD$ ; Then joyn the Points  $A$  and  $f$ , with a Right-line, and it will form the Angle required.



## PROBLEM V.

To draw a Right-line, as  $FD$ , parallel to a given Right-line  $AB$ , that shall pass through any assigned Point, as at  $x$ , viz. at any Distance required ( 31. e. 1. )

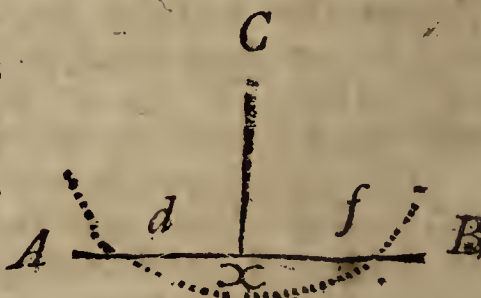
Take any convenient Point in the given Line, as at  $C$ , ( the further off  $x$  the better ); make  $Cx$  Radius, and with it upon the Point  $C$ , describe a Semi-circle, as  $HMxN$ . Then make the Arch  $HM$  equal to the Arch  $xN$ ; through the Points  $M$  and  $x$ , draw the Right-line  $FD$ , and it will be parallel to the Line  $AC$ , as was required.



## PROBLEM VI.

To let fall a Perpendicular as  $Cx$ , upon a given Right-line  $AB$ . from any assigned Point that is not in it, as from  $C$ . ( 12. e. 1. )

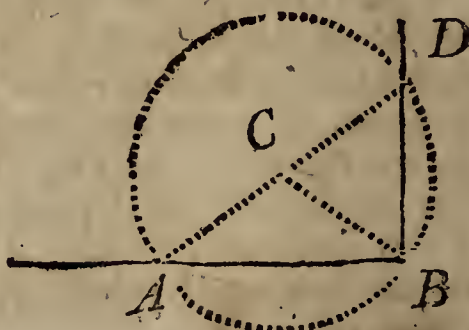
Upon the given Point  $C$ , describe such an Arch of a Circle, as will cross the given Line  $AB$  in two Points, as at  $d$ , and  $f$ ; then Bisect the Distance between those two Points  $d, f$  ( per Prob. 2. ) as at  $x$ . Draw the Right-line  $Cx$ , and it will be the Perpendicular required.



## PROBLEM VII.

To erect or raise a Perpendicular upon the End of any given Right-line, as at  $B$ ; or upon any other Point assigned in it. ( 11. e. 1. )

Upon any Point ( taken at an Adventure ( out of the given Line, as at  $C$ , describe such a Circle, as will pass through the Point from whence the Perpendicular must be raised, as at  $B$  ( viz. make  $CB$  Radius ); and from the Point where the Circle cuts the given Line, as at  $A$ , draw the Circle's Diameter  $ACD$ . Then from the Point  $D$ , draw the Right-line  $DB$ , and it will be the Perpendicular as was required.

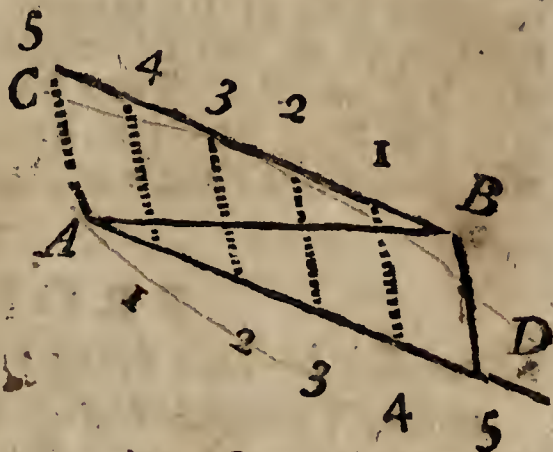




PROBLEM VIII.

*To divide any given Right-line, as  $AB$ , into any proposed Number of equal Parts (10. e. 6.)*

At the extreme Points ( or Ends ) of the given Line, as at  $A$  and  $B$ , make two equal Angles ( by Prob. 4. ) continuing their Sides  $AD$ , and  $BC$  to any sufficient Length ; then upon those Sides, beginning at the Points  $A$ , and  $B$ , set off the proposed Number of equal Parts at pleasure ( suppose them 5. ) If Right-lines be drawn ( cross the given Line ) from one Point to the other, as in the annexed Figure, those Lines will divide the given Line  $AB$  into the Number of equal Parts required.



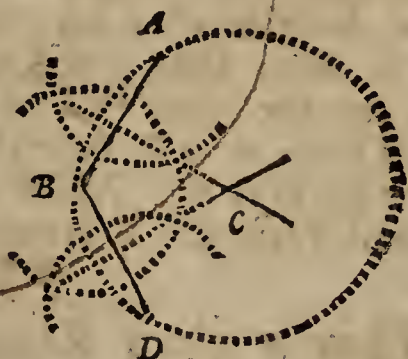
PROBLEM IX.

*To describe a Circle that shall pass ( or cut ) through any three Points given, not lying in a Right-line. As the Points  $A$ ,  $B$ ,  $D$ .*

Joyn the Points  $B A$ . and  $B D$  with Right-lines, then Bisect both those Lines ( per Problem 2. ) The Point where the bisecting Lines meet, as at  $C$ , will be the Center of the Circle required.

The Work of this Problem being well understood, it will be easy to perform the two following without any Scheme, viz. 1. To find the Center of any Circle given. ( 1. e. 3. )

By the last Prob. 'tis plain, that if three Points be any where taken in the given Circle's Periphery, as at  $A$ ,  $B$ ,  $D$ , the Center of that Circle may be found as before.



2. If a Segment of any Circle be given; To complete or describe the whole Circle. ( 25. e. 3. )

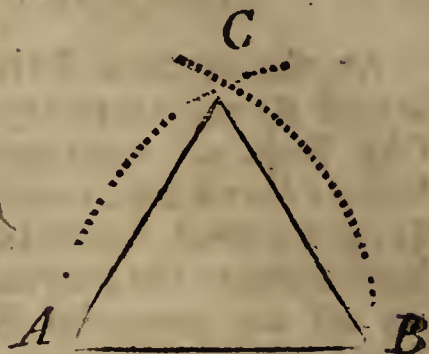
This may be done by taking any three Points in the given Segment's Arch, and then proceed as before.

PRO.

## PROBLEM X.

*Upon a Right-line given, as  $AB$ ; To describe an equilateral Triangle. (I. e. I.)*

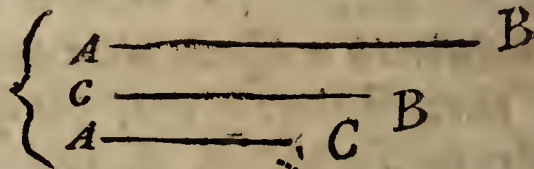
Make the given Line Radius, and with it, upon each of its extreme Points, or Ends, as at  $A$ , and  $B$ , describe an Arch, viz.  $AC$ , and  $BC$ . Then joyn the Points  $AC$ , and  $BC$  with Right-lines, and they will make the Triangle required.



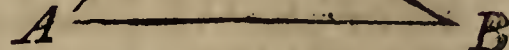
## PROBLEM XI.

*Three Right-lines being given; To form them into a Triangle, (provided any two of them taken together be longer than the third.) (22. e. I.)*

Let the given Lines be



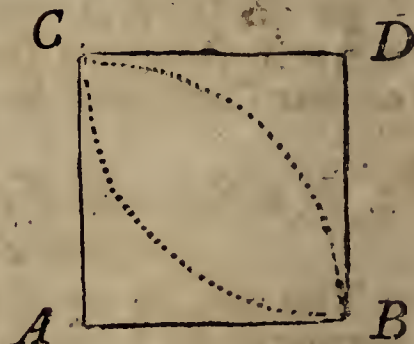
Make either of the shorter Lines, as  $AC$ , Radius, and upon either End of the longest Line, as at  $A$ , describe an Arch; then make the other Line  $CB$  Radius, and upon the other End of the longest Side, as at  $B$ , describe another Arch to cross the first Arch, as at  $C$ ; join the Points  $CA$ , and  $CB$  with Right-lines, and they will form the Triangle required.



## PROBLEM XII.

*Upon a given Right-line, as  $A, B$ , To form a Square (46. e. I.)*

Upon one End of the given Line, as at  $B$ , erect the Perpendicular  $BD$ , equal in Length with the given Line, viz. make  $BD = AB$ ; that being done, make the given Line Radius, and upon the Points,  $A$ , and  $D$ , describe equal Arches to cross each other, as at  $C$ ; then joyn the Points  $CA$ , and  $CD$  with Right-lines, and they will form the Square required.



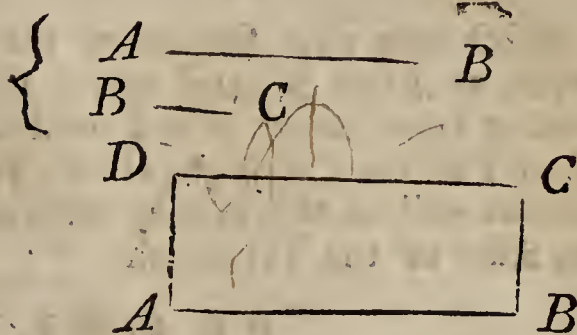
PRO-



PROBLEM XIII.

*Two unequal Right-lines being given; to form or make of them a Right-angled Parallelogram.*

Let the given Lines be



Upon one End of the longest Line, as at  $B$ , erect a Perpendicular of the same Length with the shortest Line  $BC$ ; then from the Point  $C$  draw a Line Parallel, and of the same Length to  $AB$ , viz. make  $DC = AB$ . Joyn  $DA$  with a Right-line, and it will form the Oblong or Parallelogram required.

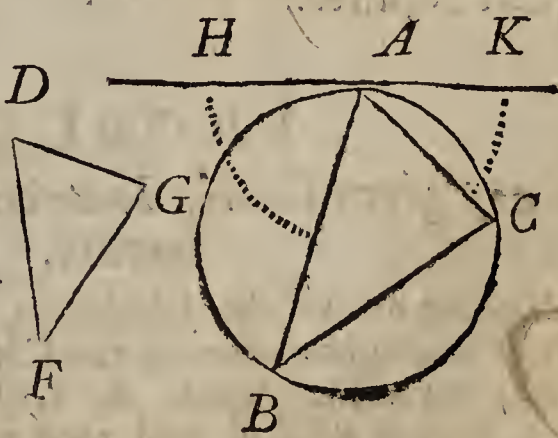
As for Rhombus's, and Rhomboides, to wit, Oblique-angled Parallelograms, they are made, or described after the same Manner with the two last Figures; only instead of erecting the Perpendiculars, you must set off their given Angles, and then proceed to draw their Sides parallel, &c. As before.

PROBLEM XIV.

*In any given Circle, to inscribe or make a Triangle, whose Angles shall be equal to the Angles of a given Triangle. As the Triangle  $FDG$ , (2. e. 4.)*

Note, Any Right-lin'd Figure is said to be inscrib'd in a Circle, when all the Angular Points of that Figure do just touch the Circle's Periphery.

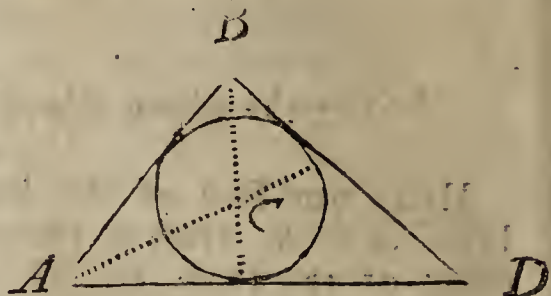
Draw any Right-line (as  $HK$ ) so as just to touch the Circle, as at  $A$ ; then make the Angle  $KAC$  equal to any one Angle of the given Triangle (as  $DFG$ ); and the Angle  $HAB$  equal to another Angle of the Triangle (as  $DGF$ ); then will the Angle  $BAC$  be equal to the Angle  $FDG$ . Joyn the Points  $B$  and  $C$  with a Right-line, and it will form the Triangle required.



**PROBLEM XV.**

*In any given Triangle, as  $ABD$ , To describe a Circle that shall touch all its Sides. (4. e. 4.)*

Bisect any two Angles of the Triangle, as  $A$  and  $B$ , and where the bisecting Lines meet, as at  $C$ , will be the Center of the Circle required; and its Radius will be the nearest Distance to the Sides of the Triangle.


**PROBLEM XVI.**

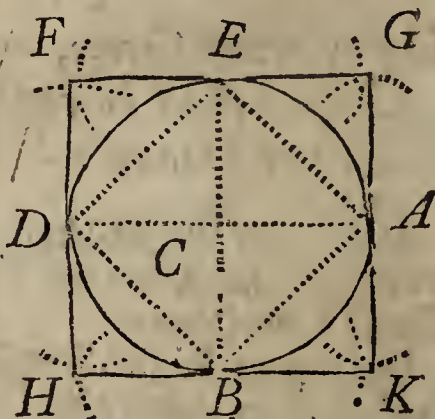
*To describe a Circle upon any given Triangle. (5. e. 4.)*

This Problem is performed in all Respects like the ninth, viz. by bisecting any two Sides of the given Triangle; the Point where those bisecting Lines meet, will be the Center of the Circle required.

**PROBLEM XVII.**

*To describe a Square about any given Circle. (7. e. 4.)*

Draw two Diameters in the given Circle, as  $DA$ , and  $EB$ , at Right-angles, in the Center  $C$ ; and with the Circle's Radius  $CA$ , describe from the extreme Points of the Diameter,  $A, B, D, E$ , cross Arches, as at  $F, G, H, K$ ; then joyn those Points where the Arches cross with Right-lines, and they will form the Square required.


**PROBLEM XVIII.**

*In any given Circle, To describe the largest Square it can contain. (6. e. 4.)*

Having drawn the Diameters, as  $DA$ , and  $EB$ , bisecting each other at Right-angles in the Center  $C$ , (as in the last Scheme;) Then joyn the Points  $A, B, D$ , and  $E$ , with Right-lines viz.  $AB, BD, DE, EA$ , and they will be the Sides of the Square required.

PRO-



PROBLEM XIX.

*Upon any given Right-line, as  $AB$ , To describe a regular Pentagon, or five-sided Polygon.*

Make the given Line Radius, and upon each End of it describe a Circle, and through those Points where the Circles cross each other, as at  $G$ ,  $\infty$ , draw the Right-line  $Ge\infty$ , upon the Point  $G$ , with the same Radius describe the Arch  $HAeBD$ . Then lay a Ruler upon the Points  $De$ , and mark where it crosses the other Circle as at  $F$ . Again, lay the Ruler upon the Points  $He$ , and mark where it crosses the other Circle as at  $C$ . Then from the Points  $F$  and  $C$ , (with the same Radius as before) describe cross Arches, as at  $K$ , Join the Points  $AF$ ,  $FK$ ,  $KC$ , and  $CB$ , with Right-lines, and they will form the Pentagon required.



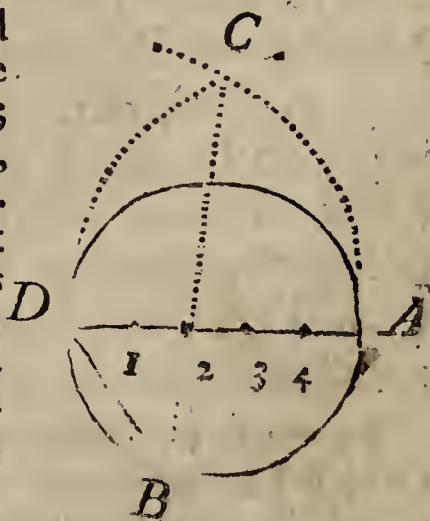
Viz.  $AF = FK = KC = CB = AB$ . And the Angles at  $A, B, C, K, F$ , will be equal.

PROBLEM XX.

*In any given Circle, To describe a regular Pentagon. ( II. e. 4. and IO. e. 3. )*

*Or, in general Terms, to describe any regular Polygon in a Circle.*

Draw the Circle's Diameter  $DA$ , and divide it into so many equal Parts, as the proposed Polygon hath Number of Sides; then make the whole Diameter a Radius, and describe the two Arches  $CA$ , and  $CD$ . If a Right-line be drawn from the Point  $C$ , through the second of those equal Parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's Periphery, as at  $B$ . Join  $DB$  with a Right-line. and it will be the true Side of the Pentagon required,



These twenty Problems are sufficient to exercise the young Practitioner, and bring his Hand to the right Management of a Ruler and Compasses, wherein I would advise him to be very ready and exact.

As to the Reason why such Lines must be so drawn, as directed at each Problem; That, I presume, will fully and clearly appear from the following Theorems; and therefore I have (for Brevity's sake) omitted giving any Demonstrations of them in this Chapter, desiring the Learner to be satisfied with the bare Knowledge of doing them only, until he hath fully considered the Contents of the next Chapter; and then I doubt not but all will appear very plain and easy.

### C H A P. III.

#### *A Collection of the most useful Theorems in plane Geometry Demonstrated.*

*Note,* In order to shorten several of the following Demonstrations, it will be necessary to premise, That

1. The Periphery ( or Circumference ) of every Circle ( whether great or small ) is supposed to be divided into 360 equal Parts, called Degrees; and every one of those Degrees are divided into 60 equal Parts, called Minutes, &c.

2. All Angles are measured by the Arch of a Circle described upon the Angular Point (*see Defin. 9. Page 287.*) and are esteemed greater or less, according to the Number of Degrees contained in that Arch.

3. A Quadrant or Quarter Part of any Circle, is always 90 Degrees, being the Measure of a Right-angle, (*Defi. 6. Page 287.*) And a Semicircle is = 180 Degrees, being the Measure of two Right-angles.

4. Equal Arches of a Circle, or of equal Circles, measure equal Angles.

To those five general Axioms already laid down in *Page 146.* (which I here suppose the Reader to be very well acquainted with) it will be convenient to understand these following, which begin their Number where the other ended.

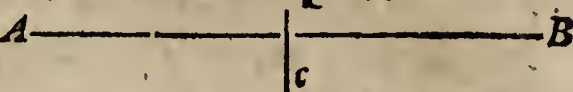
**Axioms.**



Axioms.

6. *Every whole Thing is Greater than its Part.*

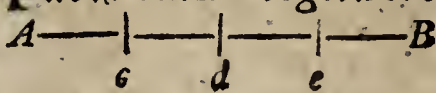
That is, the whole Line  $AB$  is greater than its Part  $Ac$ , &c.



The same is to be understood of Superficies and Solids.

7. *Every Whole is Equal to all its Parts taken together.*

That is, the whole Line  $AB$  is equal to its Parts  $AC + cd + de + eB$ .



The same is also true in Superficies and Solids.

8. *Those Things which being laid one upon another, do agree, or meet in all their Parts, are equal one to the other.*

But the Converse of this Axiom; to wit, that equal Things being laid one upon the other, will meet, is only true in Lines and Angles, but not in Superficies, unless they be alike, viz. of the same Figure or Form; as for Instance, a Circle may be equal in Area to a Square; but if they are laid one upon the other, 'tis plain they cannot meet in all their Parts, because they are unlike Figures. Also a Parallelogram, and a Triangle may be equal in their Area's one to another; and both of them may be equal to a Square; but if they are laid one upon the other, they will not meet in all their Parts, &c.

Note, Besides the Characters already explain'd in Part I. And in other Places of this Tract, these following are added.

Viz.  $\angle$  denotes an Angle in general, and  $\angle \angle$  signifies Angles;  $\triangle$  signifies a Triangle;  $\square$  signifies a Square; and  $\square$  denotes a Parallelogram. And when an Angle is denoted by any three Letters, (as  $ABC$  &c.) the middle Letter (as  $B$ ) always denotes the Angular Point; and the other two Letters (as  $AB$  and  $BC$ ) denote the Lines, or Sides of a Triangle which include that Angle.

These things being premised, the young Geometer may proceed to the Demonstrations of the following Theorems: Wherein he may perceive an absolute Necessity of being well versed in several things that have been already delivered: And also it will be very advantageous to store up several useful Corollaries, and Lemmas, as they become discovered Truths; for it often happens, that a Proposition cannot be clearly demonstrated, *a priori* or of it self, without a great deal of Trouble. Therefore it will be useful to have Recourse to those Truths that may be assisting to the Demonstration then in Hand.

T H E O.

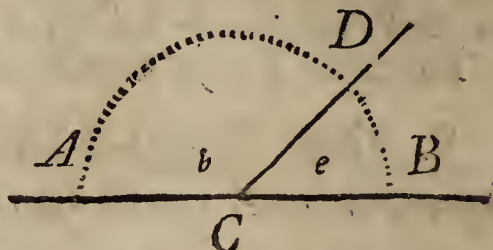
## THEOREM I.

If a Right-line stand upon ( or meet with ) another Right-line, and make Angles with it, they will either be two Right-angles, or two Angles equal to two Right-angles.

(13. e. 1.)

## Demonstration.

Suppose the Lines to be  $AB$  and  $DC$ , meeting in the Point at  $C$ ; upon  $C$  describe any Circle at Pleasure. Then will the Arch  $AD$  be the Measure of the  $\angle b$ , and the Arch  $DB$  the Measure of the  $\angle e$ ; but the Arches  $AD + DB = 180^\circ$ . viz. they complete the Semicircle.



Consequently, the  $\angle b + \angle e = 180^\circ$ . Which was to be proved.

## Corollaries.

I. Hence it follows, that if the  $\angle b = 90^\circ$ . then  $\angle e = 90^\circ$ . but if  $\angle b$  be obtuse, then the  $\angle e$  will be acute, &c.

From hence it will be easy to conceive, that if several Right-lines stand upon, or meet with any Right-line, at one and the same Point, all the Angles taken together will be  $= 180^\circ$ . viz. Two Right-angles.

## THEOREM II.

If two Angles intersect ( viz. cut or cross ) each other, the two opposite Angles will be equal. (15. e. 1.)

## Demonstration.

Let the two Lines be  $AB$  and  $DE$ , Intersecting each other in the Center  $C$ .

Then  $\angle b + \angle e = 180^\circ$  } per Last.  
And  $\angle b + \angle a = 180^\circ$  }

Consequently,  $\angle b + \angle e = \angle b + \angle a$ , per Axiom 5.

Subtract  $\angle b$  on both Sides of the Equation; and it will leave  $\angle e = \angle a$ .

Again  $\angle b + \angle e = 180^\circ$ . as before; and  $\angle e + \angle c = 180^\circ$ . Consequently  $\angle e + \angle c = \angle b + \angle e$ . Subtract  $\angle e$  and then  $\angle c = \angle b$ . Q. E. D.

Corol-





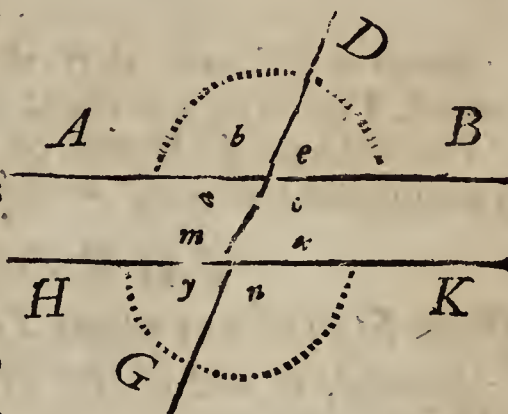
Corollary.

From hence 'tis evident, that if two Lines intersect each other, they will make four Angles, which being taken together, will always be equal to four Right-angles.

THEOREM III.

*If a Right-line cut ( or cross ) two parallel Lines, it will make the opposite Lines equal one to another. ( 29. e. 1. )*

Suppose the two Lines  $AB$ , and  $HK$  to be parallel, and the Right Line  $DG$ , to cut them both at  $C$  and  $n$ . Upon the Point  $C$  (with any Radius) describe a Semicircle, and with the same Radius, upon the Point at  $n$ , describe another Semicircle opposite to the first, as in the Figure. Then 'tis plain, and I suppose very easy to conceive, that if the Center  $C$  were moved along upon the Line  $DG$ , until it came to the Center at  $n$ , the two Lines  $AB$ , and  $HK$  would meet and concur, *viz.* become one Line ( for parallel Lines are as it were but one broad Line. ) Consequently the two Semicircles would also meet, and become one entire Circle, like to that in the last Demonstration.



And therefore the  $\angle y = \angle x = \angle a = \angle e$  } } As before, per  
 And  $\angle m = \angle n = \angle b = \angle c$  } } last Theorem  
 Q. E. D

Corollary.

Hence it follows, that if three, four, or never so many parallel Lines are cut or cross'd by one Right-line, all their opposite Angles will be equal.

THEOREM IV.

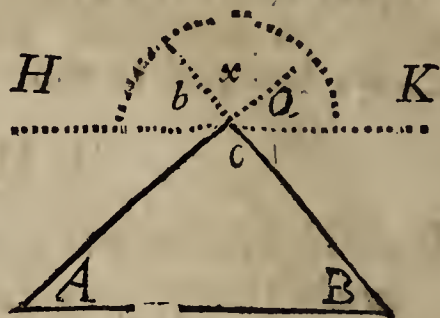
*The three Angles of every plane Triangle, are equal to two Right-angles. ( 32. e. 1. )*

*Consequently any two Angles of any plane Triangle, must needs be less than two Right-angles. ( 17. e. 1. )*

Demon.

**Demonstration.**

Let the  $\triangle ABC$  be proposed; draw the Right-line  $HK$  parallel to the Side  $AB$ , just touching the Vertical Angle  $C$ ; and upon the same Angular Point  $C$ , describe any Semicircle, and produce the Sides  $AC$ , and  $BC$  to its Periphery. Then will  $\angle b = \angle B$ ,  $\angle a = \angle A$ , and  $\angle x = \angle C$ , per last Theorem.



But  $\angle b + \angle a + \angle x = 180^\circ$ , or two Right-angles. Consequently  $\angle B + \angle A + \angle C = 180^\circ$ . Per Axiom 5. Q. E. D.

**Corollary.**

Hence it follows, that the two Acute Angles of every Right-angled Triangle, are equal to a Right-angle or  $90^\circ$ .

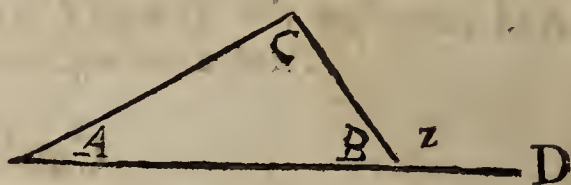
Consequently, if one of the Acute Angles be given, the other is also given; viz.  $90^\circ$  — the given  $\angle$  leaves the other  $\angle$

**THEOREM V.**

*If one Side of any plane Triangle, be continued or produced beyond, or out of the Triangle; the outward Angle will be equal to the two inward opposite Angles. (32. e. 1.)*

**Demonstration.**

Let the Side  $AB$  of the  $\triangle ABC$  be produced out of the  $\triangle$  suppose to  $D$ , &c. as in the Figure. Then  $\angle z = \angle A + \angle C$ , for the  $\angle B + \angle z = 180^\circ$ . Per Theorem I. And the  $\angle B + \angle A + \angle C = 180^\circ$ . Per last Theorem.



Therefore  $\angle B + \angle z = \angle B + \angle A + \angle C$ . Per Axiom 5. Subtract  $\angle B$  on both Sides the  $\text{\AA}equation$ , and it will leave  $\angle z = \angle A + \angle C$ . (Per Axiom 2.) Q. E. D.

Consequently, the outward Angle (at  $z$ ) of any plane Triangle, must needs be greater than either of the inward opposite Angles, viz. greater than  $\angle A$ . or  $\angle c$ . (16. e. 1.)

**Corollary.**

Hence it follows, That if one Angle of any plane Triangle be given, the Sum of the other two Angles is also given, for  $180^\circ$ . — the given  $\angle =$  the other two  $\angle$   $\angle$ .

*Theorem*



## THEOREM VI.

*In every plane Triangle, equal Sides subtend (viz. are opposite to) equal Angles. (5. e. 1.)*

*Consequently, Equal Angles are subtended by Equal Sides. (6. e. 1.)*

## Demonstration.

Suppose the  $\triangle BCD$  to be an *Isoceles*  $\triangle$ ; That is, let  $BC = CD$ . Bisect the  $\angle C$ , or (which is all one) make  $CA$  Perpendicular to  $BD$ ; then will the  $\angle$  on each Side of it, viz.  $\angle BAC$  and  $\angle DAC$  be Right Angles.



Therefore  $\left\{ \begin{array}{l} \frac{1}{2} \angle C + \angle B = 90^\circ \\ \frac{1}{2} \angle C + \angle D = 90^\circ \end{array} \right\}$  Per Corol. to Theorem 4.

Consequently  $\frac{1}{2} \angle C + \angle B = \frac{1}{2} \angle C + \angle D$ . Per Axiom 5. Subtract  $\frac{1}{2} \angle C$  from both Sides of the Equation, and it will leave  $\angle B = \angle D$ . Per Axiom 2. Q.E.D.

## COROLLARY.

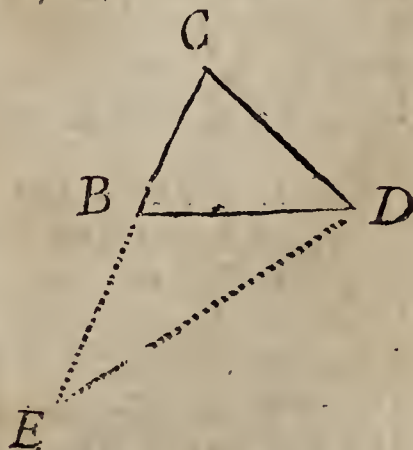
From hence it follows, that the *three Angles of an Equilateral Triangle*, are equal one to another:

## THEOREM VII.

*In every plane Triangle, the longest Side subtends the greatest Angle. (18. e. 1.)*

*Consequently, the greatest Angle of any plane Triangle, is subtended by the longest Side.*

This Theorem is evident by Inspection only; for let one of the Sides of any plane Triangle, as  $CB$ , be produced, suppose to  $E$ ; joyn  $DE$  with a Right line; then 'tis evident, that because  $CE$  is now made longer than the Side  $BC$ , therefore the  $\angle$ , at  $D$  is become larger than it was before by the  $\angle BDE$ ; and it is plain, the longer the Side  $CE$  had been made, the Angle at  $D$  would have been the more enlarged.



### T H E O R E M VIII.

*If the Sides of two Triangles are equal, the Angles opposite to those equal Sides will be equal. (8. e. 1.)*

The Truth of this *Theorem* is evident by the two included *Triangles* in the 6th *Theorem*, for they have their respective Sides equal, viz.  $BC = CD$ ,  $BA = DA$ , and  $CA$  common to both  $\triangle$ s. And it is there proved, That the  $\angle$ s opposite to those equal Sides, are equal, &c. which needs no further *Proof*.

Note, The *Converse* of this *Theorem* holds not true; for the *Angles* of two *Triangles* may be equal, and their opposite or subtending Sides unequal, as will appear at *Theorem* 12.

### C O R O L L A R Y.

Hence it follows, that  $\triangle$ s mutually *Equilateral*, are also mutually *Equiangular*. And

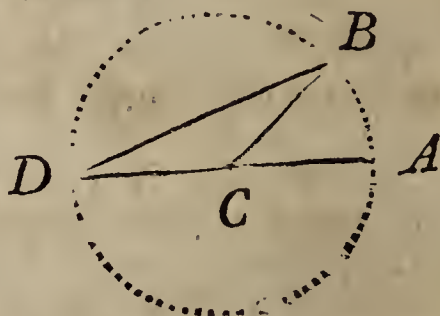
That  $\triangle$ s mutually *Equilateral*, are equal one to another. (4, and 26. e. 1.)

### T H E O R E M IX.

*An Angle at the Center of any Circle, is always double to the Angle at the Periphery, when both the Angles stand upon the same Arch. (20. e. 3.) This Theorem hath three Varieties or Cases.*

**Demonstration.**

*Case 1.* Let the Diameter  $DA$  and the Line  $DB$ , be the two Lines which form the  $\angle$  at the *Periphery*; draw the Radius  $BC$ , then is  $\angle BCA$  the  $\angle$  at the *Center*. But  $\angle BCA = \angle D + \angle B$ . Per 1b. 5. And because  $DC = BC$ , therefore  $\angle D = \angle B$ . Per *Theorem* 6. Consequently  $\angle BCA = 2 \angle D$ .



*Case 2.* Suppose the  $\angle BCF$  at the *Center*, to be within the  $\angle BDF$  at the *Periphery* (as in the annexed Figure.) Draw the Diameter  $DA$ .

Then the  $\angle BCA = 2 \angle BDA$   
And the  $\angle FCA = 2 \angle FDA$  } per *Case 1*.  
Add these two *Equations* together

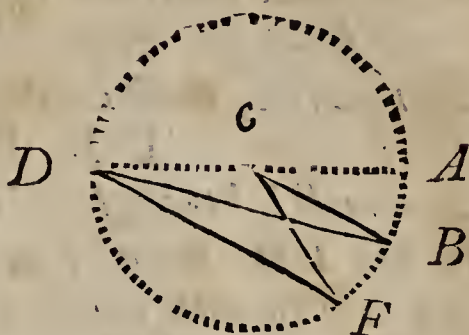


Then



Then will  $\angle BCA + \angle FCA = 2 \angle BDA + 2 \angle FDA$  per Ax. 1.  
 But  $\angle BDA + \angle FCA = \angle BCF$ .  
 And  $2 \angle BCA + 2 \angle FDA = 2 \angle BDF$ .  
 Consequently  $\angle BCF = 2 \angle BDF$ .

Case 3. Again, suppose the  $\angle BCF$  at the Center to be out of the  $\angle BDF$  at the Periphery. From the Angular Point  $D$  at the Periphery, draw the Diameter  $DA$ .



Then  $\angle FCA = 2 \angle FDA$   
 And  $\angle BCA = 2 \angle BDA$  } per Case 1.  
 Subtract this last Equation from the other, and it will leave

$\angle FCA - \angle BCA = 2 \angle FDA - 2 \angle BDA$  Per Axiom 2.  
 But  $\angle FCA - \angle BCA = \angle FCB$ . And  $2 \angle FDA - 2 \angle BDA = 2 \angle FDB$ . Consequently  $\angle FCB = 2 \angle FDB$ . Q. E. D.

### COROLLARY.

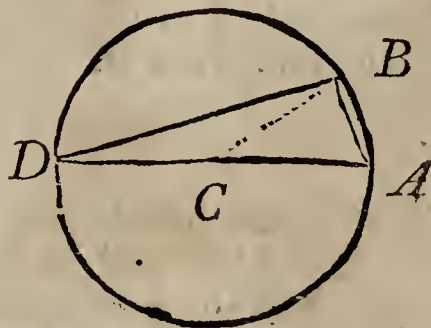
Hence it is evident, That all *Angles* at the *Periphery*, which stand on the same *Segment* or *Arch* of a *Circle*, or upon equal *Arches*, are equal one to another. (21. e. 3.)

### THEOREM X.

*An Angle in a Semi-circle, is a Right Angle. (31. e. 3.)*  
 That is, if the Diameter of any Circle be the Side of a Triangle, and the Angle opposite to that Side be any where in the Circle's Periphery, it will be a Right Angle.

#### Demonstration.

Let  $DA$  be the Diameter, and  $DBA$  the  $\Delta$ , then  $\angle B = 90^\circ$ . Draw the Radius  $BC$ , then is the  $\angle DBA = \angle D + \angle A$ .



For  $\angle CBD = \angle D$  and  $\angle CBA = \angle A$ .  
 Per Theorem 6

Therefore  $\angle DBA = \angle CBD + \angle CBA$ . Per Axiom 5.

Again,  $\angle DBA + \angle D + \angle A = 180^\circ$ . Per Theorem 4.  
 Consequently  $\angle DBA = 90^\circ$ , or a Right Angle. Q. E. D.

# COROLLARIES.

1. Hence it will be easy to conceive, that an *Angle* made in any *Segment* less than a *Semi-circle*, will be *obtuse* or greater than a *Right Angle*.

2. And an *Angle* made in any *Segment* greater than a *Semi-circle*, must consequently be *acute*.

# THEOREM XI.

*In any Right-angled Triangle, the Square which is made of the Hypothenuse, or Side subtending the Right Angle, is equal to both the Squares which are made of the Sides including the Right Angle. (47. e. 1.)*

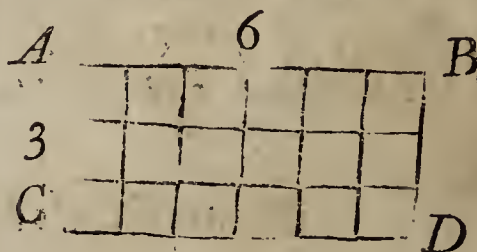
There are several Ways of demonstrating this noble and useful Theorem; but I presume none more easy to be understood by a Learner, than that which I shall here propose; And in order thereto, it will be requisite to premise the following *Lemma's*.

## Lemma 1.

A *Right-line* is said to be *multiplied* with a *Right-line* when either a *Square*, or other *Right-angled Parallelogram* is made of the two *Lines*.

That is, the *Area* of any *Right-angled Parallelogram* is equal to the *Product* of those *Numbers* which express the *Measure* of its *Sides*.

Thus, if  $AB = 6$  Inches,  
And  $AC = 3$  Inches;  
Then  $AB \times AC = 6 \times 3 = 18$  square  
Inches. which is the Area of the Parallelogram  $ABCD$ .



## Lemma 2.

If a *Right-line* be any wise cut into two *Parts*, the *Square* of the whole *Line* will be equal to the *Squares* of each *Part*, and a double *Rectangle* or *Parallelogram* made of both the *Parts*. (4. e. 2.)

That is, if the *Line*  $s$ , be cut into the two *Parts*  $B$  and  $C$



Then is  $s = B + C$  but if both the *Sides* of the *Equation* be involved, it will be  $ss = BB + 2BC + CC$ .

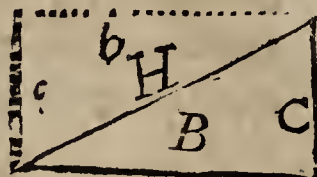
## Lemma



## Lemma 3.

The Area of every Right-angled Triangle, is half the Parallelogram made of its Base and Perpendicular.

For  $BC =$  the Area of the whole Parallelogram, by the first Lemma. And  $\triangle BCH + \triangle bch =$  the Parallelogram; But  $B = b$  and  $C = c$ .



Therefore  $\frac{1}{2} BC =$  the Area of each  $\triangle$  viz.  $\frac{1}{2} BC + \frac{1}{2} bc = BC$ .

These Things being premised, let us suppose, the Triangle  $BCH$  to be a Right angled Triangle, viz. the Side  $C$  perpendicular to the Side  $B$ . Then will  $BB + CC = HH$ .

## Demonstration

Make a Square whose Side is  $= B + C$ , and draw the included Square whose Side is  $= H$ , as in the Scheme. Then will the Area of the great Square, be equal to the Area of the four Triangles  $+ HH$ , but the Area of each  $\triangle = \frac{1}{2} BC$  Per Lemma 3. Therefore the 4  $\triangle$ 's  $= \frac{1}{2} BC \times 4 = 2 BC$ . Consequently, the Area of the great Square is  $HH + 2 BC$ . Involve  $B + C$ , and it will be  $BB + 2 BC + CC =$  the Area of the great Square. Per Lemma 2.



Consequently  $HH + 2 BC = BB + 2 BC + CC$ . Per Axiom 5. Subtract  $2 BC$  from both Sides of the Equation, and there will remain  $HH = BB + CC$ . Q.E.D.

To Illustrate this Theorem by Numbers; Let us

Suppose  $C = 3$ .  $B = 4$ . and  $H = 5$ .

Then will  $CC = 9$ .  $BB = 16$ . and  $HH = 25$ .

Consequently,  $BB + CC = HH = 16 + 9 = 25$ .

## CONSECTARY.

From this admirable Theorem (said to be first invented by Pythagoras) is deduced the Method of adding, and subtracting Squares, Parallelograms, Circles, &c.

THEO-

## T H E O R E M XII.

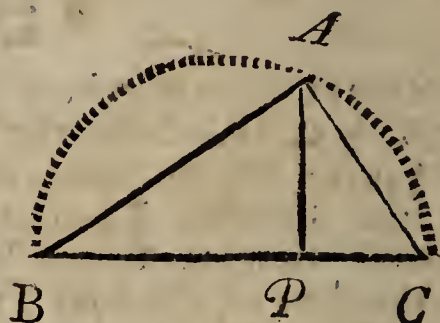
*In any Right-angled Triangle, a Perpendicular being let fall from the Right Angle upon the Hypothenuſe, will divide the Triangle into two Right-angled Triangles, which will be both ſimilar or alike to the firſt Triangle, and to each other. (8. e. 6.)*

*Note, All plane Triangles are ſaid to be ſimilar, viz. Alike when each ſingle Angle in one of the Triangles, is equal to each ſingle Angle of the other; but if any two ſingle Angles of one Triangle are equal to two ſingle Angles of the other; the third Angles will be equal. Per Theorem 4.*

1. In the Right-angled  $\triangle BAC$ ,  
Let  $AP$  be ſuppoſed Perpendicular  
to the Hypothenuſe  $BC$ , Then  $\angle BAP = \angle C$ .

For  $\angle BAP + \angle B = 90^\circ$ .

And  $\angle B + \angle C = 90^\circ$ . Per Corollary to Theorem 4.



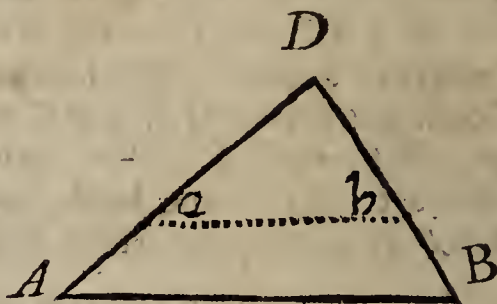
Therefore  $\angle BAP = \angle C$ . Per Axiom 5.

Again,  $\angle PAC + \angle C = 90^\circ$ . and  $\angle B + \angle C = 90^\circ$ .

Therefore  $\angle PAC = \angle B$ , &c. Conſequently the  $\triangle BAP$  is alike to the  $\triangle ACP$ ; And each is alike to the whole  $\triangle BAC$ .

2. Or if a Right-line be drawn parallel to one of the Sides of any plane Triangle (viz. within it) it will cut off a Triangle ſimilar or alike to the whole Triangle. Thus,

In the  $\triangle ABD$ , draw the Right-line  $ab$  Parallel to the Side  $AB$ ; Then will the included  $\triangle adb$ , be like the  $\triangle ADB$ . For  $\angle a = \angle A$ , and  $\angle b = \angle B$ . Per Theorem 3. and  $\angle D$  is common to both the Triangles; Ergo, &c.



## T H E O R E M XIII.

*If two Triangles are alike, their like Sides will be Proportional.*

That is, thoſe Sides which ſubtend the equal Angles, as alſo thoſe Sides which are about the equal Angles, will be Proportional



nal to each other; And consequently, if any two Triangles have their Sides Proportional, their Angles are equal. (4, 5, 6, 7. e. 6.)

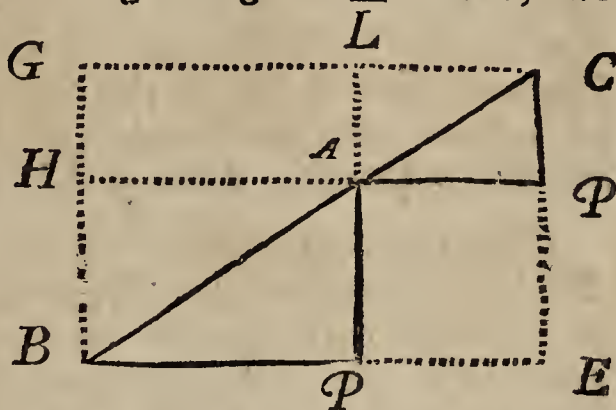
**Demonstration.**

Let the Similar Triangles in the Scheme of the last Theorem be here propos'd again.

Then it will be  $BP : AP :: AP : CP$ . According to this Theorem, Ergo  $BP \times CP = AP \times AP$ .

**Demonstration.**

Let us suppose the afore said Right-angled  $\triangle BAC$ , cut through the Perpendicular  $AP$ , and there open'd until the Sides  $BA$  and  $CA$  become one Right-line: Let the Sides  $BP$  and  $CP$  be continued until they meet in  $E$ : Then complete the Parallelograms by drawing the Parallel Lines  $GLC$ ,  $HAP$ ,  $GHB$ , and  $LAP$ , as in the Figure.



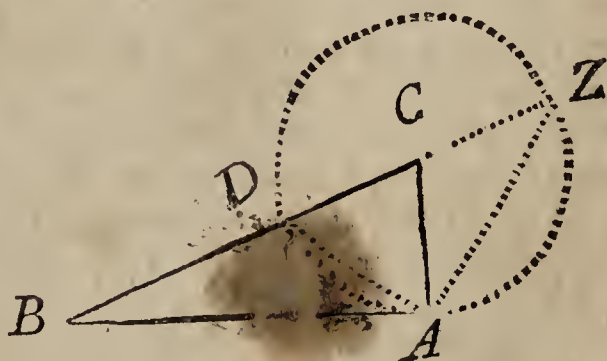
Then it is evident, that the  $\triangle BHA = \triangle BPA$ , and the  $\triangle CPA = \triangle CLA$ ; also that the  $\triangle BEC = \triangle BGC$ , because all their respective Sides are equal.

But the  $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA + \triangle CPA + \square APEP$ . Now, if from both Sides of this Equation there be subtracted the equal  $\triangle \triangle$ , there will remain  $\square HGLA = \square APEP$ .

But  $\square HGLA = BP \times CP$ , and  $\square APEP = AP \times AP$ . Consequently  $BP : AP :: AP : CP$ . Which was to be prov'd.

Or otherwise, thus:

Suppose the  $\triangle BAC$  to be Right-angled at  $A$ , upon the  $\angle$  Point  $C$  with the Radius  $CA$ , describe a Circle, and continue the Hypotenuse  $BC$  to  $Z$ ; join  $ZA$  and  $AD$  with Right Lines; then will the  $\triangle BAD$  be like to the  $\triangle BZA$ .



For  $\angle DAB + \angle DAC = 90^\circ$ . By Construction.

And  $\angle ZAC + \angle DAC = 90^\circ$ . By Theorem 10.

Therefore  $\angle DAB + \angle DAC = \angle ZAC + \angle DAC$ . By Axiom 5. Subtract  $\angle DAC$  from both Sides of the Equation and there will remain  $\angle DAB = \angle ZAC$ . But  $\angle ZAC = \angle CZA$ .

By

By *Theorem 6.* And  $\angle B$  is common to both  $\triangle \triangle$ .

Therefore  $\angle BDA = \angle BAZ$ . By *Theorem 4.*

Consequently  $\triangle BAD$  is like to  $\triangle BZA$ .

Let the Sides  $\left\{ \begin{array}{l} BA = b \\ BC = h \\ CA = c \end{array} \right\} \left\{ \begin{array}{l} \text{Then } bb + cc = hh, \text{ Per Theor. II.} \\ \text{Consequently } bb = hh - cc. \\ \text{Which gives the following Analogy,} \end{array} \right.$

*viz.*  $b : b + c :: h - c : b$ . That is,  $BA : BZ :: BD : BA$ .

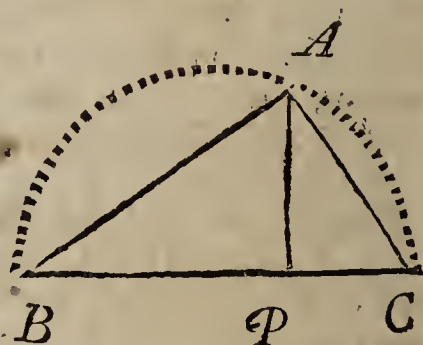
Q. E. D.

## COROLLARIES.

1. Hence it is evident, that in any Right-angled Triangle, a Perpendicular being let fall from the Right Angle upon the Hypotenuse, will be a mean Proportional between the Segments of the Hypotenuse. That is,  $BP : PA :: PA : PC$ .

2. The Base ( $BA$ ) is a mean Proportional between the Hypotenuse ( $BC$ ) and that Segment of the Hypotenuse next to the Base, (*viz.*  $BP$ .) That is,  $BC : BA :: BA : BP$ .

3. The Cathetus ( $AC$ ) is a mean Proportional between the Hypotenuse ( $BC$ ) and that Segment of the Hypotenuse next to the Cathetus, (*viz.*  $PC$ .) That is,  $BC : AC :: AC : PC$ .



## SCHOLIUM.

I have been larger upon this most excellent *Theorem*, in giving a double Demonstration of it, because it is so universally useful in all Parts of the Mathematicks : For the Business of *Trigonometry* (both Plane and Spherical) wholly depends upon it ; and therefore one may truly say, that *Astronomy*, *Dialling*, *Navigation*, *Surveying*, *Optics*, &c. depend upon a due Application of it.

And of its Use in *Geometry*, *Des Cartes* takes particular Notice, as you may find in *Dr. Pell's Algebra*, Page 65, whose Words are these,

“ *Des Cartes*, in a Letter not yet printed, writes thus : In  
“ searching the Solution of Geometrical Questions, I always  
“ make use of Lines Parallel and Perpendicular, as much as  
“ is possible, (he means as many Lines as are useful) and I con-  
“ sider no other *Theorems* but these two. [the Sides of like  
“ Triangles have like Proportion]. And [in Rectangle Triangles,  
“ the



“ the Square of the greatest Side is equal to the Squares of the two  
 “ other Sides. ] And I am not afraid to suppose many unknown  
 “ Quantities, that I may reduce the propos'd Question to such  
 “ Terms, as to depend on no other Theorems but these two.

This I thought convenient to insert, that the young Learner  
 may see how the Great *Des Cartes* esteem'd these two Theorems, viz.  
 the last, and Theorem II. For, in Truth, all the preceding Theo-  
 rems are only (as it were) Preparatives to these two.

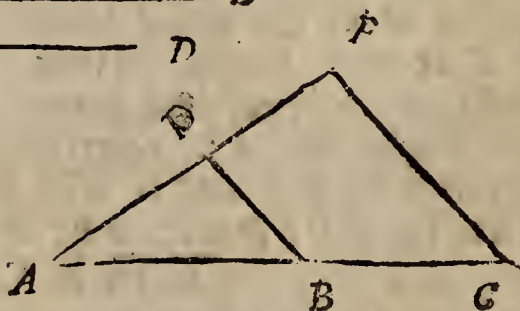
This last Theorem demonstrates the Reason of the Method used  
 in finding out proportional Lines, as in the three following  
 Problems.

### PROBLEM I.

*Two Right-lines being given, to find a third Line in Pro-  
 portion to them. (II. e. 6.)*

Let the two Lines be  $\left\{ \begin{array}{l} A \text{ ————— } B \\ A \text{ ————— } D \end{array} \right.$

Set the two given Lines at an-  
 ny Angle in the Point *A*, and  
 produce the Line *AB* to *C*,  
 making *BC* = *AD*; join  
 the Points *B* *D* with a Right-  
 line, and draw *CF* parallel  
 to *BD*; then will the  $\Delta$   
*ABD* be like the  $\Delta$  *ACF*.



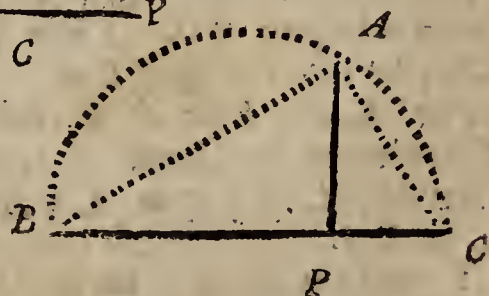
Therefore  $AB : BC (= AD) :: AD : DF$ , which is the third  
 Proportional requir'd.

### THEOREM II.

*Two Right-lines being given, to find a Mean proportional  
 Line between them. (13. e. 6.)*

Let the given Lines be  $\left\{ \begin{array}{l} B \text{ ————— } P \\ P \text{ ————— } C \end{array} \right.$

Join the two given Lines  
 into one, viz. make *BC*  
 = *BP* + *PC*, and upon  
*BC*, as Diameter, describe  
 a Semicircle; then upon  
 the Point *P*, where the 2



sf

lines

Lines meet, erect a Perpendicular to touch the Circle's Periphery, as  $PA$ , and it will be the Mean Proportional requir'd.

$$\text{Viz. } BP : AP :: AP : PC.$$

By this Problem it is easy to conceive how to make a Square equal to any given Parallelogram, (14. e. 6.)

For if  $BP$  be the Length, and  $PC$  the Breadth of the given Parallelogram, then will  $AP$  be the Side of the Square, equal in Area to that Parallelogram.

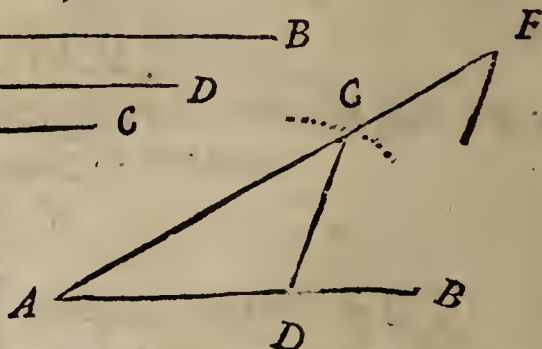
### PROBLEM III.

*Three Right-lines being given, to find a fourth Proportional Line, (12. e. 6.)*

Suppose the three Lines  $\left\{ \begin{array}{l} A \text{ --- } B \\ A \text{ --- } D \\ D \text{ --- } C \end{array} \right.$

Upon the longest Line  $AB$  set off the next longest Line  $AD$ , viz. make  $DB = AB - AD$ ; then upon the Point  $D$  set the other Line  $DC$  at any Angle, either right or oblique, and draw

the Right-line  $AC$ , continuing it a sufficient Length; make  $BF$  parallel to  $DC$ , and it will be the fourth Proportional requir'd. That is,  $AD : DC :: AB : BF$ .



### THEOREM XIV.

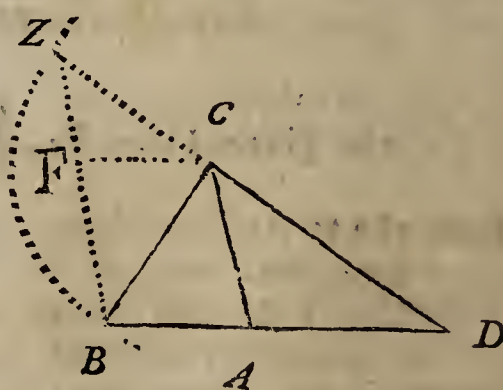
*If any Angle of a plane Triangle be bisected (viz. divided into two equal Angles) with a Right-line, (viz. as  $CA$  is suppos'd to do the Angle  $BCD$  it will cut the opposite Side (viz.  $BD$ ) in Proportion to the other two Sides of the Triangle. (3. e. 6.)*

#### Demonstration.

Produce the Side  $DC$ , until  $CZ = CB$ ; join the Points  $ZB$  with a Right-line, and draw the Line  $FC$  parallel to  $BD$ . Then will  $\triangle CZE$ , be like to  $\triangle DCA$ .

For  $\angle ZCF = \angle D$  and  $\angle Z$  is common to both  $\triangle$ s, consequently,  $\angle ZFC = \angle CAD$  and  $FC = BA$ .

Therefore  $BA (= FC) : BC (= ZC) :: AD : CD$ . Q. E. D.



THE-



THEOREM XV.

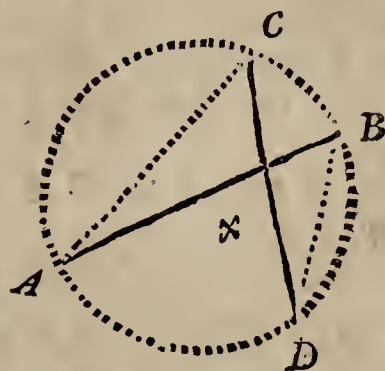
If two Right-lines (howsoever drawn) within a Circle, do cut each other, the Rectangle made of the Segments (or Parts) of the one Line, will be equal to the Rectangle made of the Segments (or Parts) of the other Line, (35. e. 3.)

That is, if two Lines, as  $AB$  and  $CD$  do cut each other in any Point, as at  $x$ , then will  $Ax \times Bx = Dx \times Cx$ .

Demonstration.

Join the Points  $AC$  and  $BD$  with Right-lines; then will the  $\triangle Cx A$  be like to the  $\triangle Bx D$ . For  $\angle B = \angle C$ , and  $\angle A = \angle D$ . By Corollary to Theorem 9.

And  $\angle AxC = \angle BxD$  by Theor. 2. Therefore it will be  $Ax : Dx :: Cx : Bx$ . By Theorem 13. Consequently,  $Ax \times Bx = Dx \times Cx$ . Q. E. D.



THEOREM XVI.

If two Right-lines are so drawn within a Circle, as being continued they will meet in a Point out of the Circle's Periphery, the Rectangle made of one whole Line, and its Part out of the Circle, will be equal to the Rectangle of the other whole Line, and its Part out of the Circle, (36, 37, e. 3.)

That is, if the Lines  $AC$  and  $DB$  be continued unto the Point  $Z$ ;

Then will  $AZ \times CZ = DZ \times BZ$ .

Demonstration.

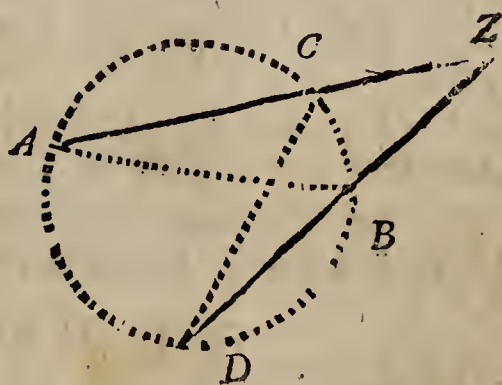
Draw the Lines  $AB$  and  $CD$ ; then will  $\triangle CZD$  be like to the  $\triangle BZA$ , for  $\angle A = \angle D$ , and  $\angle Z$  is common to both  $\triangle$ s. Consequently,  $\angle ABZ = \angle DCZ$ . By Theorem 4.

Therefore  $AZ : BZ :: DZ : CZ$ . Ergo,  $AZ \times CZ = DZ \times BZ$ .

THEOREM XVII.

If from any Angle of a plane Triangle inscrib'd in a Circle, there be let fall a Perpendicular upon the opposite Side;

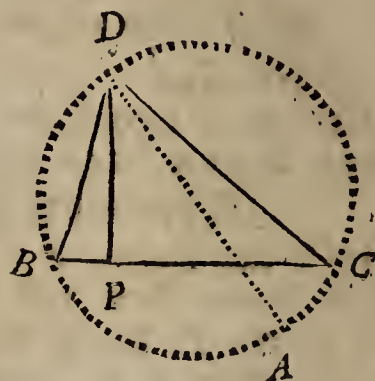
as



(as  $DP$ ) as that Perpendicular is in Proportion to one of the Sides including the Angle, so is the other Side including the Angle to the Diameter of the Circle.

### Demonstration.

Let  $BCD$  be the propos'd  $\Delta$ . From the  $\angle$  at  $D$  draw the Diameter  $DA$ ; then will  $\angle A = \angle B$ , because they both stand upon the same Arch  $DC$ , and  $\angle DCA = 90^\circ$ . By Theorem 10. Consequently the  $\angle ADC = \angle BDP$ . By Theorem 4. Therefore  $\Delta DCA$  is like to the  $\Delta DPB$ ; and therefore,



$$DP : DB :: DC : DA; \text{ Or, } DP : DC :: DB : DA.$$

Q. E. D.

### THEOREM XVIII.

If any Quadrangle (that is, a Trapezium) be inscribed within a Circle; the two opposite Angles taken together are equal to two Right-angles, viz.  $180^\circ$ . (22. e. 3.)

That is, in the Quadrangle  $ABCD$ , the  $\angle A + \angle C = 180^\circ$ . And the  $\angle B + \angle D = 180^\circ$ .

### Demonstration.

Draw the two Diagonals  $AC$  and  $BD$ ; then will the  $\angle BDA = \angle BCA$ , and the  $\angle BDC = \angle BAC$ .

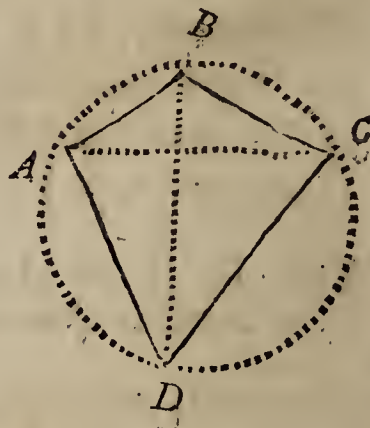
By Corol. to Theorem 9.

$$\text{But } \angle ABC + \angle BCA + \angle BAC = 180^\circ.$$

By Theorem 4.

$$\text{And the } \angle BDA + \angle BDC = \angle ADC.$$

$$\text{Therefore the } \angle ABC + \angle ADC = 180^\circ.$$



And by the same Way of Arguing, it may be prov'd, that the  $\angle BAD + \angle BCD = 180^\circ$ . Q. E. D.

### THEOREM XIX.

If in any Quadrangle inscrib'd within a Circle, there be drawn two Diagonals, as  $AC$  and  $BD$ , the Rectangle made of the two Diagonals will be equal to both the Rectangles made of the opposite Sides of the Quadrangle.

$$\text{That is, } AC \times BD = AB \times CD + AD \times BC.$$

Demon-

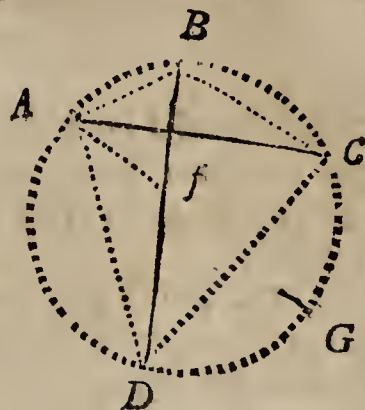


Demonstration.

Make the Arch  $DG = \text{Arch } BC$ , and from the Points  $A$   $G$  draw the Line  $Af$ , and it will form the  $\triangle AfD$ , like to the  $\triangle ABC$ . For the  $\angle fAD = \angle BAC$ , because the Arches  $DG$  and  $BC$  are equal.

Again, the  $\angle fDA = \angle BCA$  because they stand both upon the Arch  $AB$ . Consequently the  $\angle AfD = \angle ABC$ . By Theorem 4.

Therefore it will be  $AC : BC :: AD : Df$ . By Theorem 13.



$$\text{Ergo } \frac{BC \times AD}{AC} = Df.$$

Again, the  $\triangle Baf$ , and  $\triangle ACD$ , are alike. For  $\angle ABf = \angle ACD$ , and  $\angle Baf = \angle CAD$ , because the  $\angle fAD = \angle BAC$ . And the  $\angle Caf$  is common to both  $\triangle \triangle$ . Consequently the  $\angle AfB = \angle ADC$ .

Therefore  $AC : CD :: AB : Bf$ . By Theorem 13.

$$\text{Ergo } \frac{CD \times AB}{AC} = Bf. \text{ But } Df + Bf = BD.$$

Consequently,  $BC \times AD + CD \times AB = BD \times AC$ .

Q. E. D.

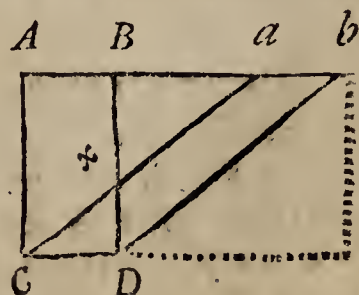
THEOREM XX.

*All Parallelograms (whether right or oblique-angled) that stand upon the same Base, or upon equal Bases, and betwixt the same Parallels, are equal one to another, (35 and 36 e. 1.)*

That is,  $\square ABCD = \square abCD$ .

Demonstration

Because  $AB = CD = ab$  by Supposition; therefore  $Aa = Bb$ ; for  $Ba$  is common to both. And because  $AC = BD$ , and the  $\angle A = \angle B$ , therefore the  $\triangle ACa = \triangle BDb$ ; and if from both  $\triangle \triangle$  there be taken the  $\triangle Bxa$  common to both, there will remain the Trapeziums  $AB \times C = ab \times D$ . Per Axiom 5.



But

But Trapezium  $AB \times C + \triangle C \times D = \square ABCD$ .

And Trapezium  $ab \times D + \triangle C \times D = \square abCD$ .

Consequently,  $\square ABCD = \square abCD$ .

Q. E. D.

### Corollary.

Hence it will be easy to conceive, that all Triangles which stand upon the same Base, or upon equal Bases, and between the same Parallels, (*viz.* having the same Height) are equal one to another, (37 and 38 e. I.)

For all Triangles are the Halves of their circumscribing Parallelograms; and therefore, if the Wholes be equal, their Halves will also be equal.

## THEOREM XXI.

*Parallelograms (and consequently Triangles) which have the same Height, have the same Proportion one to another as their Bases have. (1. e. 6.)*

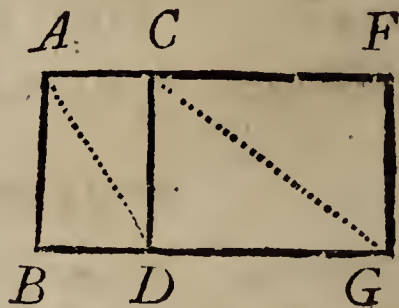
### Demonstration.

Draw  $AF$  parallel to  $BG$ , and draw  $AB, CD, FG$  Perpendiculars to them.

Then will  $BD \times AB = \square ABCD$ .

And because  $CD = AB$ , therefore  $DG \times AB = \square CD FG$ . But  $BD : DG :: BD \times AB : DG \times AB$ .

And consequently  $\triangle ABD : \triangle CDG :: BD : DG$ , &c. Q. E. D.



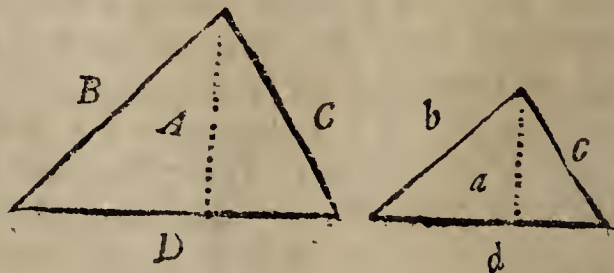
## THEOREM XXII.

*Like Triangles are in duplicate Ratio to that of their homologous Sides. (19. e. 6.)*

That is, the Area's of like Triangles, are in Proportion one to another, as are the Squares of their like Sides.

### Demonstration.

Suppose the  $\triangle BCD$  and  $\triangle bcd$  to be alike, and their like Sides to be those mark'd with the same Letters.



Let



Let  $A$  and  $a$  be Perpendiculars to the two Bases  $D$  and  $d$ .

Then  $\frac{1}{2} D A =$  the Area of  $\triangle B C D$  } by Lemma 3. Page 303.  
And  $\frac{1}{2} d a =$  the Area of  $\triangle b c d$  }

But 1  $B : b :: D : d$  } &c. By Theorem 13.  
And 2  $B : b :: A : a$  }

Conseq. 3  $D : d :: A : a$

3 4  $D a = d A$

4  $\times \frac{1}{2} D d$  5  $\frac{1}{2} D D d a = \frac{1}{2} D d d A$ . By Axiom 3.

5, hence 6  $D D : d d :: \frac{1}{2} D A : \frac{1}{2} d a$ . And so for other Sides.

Q. E. D.

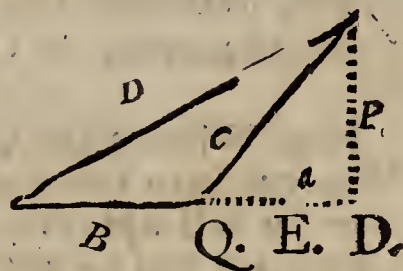
### THEOREM XXIII.

In every Obtuse-angled Triangle, (as  $BCD$ ) the Square of the Side subtending the obtuse Angle (as  $D$ ) is greater than the Squares of the other two Sides ( $B$  and  $C$ ) by a double Rectangle made of one of the Sides (as  $B$ ) and the Segment or Part of that Side produc'd, (as  $a$ ) until it meet with the Perpendicular ( $P$ ) let fall upon it. (12. e. 2.)

That is,  $DD = BB + CC + 2Ba$ .

Demonstration.

First	1	$DD = PP + aa + 2Ba + BB$
And	2	$CC = PP + aa$
1 — 2	3	$DD - CC = 2Ba + BB$
1 + CC	4	$DD = BB + CC + 2Ba$



Corollary.

Hence it is evident, that if the Sides of any obtuse-angled Triangle are given, the Segment ( $a$ ) of the Side produc'd, or the Perpendicular ( $P$ ) may be easily found.

### THEOREM XXIV.

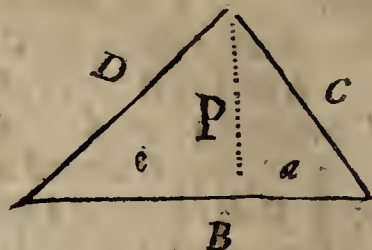
If a Perpendicular (as  $P$ ) be let fall in any acute-angled Triangle, (as  $BCD$ ) the Square of either of the two Sides (as  $D$ ) is less than the Squares of the other Side, and that Side upon which the Perpendicular falls (viz.  $C$  and  $B$ ) by a double Rectangle made of the Side  $B$ , and that Segment or Part of it (viz.  $a$ ) which lies next to the Side  $C$ . (13. e. 2.)

That is,  $DD + 2Ba = BB + CC$ .

Demon-

## Demonstration.

First	1	$DD = PP + ee$	} By Theo. II.		
And	2	$CC = PP + aa$			
But	3	$B - a = e$	by Figure.		
3	⊙	2	4	$BB - 2Ba + aa = ee$	
4	—	a	a	5	$BB - 2Ba = ee - aa$
1	—	2	6	$DD - CC = ee - aa$	
5,		6	7	$DD - CC = BB - 2Ba$	
7	+	8	$DD + 2Ba = BB + CC$		



Q. E. D.

## Corollary.

Hence it follows, that if the Sides of any acute-angled Triangle be known, the Perpendicular (P) and the Segments of the Side whereon it falls, (*viz.* a. e.) may be easily found.

## C H A P. IV.

*The Solution of several Easy Problems in Plane Geometry, whereby the Learner may (in Part) perceive the Application, or Use of the foregoing Theorems.*

Note, When a Line, or the Side of any plane Triangle, is any Way cut into two (or more) Parts, either by a perpendicular Line let fall upon it, or otherwise, those Parts are usually call'd Segments; and so much as one of those Parts is longer than the other, is call'd the Difference of the Segments.

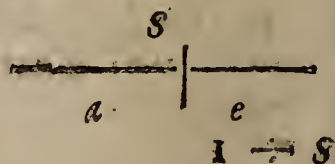
And when any Side of a Triangle, or any Segment of its Side is given, it is usually mark'd with a small Line cross it, thus —|— and those Sides, or Parts of Sides, that are sought, are mark'd with four Points, thus —::—

## P R O B L E M I.

*To cut or divide a given Right-line (as S) into extreme and mean Proportion, (II. e. 2.)*

That is, to divide a Line so, that the Square of the greater Segment (or Part) a, may be equal to the Rectangle made of the whole Line S, and the lesser Segment e.

*Viz.* | 1 |  $Se = aa$ , by the Problem.  
And | 2 |  $S - a = e$ , for  $S = a + e$





1	$\div S$	3	$\frac{aa}{S} = e$
2 and 3		4	$\frac{aa}{S} = S - a$ . By Axiom 5.
4	$\times S$	5	$aa = SS - Sa$
5	$+ Sa$	6	$aa + Sa = SS$
6, Solved		7	$a = \sqrt{SS + \frac{1}{4}SS} : \frac{1}{2} S$ . See Pages 195. 196.

Note, The last Problem cannot be truly answer'd by Numbers; but Geometrically it may be perform'd thus.

1. Make a Square; whose Side is  $= S$  the given Line, and bisect one of its Sides in the Middle, as at C; upon the Point C describe such a Semi-circle, as will pass through the remotest Points of the Square, and complete its Diameter.



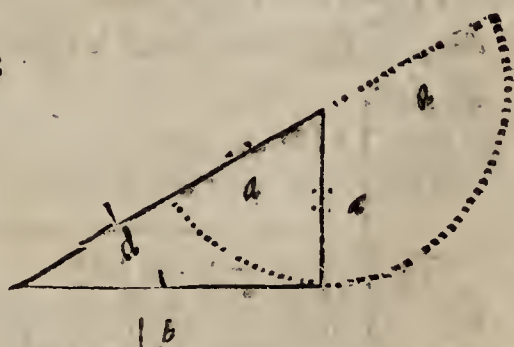
2. Then will either Part of the Diameter, on each End of the Side S, be  $= a$ , the greater Segment sought.

But  $a + S : S :: S : a$ . By Theorem 13.  
Ergo,  $aa + Sa = SS$ . Which was to be done.

## P R O B L E M II.

The Base of any Right-angled Triangle, and the Difference between the Hypotenuse and Cathetus or Perpendicular being given, to find the Cathetus, &c.

Let	1	$b = 72$
	2	$d = 32$
And	3	$a = \text{Cathetus sought}$
Then	4	$bb + aa = dd + 2da + aa$
		By Theorem II.
4 — aa	5	$bb = dd + 2da$
5 — dd	6	$2da = bb - dd$
6 $\div 2d$	7	$a = \frac{bb - dd}{2d} = 65$



Or 8  $b : d + 2a :: d : b$ . By Theorem 13.  
8  $\therefore$  9  $bb = dd + 2da$ . As before at the 5th Step.

T :

Here

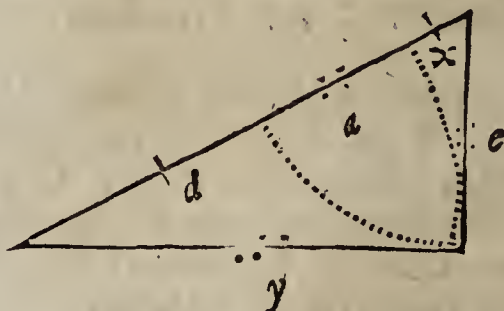
# 322 Elements of Geometry. Part III.

Here you see, that either Way raises the same *Æquation*; neither is there any constant Method or Road to be observ'd in solving Geometrical Problems, but every one makes use of such Ways and *Theorems* as happen to come first into their Mind, the Result being every Way the same.

## P R O B L E M III.

*The Difference between the Base and Hypothenufe of any Right-angled Triangle, and the Difference between the Cathetus and Hypothenufe being both given, to find the Triangle.*

Let  $\begin{cases} 1 & d = 32 \\ 2 & x = 25 \end{cases}$   
 And  $3 \quad d + x + a = \text{Hypot.}$   
 Then  $\begin{cases} 4 & d + a = y \\ 5 & x + a = e \end{cases}$  by the Prob.



$4 \odot 2 \quad 6 \quad dd + 2da + aa = yy$   
 $5 \odot 2 \quad 7 \quad xx + 2xa + aa = ee$   
 $3 \odot 2 \quad 8 \quad dd + 2dx + 2da + 2xa + xx + aa = \square \text{ Hypothenufe.}$   
 $6 + 7 \quad 9 \quad dd + 2da + 2xa + xx + 2aa = yy + ee$

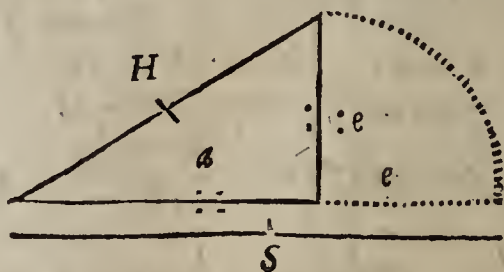
The two last Steps are equal; by *Theorem II*. Consequently, if those Things that are equal in both be taken away, the Remainders will be equal. By *Axiom 2*.

That is,  $10 \quad aa = 2dx = 1600$   
 $10 \text{ w } 2 \quad 11 \quad a = \sqrt{2dx} = 40$   
 $1 + 11 \quad 12 \quad d + a = 72 = y \quad \text{The Base.}$   
 $2 + 11 \quad 13 \quad x + a = 65 = e \quad \text{The Cathetus.}$   
 $1 + 2 + 11 \quad 14 \quad d + x + a = 97 \quad \text{The Hypothenufe.}$

## P R O B L E M IV.

*The Hypothenufe, and the Sum of the other two Sides of any Right-angled Triangle being given, thence to find the Sides.*

Let  $1 \quad H = 97$   
 And  $2 \quad a + e = S = 137$   
 By Fig.  $3 \quad aa + ee = HH$   
 $2 \odot 2 \quad 4 \quad aa + 2ae + ee = SS$   
 $4 - 3 \quad 5 \quad 2ae = SS - HH$   
 $3 - 5 \quad 6 \quad aa - 2ae + ee = 2HH - SS$   
 $6 \text{ w } 2 \quad 7 \quad a - e = \sqrt{2HH - SS}$



$2 + 7$

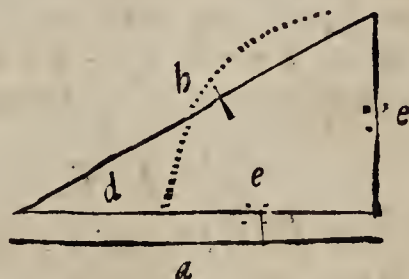


$2 + 7$	$8$	$2a = S + \sqrt{2HH - SS} = 144$	
$8 \div 2$	$9$	$a = \frac{S + \sqrt{2HH - SS}}{2} = 72$	The Base requir'd.
$2 - 9$	$10$	$e = \frac{S - \sqrt{2HH - SS}}{2} = 65$	The Cathetus.

### PROBLEM V.

*The Hypothenuse, and the Difference of the other two Sides of any Right-angled Triangle being given, to find the Sides.*

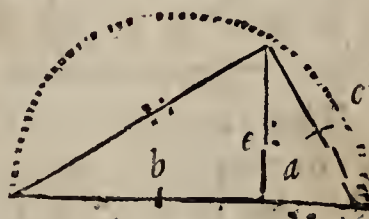
Let	1	$h = 97$	As before.
And	2	$a - e = d = 7$	Quere $a$ .
By Fig.	3	$aa + ee = hh$	
$2 \odot 2$	4	$aa - 2ae + ee = dd$	
$3 - 4$	5	$2ae = hh - dd$	
$3 + 5$	6	$aa + 2ae + ee = 2hh - dd$	
$6 \div 2$	7	$a + e = \sqrt{2hh - dd}$	
$2 + 7$	8	$2a = d + \sqrt{2hh - dd} = 144$	
$8 \div 2$	9	$a = 72$	
$7 - 2$	10	$2e = \sqrt{2hh - dd} - d = 130$	
$1 \div 2$	11	$e = 65$	



### PROBLEM VI.

*In any Right-angled Triangle, either the Base, or Cathetus, and the Alternate Segment of the Hypothenuse (made by a Perpendicular let fall from the Right Angle) being given, to find the other Segment.*

Let	1	$c = 45$	The Cathetus.
And	2	$b = 48$	The alternate Seg.
Then	3	$b : e :: e : a$	Quere $a$
$3 \therefore$	4	$ba = ee$	
Again	5	$cc - aa = ee$	By Theor. II.
$4, 5$	6	$ba = cc - aa$	
$6 + aa$	7	$aa + ba = cc$	
$7, C \square$	8	$aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb$	
$8 \div 2$	9	$a + \frac{1}{2}b = \sqrt{cc + \frac{1}{4}bb}$	
$9 - \frac{1}{2}b$	10	$a = \sqrt{cc + \frac{1}{4}bb} - \frac{1}{2}b = 27$	And so on for $e$ , &c.



I shall now shew the Geometrical Construction, or Solution of the three Cases of Quadratic Equations, promis'd in Page 202. Let the first Example be that above, viz.  $aa + ba = cc$ . Case 1.

Make the Co-efficient  $b$ , and the Root of the Resolvend, (which is here)  $c$ , into a Right-angled Parallelogram. Per Problem 13. Chap. 2.

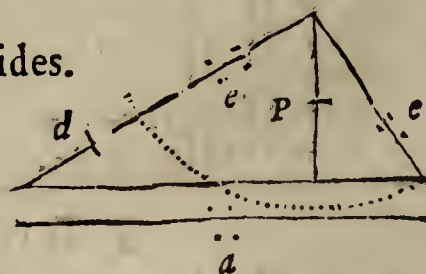
And upon the middle Point of the Side  $= b$  describe such a Semi-circle, as will pass through the remotest Points (or Angles) of the Parallelogram, completing its Diameter, as in the annexed Scheme. Then will either Part of the Diameter on each End  $b$ , be equal to  $a$ , the other Part will be  $a + b$ , and the Side  $c$  will be a mean Proportional between them. That is,  $a + b : c :: c : a$ . By Theorem 13. Consequently  $aa + ba = cc$ . Which was to be done.



### PROBLEM VII.

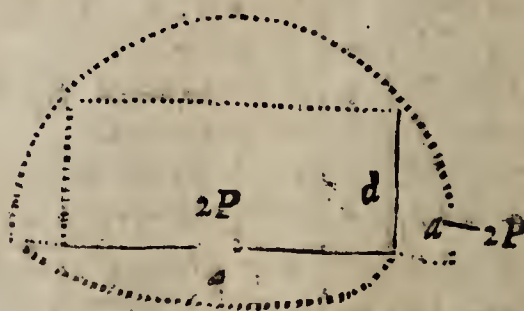
The Difference between the Base and Cathetus of any Right-angled Triangle, and the Perpendicular let fall from the Right Angle upon the Hypothenuse, being given; Thence to find the Hypothenuse, &c.

Let	1	$d = 15$	The Differ. of the Sides.
And	2	$p = 36$	
Quere a	3	$a =$	The Hypothenuse.



By Fig.	4	$d + e : p :: a : e$	
4 ::	5	$le + ee = pa$	
Again	6	$dd + 2de + 2ee = aa$ .	By Theorem 11.
5 x 2	7	$2de + 2ee = 2pa$	
6 - 7	8	$dd = aa - 2pa$	Case 2.
8 C □	9	$aa - 2pa + pp = dd + pp = 1521$ .	
9 ws 2	10	$a - p = \sqrt{dd + pp} = 39$	
10 + p	11	$a = p + \sqrt{dd + pp} = 75$ .	&c. for e per Step 5.

The Geometrical Construction of this Case 2. viz.  $aa - 2pa = dd$ . may be perform'd in the very same Manner as the last Case was; that is, by making a Right-angled Parallelogram of the Co-efficient  $2p$ , and the  $\sqrt{dd}$ , viz.  $d$ , &c. as in the annexed Figure.



Then



Then will the Greater Part of the Diameter to one End of the Parallelogram be  $= a$ , and the lesser Part will be  $a - 2p$ .

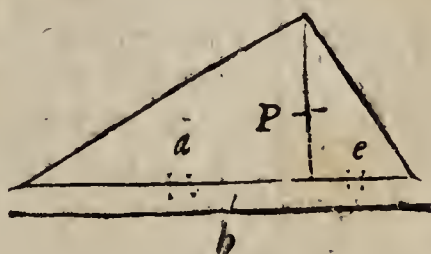
For  $a : d :: d : a - 2p$ . By *Theorem 13*.

Consequently,  $aa - 2pa = dd$ . Which was to be done.

## P R O B L E M VIII.

*The Hypothenuse of any Right-angled Triangle, and the Perpendicular let fall from the Right Angle upon the Hypothenuse, being given, To find the Greater Segment of the Hypothenuse, &c.*

Let	1	$h = 75$	The Hypothenuse
And	2	$p = 36$	
Then	3	$a + e = h$	Quere $a$
<hr/>			
per Fig.	4	$a : p :: p : e$	
4 ::	5	$\frac{pp}{a} = e$	
3 — $a$	6	$h - a = e$	
5, 6	7	$h - a = \frac{pp}{a}$	
7 $\times a$	8	$ha - aa = pp$	Case 3.
8 $\pm$	9	$aa - ha = -pp$	
9 $C \square$	10	$aa - ha + \frac{1}{4}bb = \frac{1}{4}bb - pp = 110, 25$	
10 w 2	11	$a - \frac{1}{2}h = \sqrt{\frac{1}{4}bb - pp} = 10, 5$	
11 $+$ $\frac{1}{2}h$	12	$a = \frac{1}{2}h + \sqrt{\frac{1}{4}bb - pp} = 48$	Or $a = 27$



The Geometrical Construction of *Case 3*. viz.  $ha - aa = pp$ , may be thus perform'd. Draw a Right Line (of any convenient length at Pleasure) and near its Middle erect a Perpendicular  $= p$ . viz. of the same length with the Root of the Resolvend. From the Top Point or upper End of that Perpendicular, set off half the Length of the Co-efficient, viz.  $\frac{1}{2}h$ , and upon the Point where  $\frac{1}{2}h$  just touches the first Line (with the same Distance) describe a Semi-circle; then will its Diameter  $b$  be cut by the Perpendicular  $p$  into two Segments, which are the two Values of the Root  $a$ , viz. the greater and lesser Roots, both taken together being always equal to the Co-efficient, (vide Page 201.)



For  $h - a : p :: p : a$  by *Theorem 13*.

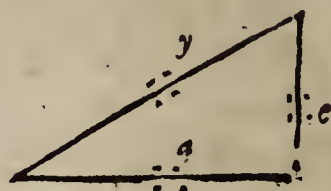
Ergo  $ha - aa = pp$ . Which was to be done.

PROBLEM

## PROBLEM IX.

The Perimeter, viz. the Sum of all the three Sides of any Right-angled Triangle, and its Area being given, thence to find each Side.

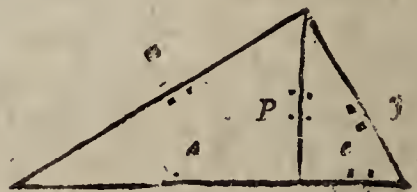
Viz. Let	1	$a + e + y = s = 234$	The Sum of the Sides.
And	2	$\frac{1}{2}ae = A$	The Area = 2340
Again	3	$aa + ee = yy$	By Figure
2 × 4	4	$2ae = 4A$	
3 + 4	5	$aa + 2ae + ee = yy + 4A$	
1 - y	6	$a + e = s - y$	
6 ⊙ 2	7	$aa + 2ae + ee = ss - 2sy + yy$	
5, 7	8	$yy + 4A = ss - 2sy + yy$	
8 +	9	$2sy = ss - 4A = 45396$	
9 ÷ 2s	10	$y = \frac{ss - 4A}{2s} = \frac{ss}{2s} - \frac{2A}{s} = 97$	The Hypothenufe
6, 10	11	$a + e = s - y = 137$	
3 - 4	12	$aa - 2ae + ee = yy - 4A = 49$	
12 √ 2	13	$a - e = \sqrt{49} = 7$	
11 + 13	14	$2a = 137 + 7 = 144$	
13 ÷ 2	15	$a = 72$	The Base.
11 - 15	16	$e = 137 - 72 = 65$	The Cathetus.



## PROBLEM X.

In any Right-angled Triangle a Perpendicular being let fall from the Right Angle upon the Hypothenufe, if the Sum of each Segment, when added to its adjacent or next Side, be given, thence to find each Side, and the Segments.

Viz. If	1	$a + u = s = 108$	
And	2	$e + y = z = 72$	
To find		$a, e, u, y, \text{ and } p$	
1 - a	3	$u = s - a$	
3 ⊙ 2	4	$uu = ss - 2sa + aa$	
4 - aa	5	$uu - aa = ss - 2sa = pp$	
2 - e	6	$z - e = y$	
6 ⊙ 2	7	$zz - 2ze + ee = yy$	
7 - ee	8	$zz - 2ze = yy - ee = pp$	
5, 8.	9	$zz - 2ze = ss - 2sa$	
By Fig.	10	$a : p :: p : e$	
10 ∴	11	$ae = pp$	
5, 11	12	$ae = ss - 2sa$	



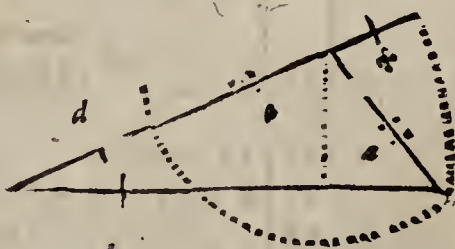


12 $\div a$	13	$e = \frac{ss - 2sa}{a}$
13 $\times 22$	14	$2ze = \frac{2zss - 4zsa}{a}$
9 $\div 14$	15	$zz = ss - 2sa + \frac{2zss - 4zsa}{a}$
15 $\times a$	16	$zza = ssa - 2saa + 2zss - 4zsa$
16 $\div$	17	$2saa + zza + 4zsa - ssa = 2zss$
17 $\div 25$	18	$aa + \frac{zza}{25} + 2za - \frac{1}{2}sa = 25$
Substitute	19	$2x = \frac{zz}{25} + 2z - \frac{1}{2}s = 114$
Then	20	$aa + 2xa = 25 = 7776$
20 $C \square$	21	$aa + 2xa + xx = 25 + xx = 11025$
21 $uw 2$	22	$a + x = \sqrt{25 + xx} = 105$
22 $- x$	23	$a = \sqrt{xs + xx} - x = 48$
1 $- 23$	24	$u = 60 = \text{The Base.}$
per 13	25	$e = \frac{ss}{a} - 2s = 27$
2 $- 25$	26	$y = 45 = \text{the Cathetus.}$
23 $+ 25$	27	$a + e = 75 = \text{the Hypothenufe.}$

## PROBLEM XI.

*The Difference of the Sides of any Oblique-angled plane Triangle, the Difference of the Segments of the Base, and the Difference between the greater Side, and the Base, being given, to find the Base, &c.*

Let	1	$d = \text{the Difference of the Sides} = 405$
	2	$b = \text{the Difference of the Segments} = 495$
	3	$x = 165 \text{ the Difference of the greater Side and Base.}$
And	4	$a = \text{the least Side}$
Then	5	$d + a + x = \text{the Base}$
And	6	$d + a + x : d + 2a :: d : b$
		By Theorem 16.
6 $\therefore$	7	$db + ba + bx = dd + 2da$
7 $\div$	8	$2da - ba = db + bx - dd$
8 $\div 2d - b$	9	$a = \frac{db + bx - dd}{2d - b} = \frac{118125}{315} = 375$
1 $\div$	10	$d + a = 780 = \text{the greatest Side.}$
3 $\div$	11	$d + a + x = 945 = \text{the Base.}$



## PROBLEM XII.

The Difference of the Sides of any plane Triangle; the Difference of the Segments of the Base, and the Perpendicular let fall from the vertical Angle, being given, thence to find all the Sides.



Let	1	$d = 405$	} as before.
And	2	$b = 495$	
Quere	3	$p = 300$	
	4	$a =$ the least Segment.	
Then	5	$b + 2a : d + 2e :: d : b$	
5	6	$bb + 2ba = dd + 2de$	
6 — dd	7	$bb - dd + 2ba = 2de$	
Substitute	8	$2x = bb - dd = 81000$	
7, 8	9	$2x + 2ba = 2de$	
÷ 2d	10	$\frac{x + ba}{d} = e$	
But	11	$pp + aa = ee$ By Theorem II.	
30 ⊙ 2	12	$\frac{xx + 2xba + bbaa}{dd} = ee$	
11, 12	13	$\frac{xx + 2xba + bbaa}{dd} = pp + aa$	
13 × dd	14	$xx + 2xba + bbaa = ppdd + ddaa$	
14 ±	15	$bbaa - ddaa + 2xba = ppdd - xx$	
8, 15	16	$2xaa + 2xba = ppdd - xx$	
16 ÷ 2x	17	$aa + ba = \frac{ppdd}{2x} - \frac{1}{2}x$	
17 C □	18	$aa + ba + \frac{1}{4}bb = \frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x$	
18 w 2	19	$a + \frac{1}{2}b = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x}$	
19 — $\frac{1}{2}b$	20	$a = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x} - \frac{1}{2}b = 225$	
20 × 2	21	$2a = 450$	
2 + 21	22	$b + 2a = 945$ the Base.	
10, Num.	23	$e = 375 =$ the lesser Side.	
1 + 23	24	$d + e = 780 =$ the greater Side.	

## PROBLEM XIII.

The Sum of the two Sides of any plane Triangle, the Difference of the Segments of the Base, and the Perpendicular let



# Chap. 4. Of Resolving Problems. 329

let fall from the Vertical Angle upon the Base, being given,  
thence to find the Base, and the Sides.

Let  $\left\{ \begin{array}{l} 1 \ s = 1155 \text{ the Sum of the Sides.} \\ 2 \ d = 495 \text{ the Difference of the Segments.} \\ 3 \ p = 300 \text{ the Perpendicular.} \end{array} \right.$   
Put  $\left\{ \begin{array}{l} 4 \ a = \text{the least Segment.} \\ 5 \ e = \text{the least Side.} \end{array} \right.$   
Then  $6 \ d + 2a = \text{the Base.}$   
And  $7 \ s - 2e = \text{the Difference of the Sides.}$

PerFig.  $\left\{ \begin{array}{l} 8 \ d + 2a : s :: s - 2e : d \\ 9 \ aa + pp = ee \end{array} \right.$

9 w 2 10  $\sqrt{aa + pp} = e$   
8  $\therefore$  11  $dd + 2da = ss - 2se$

11  $+$  12  $2se = ss - dd - 2da$   
Suppose 13  $2x = ss - dd$

Then 14  $2se = 2x - 2da$   
14  $\div$  25 15  $e = \frac{x - da}{s}$

10; 15 16  $\frac{x - da}{s} = \sqrt{aa + pp}$

16  $\odot$  2 17  $\frac{xx - 2xda + ddaa}{ss} = aa + pp$

17  $\times$  ss 18  $xx - 2xda + ddaa = ssaa + sspp$

18  $+$  19  $ssaa - ddaa + 2xda = xx - sspp$

13, 19 20  $2xaa + 2xda = xx - sspp$

20  $\div$  2x 21  $aa + da = \frac{1}{2}x - \frac{sspp}{2x}$  &c. As before.

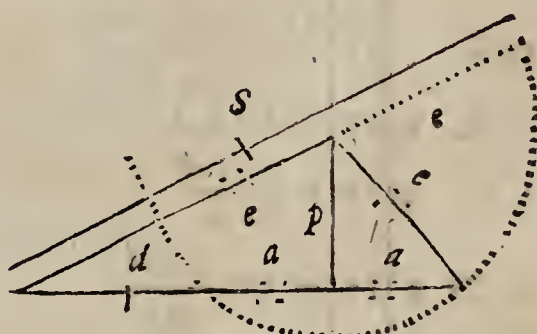
21, hence 22  $a = 225$

22  $\times$  2 23  $2a = 450$

2  $+$  23 24  $d + 2a = 945$  the Base.

10. Numb. 25  $e = 375$  the lesser Side.

1  $-$  25 26  $s - e = 780$  the greater Side.



## PROBLEM XV.

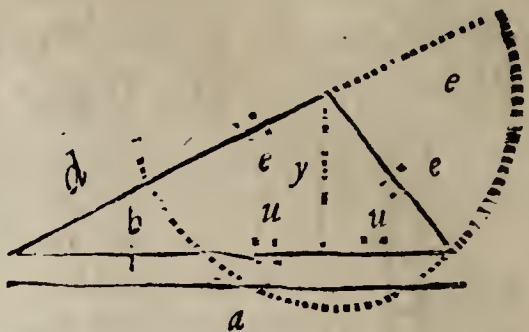
The Area of any oblique-angled Triangle; the Difference of the Sides, and the Difference of the Segments of the Base being given, thence to find the Base, &c.

Let  $\left\{ \begin{array}{l} 1 \ A = 141750 = \text{the Area.} \\ 2 \ d = 405 \\ 3 \ b = 495 \end{array} \right.$

ll u

Put

Put	5	$y = \text{the Perpendicular.}$
	5	$a = \text{the Base.}$
Then	6	$\frac{1}{2}ya = A$
Per Fig.	7	$a : d + 2e :: d : b$
7	8	$ba = dd + 2de$
8	9	$ba - dd = 2de$
9	10	$bbaa - 2ddba + dddd = 4ddee$
	11	$\frac{a - b}{2} = u \text{ the lesser Segment of the Base.}$
II	12	$\frac{aa - 2ba + bb}{4} = uu$
6	13	$ya = 2A$
13	14	$y = \frac{2A}{a}$
14	15	$yy = \frac{4AA}{aa}$
Per Fig.	16	$yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
10	17	$\frac{bbaa - 2ddba + dddd}{4dd} = ee$
16,	18	$\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
18	19	$\frac{bba^4 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bbaa}{4}$
19	20	$\{ bba^4 - 2ddba^3 + d^4aa = 16AAdd + dda^4$
20	21	$\{ - 2ddba^3 + ddbbaa$
21	22	$bba^4 - dda^4 + d^4aa - ddbbaa = 16AAdd$
	22	$aaaa - ddaa = \frac{16AAdd}{bb - dd}$
22	23	$aaaa - ddaa + \frac{1}{4}dddd = \frac{16AAdd}{bb - dd} + \frac{1}{4}d^4$
23	24	$aa - \frac{1}{2}dd = \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}d^4}$
24	25	$aa = \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd}$
25	26	$a = \sqrt{\frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd}} = 945.$

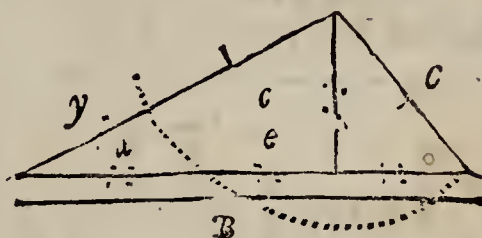




PROBLEM XV.

There is an oblique-angled plane Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base, the least Side, and the Base are given, And the Rectangle of the Difference of the Sides into the least Side is equal to the Square of the Difference of the Segments of the Base: Tis requir'd to find the Segments of the Base, &c.

Let	{	1	$c = 56 =$ the least Side.
		2	$B = 92 =$ the Base.
And		3	$a + 2e = B$
Put		4	$y =$ the Difference of the Sides.
Then		5	$cy = aa$ by the Question.
By Figure.		6	$B : 2c + y :: y : a$ , for $B = a + 2e$
6	$\therefore$	7	$Ba = 2cy + yr$
5	$\times$	8	$2cy = 2aa$
7	$-$	9	$Ba - 2aa = yy$
5	$\odot$	10	$ccyy = aaaa$
10	$\div$	11	$yy = \frac{aaaa}{cc}$
9		12	$Ba - 2aa = \frac{aaaa}{cc}$
12	$\times$	13	$ccBa - 2ccaa = aaaa$
13	$\div$	14	$ccB - 2cca = aaa$
14	$+ 2cca$	15	$aaa + 2cca = ccB$
15, in Num		16	$aaa + 6272a = 288512$



The Value of  $a$ , in this Equation, may be found as in the Examples, Page 238, viz. by putting  $r + e = a$ , &c. As in those Examples, you will find  $a = 37,55502$ , &c.

PROBLEM XVI.

The three Chords or Subtenses of three Arches completing a Semi-circle, being each given, thence to find the Diameter of that Circle.

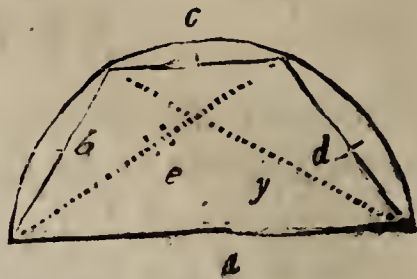
That is, any Trapezium being inscrib'd in a Semi-circle, if one of its Sides be the Diameter, and the other three Sides be given, thence to find the Diameter, or fourth Side.

U u 2

Let

Let  $\left\{ \begin{array}{l} 1 \quad b = 3 \\ 2 \quad c = 4 \\ 3 \quad d = 5 \end{array} \right\}$  the three Sides.  
 Quere  $\left\{ \begin{array}{l} 4 \quad a = \text{the Diameter sought.} \end{array} \right.$

Draw the two Diagonals  
 $e$  and  $y$



Then  $5 \quad ca + bd = ey$ . By Theorem 19.  
 And  $\left\{ \begin{array}{l} 6 \quad aa - bb = yy \\ 7 \quad aa - dd = ee \end{array} \right\}$  By Theorem 10 and 11  
 $5 \quad \odot \quad 2 \quad 8 \quad ccaa + 2bdca + bbdd = eeyy$   
 $6 \quad \times \quad 7 \quad 9 \quad aaaa - bbba - ddaa + bbdd = eeyy$   
 $8, = \quad 9 \quad 10 \quad aaaa - bbba - ddaa = ccaa + 2bdca$   
 $10 \div a \quad 11 \quad aaa - bba - dda = cca + 2bdc$   
 $11 - caa \quad 12 \quad aaa - bba - dda - cca = 2bdc$   
 $12, \text{ Num. } 13 \quad aaa - 50a = 120$

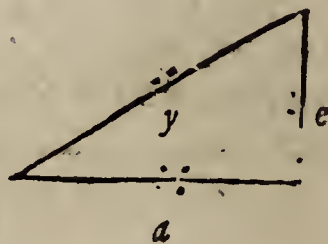
This Equation being solv'd, as in *Example 2, Page 240*, you will find  $a = 8,05581$ , &c.

## PROBLEM XVII.

In any Right-angled Triangle, the Area and the Sum of the Hypotenuse, when added to either Side, being given, thence to find the Sides, &c.

Suppose  $\left\{ \begin{array}{l} 1 \quad \frac{ae}{2} = A = 1350. \quad \text{The Area} \\ 2 \quad y + e = s = 120. \quad \text{The Sum, \&c.} \\ 3 \quad \text{Quere } a, e. \text{ and } y \end{array} \right.$

$1 \quad \times \quad 2 \quad 4 \quad ae = 2A$   
 $4 \div a \quad 5 \quad e = \frac{2A}{a}$   
 Per Fig.  $6 \quad aa + ee = yy$   
 $2 - e \quad 7 \quad y = s - e$   
 $5, \quad 7 \quad 8 \quad y = s - \frac{2A}{a}$   
 $8 \quad \odot \quad 2 \quad 9 \quad yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$   
 $5 \quad \odot \quad 2 \quad 10 \quad ee = \frac{4AA}{aa}$   
 $10 + aa \quad 11 \quad aa + ee = \frac{4AA}{aa} + aa$





6, 9, 11	14	$\frac{4AA}{aa} + aa = yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
12, That is	13	$aa = ss - \frac{4sA}{a}$
13 $\times$ a	14	$aaa = ssa - 4sA$
14 $\div$	15	$ssa - aaa = 4sA$
15, in Num.	16	$14400a - aaa = 648000$

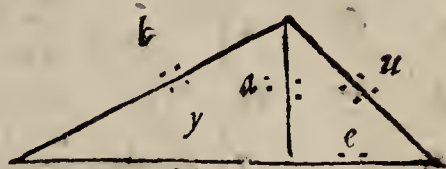
The Value of  $a$ , in this Equation, may be found as in the third Example, Page 241: That is, by making  $r + e = a$ , &c. it will be found that  $a = 60$ .

## PROBLEM XVIII.

There is an oblique-angled plane Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base, the Sum of each Segment of the Base, when added to its adjacent or next Side, and the Area of the Triangle, are given, To find the Perpendicular, and each Side.

Let	{	1	$y + b = z = 1500$	} Quere $y, b, e$ and $u$
		2	$e + u = s = 600$	
		3	$A = \text{the Area} = 141750$	
		4	$a = \text{the Perpendicular sought.}$	
		5	$y + e \times \frac{1}{2} a = A$	

5 $\times 2 \div$	a	6	$y + e = \frac{2A}{a}$	
Per Fig.	{	7	$yy + aa = bb$	
		8	$ee + aa = uu$	
1	—	y	9	$b = z - y$
2	—	e	10	$u = s - e$
9	⊙	2	11	$bb = zz - 2zy + yy$
10	⊙	2	12	$uu = ss - 2se + ee$
7,	11	13	$zz - 2zy = aa$	
8,	12	14	$ss - 2se = aa$	
13	$\div$	15	$zz - aa = 2zy$	
14	$\div$	16	$ss - aa = 2se$	
15 $\div$	2z	17	$\frac{zz - aa}{2z} = y$	
16 $\div$	2s	18	$\frac{ss - aa}{2s} = e$	
17 $+$	18	19	$\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = y + e$	



Having found the Value of  $a$ , from the 24th Step,  $e$  and  $y$  will be easily found by these 2 Steps. And  $b, u$  by the 9th and 10th Steps.

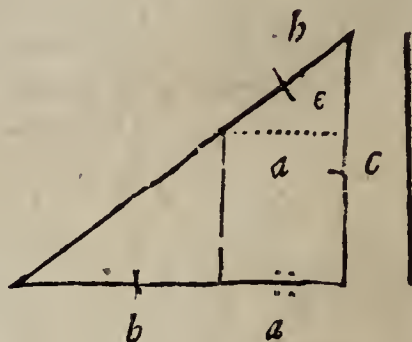
6,	19	20	$\frac{2z - aa}{2z} + \frac{ss - aa}{2s} = \frac{2A}{a}$
20	X	22	$2z - aa + \frac{2ss - 2aa}{s} = \frac{4zA}{a}$
21	X	5	$2zs - saa + 2ss - 2aa = \frac{4zAs}{a}$
22	X	a	$2zsa - saaa + 2ssa - 2aaa = 4zAs$
23, Num.		24	$900000a - aaaa = 243000000$

Here  $a = 300$  found as in the last Problem.

### PROBLEM XIX.

There is a Right-angled Triangle, wherein a Right-line is drawn parallel to the Cathetus; there is given the Cathetus, that Segment of the Hypothenufe next to the Cathetus, and the alternate Segment of the Base, thence to find the Base, &c.

Viz. Let	1	$b = 20 . c = 24 . \text{ and } h = 15$
Then	2	$b + a = \text{the Base. Quere } a$
Here	3	$b + a : c :: a : e$ Per Figure.
And	4	$aa + ee = hb$ Per Figure.
3	5	$\frac{ca}{b + a} = e$
5	6	$\frac{ccaa}{bb + 2ba + aa} = ee$
4	7	$hb - aa = ee$
6,	8	$\frac{ccaa}{bb + 2ba + aa} = hb - aa$
8	9	$ccaa = hbbb - bbba + 2hbba - 2ba^3 + hbba - a^4$
9	10	$a^4 + 2baaa + ccaa + bbba - hbba - 2hbba = hbbb$
That is	11	$aaaa + 40aaa + 751aa - 9000a = 90000$



For a Solution of this Equation, let it be made

$$aaaa + baaa + caa - da = G \quad \text{Viz. } \begin{cases} b = 40 & c = 751 \\ d = 9000 & G = 90000 \end{cases}$$

Put  $r + e = a$

Then

$$\left\{ \begin{array}{l} r + 4rrre + 6rree = a^4 \\ brrr + 3brre + 3bree = baaa \\ crr + 2cre + cee = ccaa \\ -dr - de = -da \end{array} \right\} = G = 90000$$

Let  $r = 10$

Then



Then 
$$\left\{ \begin{array}{r} + 10000 + 4000e + 600ee \\ + 40000 + 12000e + 1200ee \\ + 75100 + 15020e + 751ee \\ - 90000 - 9000e \end{array} \right\} = G = 90000$$

That is,  $35100 + 22020e + 2551ee = 90000$

Hence it will be  $22020e + 2551ee = 54900$

Consequently,  $8,63e + ee = 21,52 = D$

And  $\frac{D}{8,63+e} = e$

Operation  $8,63 \ ) \ 21,52 \ (2,1 = e$

$+ e = 2,1$

1 Divisor  $= 10$

2 Divisor  $= 10,7$

$20$

$1,52$

$1,07$

First  $r = 10$

$+ e = 2,1$

$45 \ \&c. \ r + e = 12,1 = r$  for a second

Operation, which being involv'd, and multiply'd into the Co-efficients as before, will produce these Numbers.

$$\left\{ \begin{array}{r} + 21435,8881 + 7086,24e + 878,46ee \\ + 70862,4400 + 17569,20e + 1452,00ee \\ + 109953,9100 + 18174,20e + 751,00ee \\ - 108900,0000 - 9000,00e \end{array} \right\} = G$$

Viz.  $93352,2381 + 33829,64e + 3081,46ee = 90000$

Here because  $93352,2381 > 90000$ . Therefore  $12,1 > a$ , and therefore it must needs be made  $r - e = a$ , which will produce the same Numbers, only all the second Signs must be chang'd.

Thus,  $93352,2381 - 33829,64e + 3081,46ee = 90000$  from whence will arise this Equation,

$+ 33829,64e - 3081,46ee = 3352,2381$

Consequently,  $10,9784e - ee = 1,08787332 = D$

Operation  $10,9784 \ ) \ 1,08787332 \ (0,0999 = e$

$- e = ,0999$

1. Divisor  $10,88$

2. Divisor  $10,879$

3. Divisor  $10,8785$

$9792$

$108673$

$97911$

$1076232$

$979065$

$\&c.$

Last  $r = 12,1$

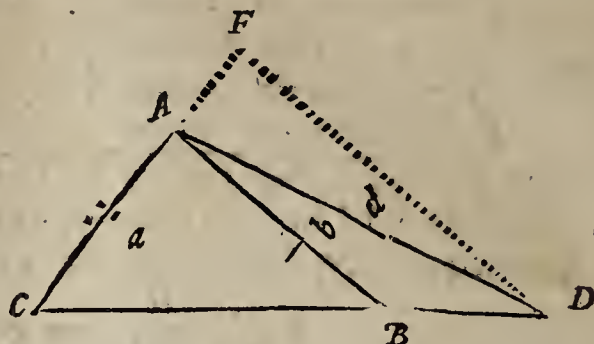
$- e = 0,0999$

$r - e = 12,0001 = a$

## PROBLEM XX.

In the oblique-angled Triangle  $CAD$ , there is given the Side  $AD$ , and the Sum of the Sides  $AC + CD$ ; also within the Triangle there is given the Line  $AB$  perpendicular to the Side  $CA$ , thence to find the Side  $CA$ , &c. Let

Let  $\left\{ \begin{array}{l} 1 \text{ } CA + CD = s = 51 \\ 2 \text{ } AD = d = 32 \\ 3 \text{ } AB = b = 21 \\ \text{And } 4 \text{ } CA = a \text{ sought.} \\ \text{Then } 5 \text{ } s - a = CD \end{array} \right.$



Suppose the Line  $DF$

Parallel to  $AB$ , and  $CA$  produc'd to  $E$ .

Then  $\triangle CAB$ , and  $\triangle CFD$  will be alike.

And  $6 \text{ } BC : CA :: DC : CF$ .

But  $7 \text{ } BC = \sqrt{bb + aa}$ . Let  $AF = e$ , and  $FD = y$

6, 7  $8 \text{ } \sqrt{bb + aa} : a :: s - a : a + e$

8  $\therefore 9 \text{ } \frac{sa - aa}{\sqrt{bb + aa}} = a + e$

5  $\odot 2 \text{ } 10 \text{ } ss - 2sa + aa = \square CD$

Per Fig.  $11 \text{ } ss - 2sa + aa = aa + 2ae + ee + yy = \square CF + \square FD$

11  $- aa \text{ } 12 \text{ } ss - 2sa = 2ae + ee + yy$

But  $13 \text{ } dd = ee + yy = \square AE + \square FD$

12  $- 13 \text{ } 14 \text{ } ss - 2sa - dd = 2ae$

Let  $15 \text{ } 2x = ss - dd$

14, 15  $16 \text{ } x - sa = ae$

16  $\div a \text{ } 17 \text{ } \frac{x - sa}{a} = e$

17  $+ a \text{ } 18 \text{ } \frac{x - sa + aa}{a} = a + e$

9  $\odot 2 \text{ } 19 \text{ } \frac{ssaa - 2saaa + a^4}{bb + aa} = \square a + e$

18  $\odot 2 \text{ } 20 \text{ } \frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} = \square a + e$

18, 19  $21 \text{ } \left\{ \begin{array}{l} \frac{ssaa - 2saaa + a^4}{bb + aa} = \\ \frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} \end{array} \right.$

This Equation being brought out of the Fractions, and into Numbers will become,

$$-2018aaaa + 125409aaa - 2464230,25aa + 35468307a = 274183922,25$$

which being divided by 2018, the Co-efficient of the highest Power of  $a$  will

be



$$\text{be } \begin{cases} -a^4 + 62,1459 a^3 - 1221,125 aa + 17575,9697 a \\ = 135869,138875, \&c. \end{cases}$$

And from hence the Value of  $a$  may be found, as in the last Problem, due Regard being had to the Signs of every Term.

This Work of reducing, or preparing  $\mathcal{A}$ equations for a Solution by Division, hath always been taught both by the Antient and Modern Writers of *Algebra*, as a Work so necessary to be done, that they do not so much as give a Hint at the Solution of any adfectèd  $\mathcal{A}$ equation without it.

Now it very often happens, that in dividing all the Terms of an  $\mathcal{A}$ equation, some of their Quotients will not only run into a long Series, but also into imperfect Fractions, (as in this  $\mathcal{A}$ equation above) which renders the Solution both tedious and imperfect.

To remedy that Imperfection, I shall here shew how this  $\mathcal{A}$ equation (and consequently any other) may be resolv'd without such Division, or Reduction.

$$\text{Let } b = 2018. \quad c = 125409. \quad d = 2464230,25 \\ f = 35468307. \quad \text{And } G = 274183922,25$$

Then the precedent  $\mathcal{A}$ equation will stand thus,

$$-baaaa + caaa - daa + fa = G$$

Put  $r + e = a$ . As before.

$$\text{Then will } \left\{ \begin{array}{l} -br^4 - 4brrre - 6brree = -ba^4 \\ + cr^3 + 3crre + 3cree = +ca^3 \\ -drr - 2dre - dee = -daa \\ + fr + fe \dots\dots = +fa \end{array} \right\} = G.$$

This is plain and easily conceiv'd; The next thing will be how to estimate the first Value of  $r$ ; and to perform that, let  $G$  be divided by  $b$ , only so far as to determine how many Places of whole Numbers there will be in the Quotient; consequently, how many Points there must be, (according to the Height of the  $\mathcal{A}$ equation.

$$\text{Thus } b = 2018 \quad G = 274183922,25 \quad ( \begin{array}{r} 130000 \\ 2018 \\ \hline 7238 \end{array} \&c.$$

Now from hence one may as easily guess at the Value of  $r$ , as if all the Terms had been divided. That is, I suppose  $r = 10$ , which being involv'd, &c. as the Letters above direct will be

X x

—20180000

$$\begin{array}{r}
 - 20180000 - 8072000e - 1210800ee \\
 + 125409000 + 37622700e + 3762270ee \\
 - 246423025 - 49284605e - 2464230,25ee \\
 + 354683070 + 35468307e \\
 \hline
 \text{viz. } 213489045 + 15734402e + 8723975ee = 274183922,25 \\
 \text{Hence } 15734402e + 87239,75ee = 60694877,25 \\
 \text{Consequently, } 180,3e + ee = 695,72 = D
 \end{array}$$

$$\text{And } \frac{D}{180,3 + e} = e$$

$$\begin{array}{r}
 \text{Operation } 180,3 \overline{) 695,72} \quad (3,7 = e \\
 + e = 3,7 \quad 549 \\
 \text{1. Divisor} = 183 \quad 146,72 \\
 \text{2. Divisor} = 184,0 \quad 128,80 \\
 \quad \quad \quad \&c.
 \end{array}$$

$$\text{First } r = 10$$

$$+ e = 2,7$$

$$r + e = 12,7 = r \text{ for}$$

a second Operation, with which you may proceed, as in the last Problem, and so on to a third Operation, if Occasion require such Exactness. But this may be sufficient to shew the Method of resolving any affected *Æ*quation, without reducing it; which is not only very exact, but also very ready in Practice, as will fully appear in the last Chapter of this Part, concerning the Periphery and Area of the Circle, &c. wherein you will find a farther Improvement in the Numerical Solution of High *Æ*quations than hath hitherto been publish'd.

## CHAP. V.

### *Practical Problems and Rules for finding the Superficial Contents, or Area's of Right-lin'd Figures.*

**B**Efore I proceed to the following Problems, it may be convenient to acquaint the Learner, that the Superficies or Area of any Figure, whether it be right-lin'd or circular, is compos'd or made up of Squares, either greater, or less, according to the different Measures by which the Dimensions of the Figure are taken or measur'd.

That is, if the Dimensions are taken in Inches, the Area will be compos'd of Square Inches; if the Dimensions are taken in Feet, the Area will be compos'd of Square Feet; if in Yards, the Area will be Square Yards; and if the Dimensions are taken by Poles or Perches, (as in Surveying of Land, &c.) then the Area will be Square Perches, &c. These Things being understood,

and



and the Definitions in the 283 and 284 Pages well consider'd, will help to render the following Rules very easy.

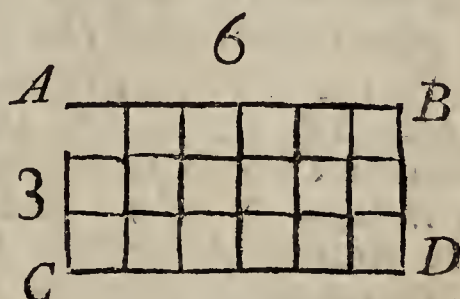
### PROBLEM I.

To find the Superficial Content, or Area of a Square; or of any right-angled Parallelogram.

Rule.  $\left\{ \begin{array}{l} \text{Multiply the Length into its Breadth, and the} \\ \text{Product will be the Area requir'd. (See Lem-} \\ \text{ma 1. Page 302.)} \end{array} \right.$

Example, Suppose the Line  $AB = 6$  Yards, and the Breadth  $AC$  or  $BD = 3$  Yards.

Then  $AB \times AC = 6 \times 3 = 18$  will be the Number of Square Yards contain'd in the Area of the Parallelogram  $ABCD$ .



This is so evident by the Figure only, that it needs no Demonstration.

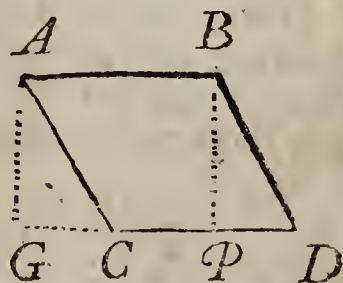
### PROBLEM II.

To find the Area of any oblique-angled Parallelogram, viz. either of a Rhombus or Rhomboides.

Rule.  $\left\{ \begin{array}{l} \text{Multiply the Length into its perpendicular Height,} \\ \text{(or Breadth) and the Product will be the} \\ \text{Area requir'd.} \end{array} \right.$

That is, the Side  $AB \times BP =$  the Area of the Rhombus  $ABCD$ . For if  $BP$  be drawn perpendicular to  $CD$ , and  $AG$  be made parallel to  $BP$ , then will  $GC = PD$ , and  $GP = CD$ . Consequently  $\triangle AGC = \triangle BPD$ , and  $\square ABGP =$  Rhombus  $ABCD$ .

But  $AB \times BP = \square ABGP$ . Therefore  $AB \times BP$ , or  $CD \times BP =$  the Area of the Rhombus  $ABCD$ .



Example, Suppose the Side  $AB = 23$  Inches, and the Perpendicular  $BP = 17,5$  Inches, (being the shortest or nearest Distance between the two Sides  $AB$  and  $CD$ .)

Then  $AB \times BP = 23 \times 17,5 = 402,5$  Square Inches, being the Area of the Rhombus requir'd.

The like may be done for any Rhomboides, whose Length and Perpendicular Breadth are given.

### P R O B L E M III.

*To find the superficial Content, or Area of any plane Triangle.*

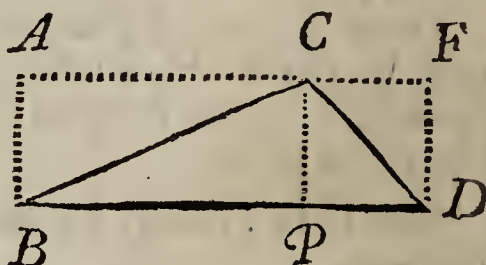
*Every plane Triangle is equal to half its circumscribing Parallelogram, (41. e. I.) which affords the following*

#### Rule.

*Multiply the Base of the given Triangle into Half its perpendicular Height, or Half the Base into the whole Perpendicular, and the Product will be the Area.*

That is,  $BD \times \frac{1}{2} CP$ , or  $\frac{1}{2} BD \times CP = \text{Area of } \triangle BCD$ . For  $AC = BP$ ,  $AB = CP$ , and  $BC$  is common to both  $\triangle \triangle$ , therefore  $\triangle ABC = \triangle BCP$ . And for the like Reasons  $\triangle CFD = \triangle CPD$ .

Therefore  $\triangle BCP + \triangle CPD = \frac{1}{2} \square ABFD$ . Consequently  $\frac{1}{2} BD \times CP$ , or  $BD \times \frac{1}{2} CP$  will be the Area of the  $\triangle BCD$ .



*Example, Suppose the Base  $BD = 32$  Inches, and the perpendicular Height  $CP = 14$  Inches.*

Then  $\frac{1}{2} BD \times CP = 16 \times 14 = 224$ . Or  $BD \times \frac{1}{2} CP = 32 \times 7 = 224$ .

Or thus,  $32 \times 14 = 448$ . Then  $2 \times 224 = 448$  ( $224 = \text{the Area of the Triangle } BCD \text{ in Square Inches}$ ).

### P R O B L E M IV.

*To find the Superficies, or, Area of any Trapezium.*

First, Divide the given Trapezium into two Triangles, by drawing a Diagonal from one of its acute Angles to the opposite Angle; and let fall two Perpendiculars (from the other two Angles) upon the Diagonal, as in the following Figure. Then

**Rule.**  $\left\{ \begin{array}{l} \text{Multiply Half the Diagonal into the Sum of the} \\ \text{two Perpendiculars, or Half the Sum of the} \\ \text{Perpendiculars into the Diagonal, and the} \\ \text{Product will be the Area.} \end{array} \right.$

That is,  $\frac{1}{2} AC \times BP + ED$ . Or  $AC \times \frac{1}{2} BP + \frac{1}{2} ED = \text{Area of the Trapezium } ABCD$ .

For the  $\triangle ABC$  is Half its circumscribing Parallelogram; And the  $\triangle ACD$  is also Half of its circumscribing Parallelogram, as hath been prov'd at the last Problem.

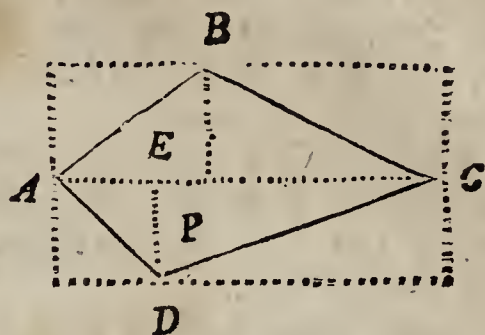
Consequently,



Consequently,  $BP + ED \times \frac{1}{2} AC$ , or  $\frac{1}{2} BP + \frac{1}{2} ED \times AC$  will be the Area of the Trapezium, As above.

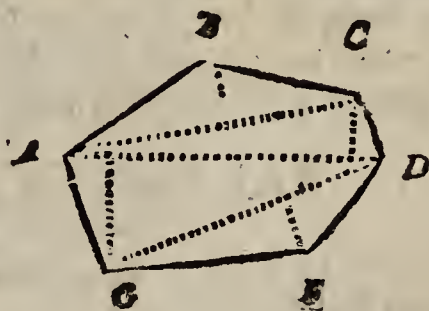
*Example,* Suppose the Diagonal  $AC = 33$  Feet, the Perpendicular  $BP = 15$  Feet, and the Perpendicular  $ED = 14$  Feet. Then  $BP + ED = 29$  Feet, and  $BP + ED \times \frac{1}{2} AC = 29 \times 16,5 = 478,5$ . Or  $AC \times \frac{1}{2} BP + \frac{1}{2} ED = 33 \times \frac{29}{2} = 478,5$ . Or thus,  $29 \times 33 = 957$ . Then

2) 957 ( 478,5 any of these Products are the Area of the Trapezium  $ABCD$ .



## PROBLEM V.

To find the Superficial Content or Area of any irregular Polygon or many-sided Figure, which by some Authors is called a Triangulate, because (as I suppose) it must be divided into Triangles, as in the annexed Figure  $ABCDEFG$ ; by which it is evident that the Sum of the Area's of all those Triangles, found as in the last Problem, &c. will be the Area of their circumscribing Polygon.



## PROBLEM VI.

To find the Superficies, or Area of any regular Polygon, viz. of any regular Pentagon, Hexagon, Heptagon, Octagon, &c.

### General Rule.

Multiply Half the Sum of its Sides into the Radius of the inscrib'd Circle, or Half the said Radius into the Sum of the Sides, and the Product will be the Area requir'd.

That is,  $\frac{AB + BD + DE + EF + FG + GH + HK + KA}{2} : \times CP$

= the Area of the annexed Octagon; wherein it is evident, that its Area is compos'd of so many equal Isosceles Triangles as there are Number of Sides in the Polygon, viz. of eight Isosceles Triangles, whose Bases are the Sides of the Octagon, viz.  $AB = BD = DE$ , &c. And the Sides of those Triangles,  $CA, CB, CD$ , &c. are the Radius's of the circumscribing Circle; and their perpendicular Heights, viz.  $CP$ , is the Radius of the inscrib'd Circle. But

But the Area of any one of those Triangles, is  $\frac{1}{2} AB \times CP$ , By Problem 3. Consequently the Sum of all their Area's will be,  $CP$  into Half the Sum of all their Bases, as above.

This being equally evident in all regular Polygons whatsoever, makes the Rule general for finding their Area's.

Now because it is requir'd to have the Radius of the propos'd Polygon's inscrib'd Circle, I shall here insert (and demonstrate) the Proportions that are between the Sides of several regular Polygons, and the Radius's both of their inscrib'd and circumscribing Circles; the one will help to delineate or project the Polygon, (if Occasion require it) and the other will help to find its Area.



### And First, Of an Equilateral Triangle.

The Side of any equilateral plane Triangle, is in Proportion to the Radius of

its { Circumscribing Circle, As 1 : To 0,57735027 &c.  
       { Inscrib'd Circle, As 1 : To 0,28867513 &c.

And to its Perpendicular Height, As 1 : To 0,86602540 &c.

That is, {  $AB : CD :: 1 : 0,57735027$   
           {  $AB : CG :: 1 : 0,28867513$

And  $AB : AG :: 1 : 0,86602540$

### Demonstration.

Let  $AB = BD = 1$ .

Then will  $BG = GD = 0,5$

But  $\square AB = \square BG = \square AG$

By Theorem 11.

That is,  $1 - 0,25 = 0,75 = \square AG$ .

Consequently,  $\sqrt{0,75} = 0,86602540$

$= AG$ :

Then  $AG : AB :: AB : AH$ . By Theorem 13.

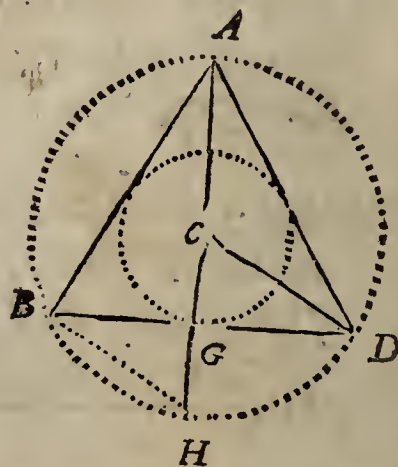
That is,  $0,8660254 : 1 :: 1 : 1,15470054 \text{ \&c.} = AH$ .

Then  $\frac{1}{2} AH = 0,57735027 = AC$ . Again,  $AG : DG :: DG : CG$ .

That is,  $0,8660254 : 0,5 :: 0,5 : 0,28867513 = CG$ . Q. E. D.

Now, by the Help of the first of these Proportions, it will be easy to resolve the following Problem.

P R O-





P R O B L E M VII.

The Side of any equilateral plane Triangle being given, To find its Area.

Example, Suppose the Side of the propos'd Triangle  $ABC$  to be 25 Inches, viz.  $AB = BC = CA = 25$  First  
 $1 : 0,8660254 :: AB = 25 : 21,650635 = BP$ . By Theorem 13.

Then  $AP (= \frac{1}{2} CA) \times BP =$  the Area of  $\triangle ABC$ . By Rule to Problem 3.

That is,  $12,5 \times 21,650635 = 270,6329$  the Area in Square Inches.

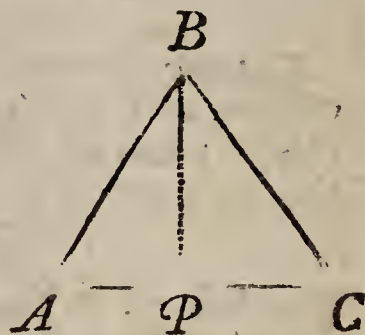
Or this Problem may be otherwise resolv'd, thus:

Let  $b = AP = \frac{1}{2} AC$ . Then  $2b = AB$ .

But  $\square AB - \square AP = \square BP$ . Per Theorem 11.

That is,  $4bb - bb = 3bb = \square BP$ . Consequently,  $\sqrt{3bb} = BP$

Then  $b\sqrt{3bb} = BP \times \frac{1}{2} AC$ . Viz.  $\sqrt{3bbbb} =$  the Area of the Triangle.



Secondly, For a Pentagon.

The Side of any regular Pentagon, is in Proportion to the Radius of its  
 { Circumscribing Circle, As 1 : To 0,85065080 &c.  
 { Inscrib'd Circle, As 1 : To 0,68819096 &c.  
 And to its Perpendicular Height, As 1 : To 1,53884176 &c.

Viz.  $\begin{cases} AB : AC :: 1 : 0,85065080 \\ AB : CH :: 1 : 0,68819096 \\ AB : AH :: 1 : 1,53884176 \end{cases}$

Demonstration.

Let  $AB = 1$ . And draw the Diagonals  $AD$ ,  $AF$  and  $DG$ , which will be equal to one another. Then will  
 $AG \times DF : + AD \times GF = AF \times DG$   
 By Theorem 19.

Consequently,  $AG \times DF = AF \times DG : - AD \times GF$

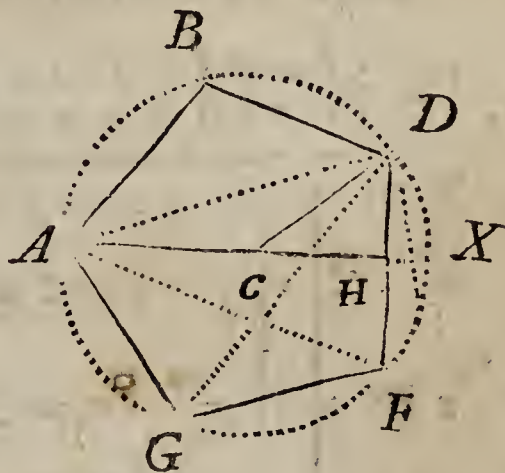
That is,  $\square AB = \square AD : - AD \times GF = 1$

Hence it will be  $AD = 1,61803398$

Then  $\square AD - \square DH = \square AH$ . By Theor. 11. But  $DH = \frac{1}{2} AB$

Therefore  $\sqrt{\square AD - \frac{1}{4} \square AB} = AH = 1,53884176$ .

Again,  $AH : AD :: AD : AX = 2 AC$ . For  $\triangle AHD$  and  $\triangle ADX$  are alike. Ergo



Ergo  $\frac{\square AD}{AH} = 2 AC = 1,707106781$ . Hence  $AC = 0,85065080$

But  $AH - AC = CH = 0,000906$  &c. Q. E. D.

From hence it will be easy to resolve the following Problem.

### P R O B L E M VIII.

*The Side of any regular Pentagon being given, To find its Area.*

*Example,* Suppose the given Side to be 15 Inches long, then it will be, as 1 : 1,53884176 :: 15 : 22,0826264 the perpendicular Height; and by the general Rule  $22,0826264 \times \frac{15}{2} = 165,619698$  the Area requir'd.

*Thirdly,* For an Octagon.

The Side of any regular Octagon is in Proportion to the Radius of its  $\left\{ \begin{array}{l} \text{Circumscribing Circle, As 1 : To 1,30656296 \text{ \&c.} \\ \text{Inscrib'd Circle, As 1 : To 1,20710678 \text{ \&c.} \end{array} \right.$

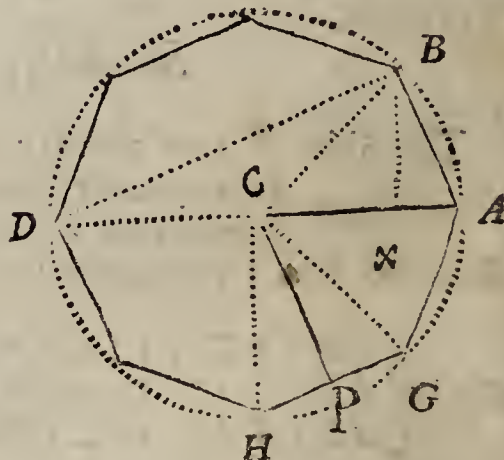
Viz.  $\left\{ \begin{array}{l} BA : CA :: 1 : 1,30656296 \\ BA : CP :: 1 : 1,20710678 \end{array} \right.$

### Demonstration.

Draw the Right-line  $DB$ , and from the Point  $B$  let fall the Perpendicular  $Bx$  upon the Diameter  $DA$ .

Then will  $\triangle DBA$  and  $\triangle DxB$  be alike. By *Theorem 10* and *12*.

Let  $\left\{ \begin{array}{l} b = BA = 1. a = CA. \\ e = DB \text{ and } y = Bx. \end{array} \right.$



Then  $1 \mid 2a : b :: e : y$  Viz.  $DA : BA :: DB : Bx$ .

1  $\therefore 2 \mid \frac{2ay}{b} = e = DB$

2  $\odot 2 \mid 3 \mid \frac{4aayy}{bb} = ee = \square DB$

But  $4 \mid 4aa - \frac{4aayy}{bb} = bb$

That is,  $\square DA - \square DB = \square BA$ . By *Theorem 11*.

4  $\times bb \mid 5 \mid 4bbax - 4aayy = bbbb$

Again,  $6 \mid \left\{ \begin{array}{l} \frac{1}{2} aa = yy. \text{ For } Cx = Bx \\ \text{and } \square Cx + \square Bx = \square CB = aa \end{array} \right.$



5,	6	7	$4bbaa - 2a^4 = b^4$	Or $2a^4 - 4bbaa = -b^4$
7	$\div$	2	$aaaa - 2bbaa = -\frac{1}{2}b^4$	
8	C	□	$a^4 - 2bbaa + b^4 = b^4 - \frac{1}{2}b^4 = \frac{1}{2}b^4$	
9	w	2	$aa - bb = \sqrt{\frac{1}{2}b^4}$	
10	+	bb	$aa = bb + \sqrt{\frac{1}{2}b^4}$	
11	w	2	$a = \sqrt{bb + \sqrt{\frac{1}{2}b^4}} = 1,30656296 \text{ \&c.} = CA$	
	Then	13	$aa - \frac{1}{4}bb = \square CP$	Viz. $\square CH - \square HP = \square CP$
13	w	2	$\sqrt{aa - \frac{1}{4}bb} = 1,20710678 \text{ \&c.} = CP$	

From hence it will be easy to find the Area of any Octagon.

### PROBLEM IX.

*The Side of any regular Octagon being given to find its Area.*

Example, Suppose the Side given to be 12 Inches long; First, As 1 : 1,20710678 :: 12 : 14,48528136 = the Radius of its inscrib'd Circle. Then  $12 \times 4 = 48$  is Half the Sum of its Sides; and  $48 \times 14,48528136 = 695,2935$  the Area requir'd.

Fourthly, for a Decagon.

*The Side of any regular Decagon, (viz. a Polygon of Ten equal Sides) is in Proportion to the Radius of*

Its { Circumscribing Circle, As 1 : To 1,61803398 &c.  
 { Inscrib'd Circle, As 1 : To 1,53884176 &c.

Viz. {  $BA : CA :: 1 : 1,61803398$   
 {  $BA : CP :: 1 : 1,53884166$

### Demonstration.

Let {  $b = BA = 1$ ,  $a = CA$   
 {  $e = DB$ , and  $y = Bx$

Then 1  $2a : b :: e : y$   
 That is  $DA : BA :: DB : Bx$

1  $\therefore$  2  $\begin{cases} 2ay = be \\ \text{and } 2y = \frac{be}{a} \end{cases}$

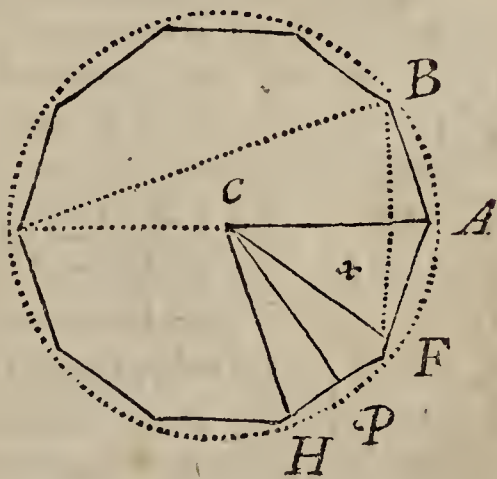
But 3  $2y : e :: 1 : 1,61803398$   
 See Pentagon.

3  $\therefore$  4  $\frac{1e}{1,61803398} = 2y = \frac{be}{a} = \frac{1e}{a}$

4  $\rightarrow$  5  $1,61803398 = a = CA$

Again 6  $\begin{cases} aa - \frac{1}{4}bb = \square CP \\ \text{Viz. } \square CF - \square PF = \square CP. \text{ By Theorem II.} \end{cases}$

That is, 7  $\sqrt{2,61803396} - 0,25 = 1,53884176 = CP$



P R O B L È M X.

*The Side of any regular Decagon being given, To find its Area.*

*Example,* Let the given Side be 14 Inches long: Then,  
As  $1 : 1,53884176 :: 14 : 21,543784 =$  the Radius of the inscrib'd  
Circle. And  $14 \times 5 = 70$  is Half the Sum of its Sides. Lastly,  
 $21,543784 \times 70 = 1508,06488$  the Area requir'd.

*Fifthly, For a Dodecagon.*

*The Side of any regular Dodecagon, (viz. a Polygon of twelve equal Sides) is in Proportion to the Radius of*

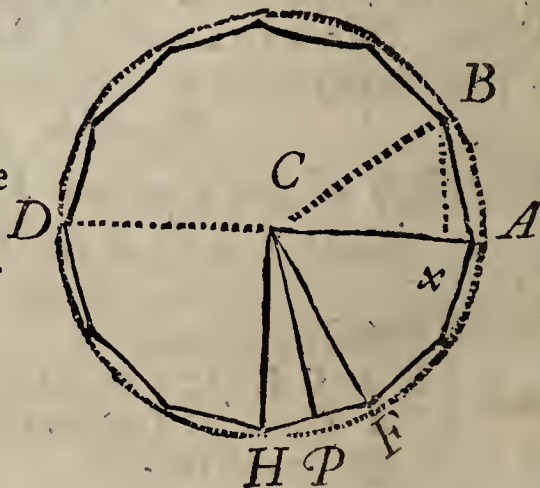
its  $\left\{ \begin{array}{l} \text{Circumscribing Circle, As 1 : To } 1,93185165 \text{ \&c.} \\ \text{Inscrib'd Circle, As 1 : To } 1,86632012 \text{ \&c.} \end{array} \right.$

*Viz.*  $\left\{ \begin{array}{l} B A : C A :: I : 1,93185165 \\ B A : C P :: I : 1,86632012 \end{array} \right.$

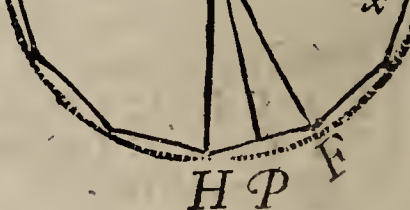
## Demonstration.

Let  $b = B$   $A = I$  .  $a = C$   $A$  as before

And  $e \equiv x A$ . Then  $a - e \equiv C x$



First	1	$\sum bb - \square Bx = ee$
		$\sum$ By Figure.
But	2	$Bx = \frac{1}{2}CA = \frac{1}{2}a$
2	3	$\square Bx = \frac{1}{4}aa$
1,	4	$bb - \frac{1}{4}aa = ee$
4	5	$\sqrt{bb - \frac{1}{4}aa} = e$
Again	6	$aa - \frac{1}{4}aa = aa - 2ae + ee$
Viz.		$\square CB - \square Bx = \square Cx.$
5	7	$2a \sqrt{bb - \frac{1}{4}aa} = 2ae$
4	8	$bb - \frac{1}{4}aa - 2a \sqrt{bb - \frac{1}{4}aa} = ee - 2ae$
7,	9	$aa - \frac{1}{4}aa = aa + bb - \frac{1}{4}aa - 2a \sqrt{bb - \frac{1}{4}aa}$
9	10	$2a \sqrt{bb - \frac{1}{4}aa} = bb$
10	11	$4bbaa - aaaa = b^4$
11	12	$aaaa - 4bbaa = -b^4$
13	13	$aaaa - 4bbaa + 4b^4 = 3b^4 = 3$
I	14	$aa - 2bb = \sqrt{3} = 1.7320508075$
14	15	$aa = 2bb + \sqrt{3} = 3.7320508075$
15	16	$1 = \sqrt{3.7320508075} = 1.93185165 = CA$
Again	17	$aa - \frac{1}{4}bb = \square CP. \text{ Viz. } \square CF - \square PF = \square CP$
17, Hence	18	$CP = \sqrt{aa - \frac{1}{4}bb} = 1.86632012$



H P E

Q. E. D.



CONSECTARY.

Hence, if the Side of any regular Dodecagon be given, the Radius of its inscrib'd Circle may be easily obtain'd, and thence the Area found, as in the last Problem.

The Work of the foregoing Polygons being well consider'd will help the young Geometer to raise the like Proportions for others, if his Curiosity, or Occasion requires them: And not only so, but they will also help to form a true Idea of a Circle's Periphery and Area, according to the Method which I shall lay down in the next Chapter for finding them both.

CHAP. VI.

*A new and easy Method of finding the Circle's Periphery and Area, to any assign'd Exactness (or Number of Figures) by one Equation only. Also a new and facile Way of making Natural Sines and Tangents.*

LET us suppose (what is very easy to conceive) the Circle's Area to be compos'd or made up of a vast Number of plane Ilosceles Triangles, having their acutest Angles all meeting in the Circle's Center: And let us imagine the Bases of those Triangles so very small, that their Sides and their perpendicular Heights, viz. the Radius's of their circumscrib'd and inscrib'd Circles, (vide Problem 6.) may become so very near in Length to each other, as that they may be taken one for another, without any sensible Error. Then will the Peripheries of their circumscribing, and inscrib'd Circles, become (although not co-incident, yet) so very near to each other, as that either of them may be indifferently taken for one and the same Circle.

But how to find out the Sides of a Polygon (viz. the Bases of those Ilosceles Triangles) to such a convenient Smallness, as may be necessary to determine and settle the Proportion betwixt a Circle's Diameter and its Periphery, (to any assign'd Exactness) hath hitherto been a Work which requir'd great Care and much Time in its Performance, as may easily be conceiv'd from the Nature of the Method us'd by all those who have made any considerable Progress in it, viz. Archimedes, Snellius, Hugenius, Marius, Van-Culen, &c. These proceeded with the Bisecting of an Arch, and found the Value of its Chord to a convenient Number of

Figures at every single Bisection, repeating their Operations until they had approach'd to the Chord design'd.

And this Method is made Choice of by the learned Dr. *Wallis*, in his Treatise of *Algebra*; wherein, after he hath given us a large Account of the different Enquiries made by several (very eminent in Mathematical Sciences) in order to find out some easier and more expeditious Way of approaching to the Circle's Periphery, as in *Chap.* 82, 84, 85, 86, and several other Places, he comes to this Result. (*Page* 321.)

“ 'Tis true. (saith he) we might in like Manner proceed by  
 “ continual Trisection, Quinisection, or other Section, if we  
 “ had for these as convenient Methods of Operation as we have  
 “ for Bisection: But because *Euclid* shews how to bisect an Arch  
 “ Geometrically, but not to trisect, &c. and the one may be done  
 “ (Algebraically) by resolving a Quadratic *Æquation*; but not  
 “ those other, without *Æquations* of a higher Composition, I  
 “ therefore make Choice of a continual Bisection, &c.

And then he lays down these following Canons.

The Subtense of  $\frac{1}{2}$

of  $\frac{1}{12}$

of  $\frac{1}{24}$

of  $\frac{1}{48}$

of  $\frac{1}{96}$

&c.

	$\sqrt{2} - \sqrt{3}$	1 into 6
	$\sqrt{2} - \sqrt{2} + \sqrt{3}$	into 12
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{3}$	&c. 24
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	48
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	96
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	192
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	384
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	768
	&c.	&c.

How tedious and troublesom the Work of these complicated Extractions is, I leave to the Consideration of those, who either have had Experience therein, or out of Curiosity will give themselves the Trouble of making Trial.

Again, in *Page* 347, the Doctor inserts a particular Method propos'd by *Leibnitius*, publish'd in the *Acta Eruditorum* at *Leipsick*, for the Month of *February* 1682. in order to find the Circle's Area, and consequently its Periphery, which is this:

As 1: To  $\frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}$  &c.

Infinitely :: So is the Square of the Diameter: to the Circle's Area. But this convergeth so very slowly, that it is not worth the Time to pursue it.

I shall here propose a new Method of my own, whereby the Circle's Periphery (and consequently its Area) may be obtain'd infinitely



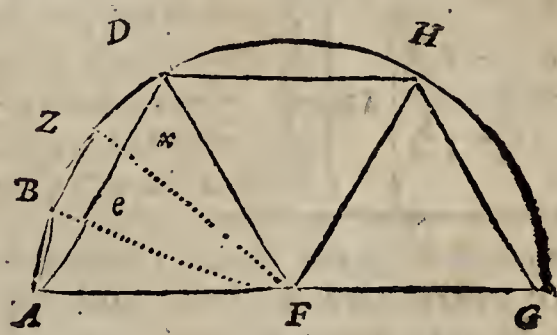
infinitely near the Truth, with much greater Ease and Expedition, than either that of Bisection, or that of *Leibnitius*, as above, or any other Method that I have yet seen, it being perform'd by resolving only one *Æquation*, deduc'd by an easy Process from the Property of a Circle, (known to every Cooper) which is this.

*The Radius of every Circle is equal to the Chord of one sixth Part of its Periphery.*

That is,  $AD = DH = HG$ , the Chords of  $\frac{1}{3}$  Part of the Semicircle, are each equal to  $AF$  its Radius.

Then if the Arch  $AD$  be trisected. it will be  $AB = BZ = ZD$ .

Let  $\begin{cases} R = AF = I \\ c = AD = I \\ a = AB \end{cases}$  Quere  $a$ .



Then	1	$R : a :: a^{\frac{aa}{R}} = B_2$
And	2	$R : a :: R : -\frac{aa}{R} : c - 2a$
That is,	3	$FB : BZ :: Fe : ex = AD - 2a$
For		$\triangle AFB$ and $\triangle BAE$ , are alike.
		And $AB = Ae = Dx$ , &c.
2	4	$Rc - 2Ra = Ra - \frac{aaa}{R}$
4 x &c.	5	$3Ra - aaa = RRc$ . That is, $3a - aaa = I$ .
		Here $a =$ the Chord of $\frac{1}{18}$ Part of the Circle.
		For $\frac{1}{3}$ of $\frac{1}{6} = \frac{1}{18}$ .

Next, To trisect the Arch  $AB$ .

Let  $\begin{matrix} 1 & \textcircled{+} & 3 \\ 1 & \times & 3 \\ 3 & - & 2 \end{matrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{cases} 3y - y^3 = a \text{ the last Chord.} \\ 27y^3 - 27y^5 + 9y^7 - y^9 = a^3 \\ 9y - 3y^3 = 3a \\ 9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = I \end{cases}$

Here  $y =$  the Chord of  $\frac{1}{4}$  Part of the Circle.

Again, To trisect the Arch, whereof  $y$  is the Chord.

Let  $\begin{matrix} 1 & \textcircled{+} & 3 \\ 1 & \textcircled{+} & 5 \end{matrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{cases} 3a - a^3 = y \\ 27a^3 - 27a^5 + 9a^7 - a^9 = y^3 \\ 243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} = y^5 \end{cases}$

1  $\textcircled{+}$  7

I	⊙	7	4	$\{ 2187a^7 - 5103a^9 + 5103a^{11} - 2835a^{13} + 945a^{15}$ $= y^7$	
I	⊙	9	5		$\{ 19683a^9 - 59049a^{11} + 78732a^{13} - 61236a^{15}$ $= y^9$
I	×	9	6	$27a - 9a^3 = 9y$	
2	×	30	7	$810a^3 - 810a^5 + 270a^7 - 30a^9 = 30y^3$	
3	×	27	8	$\{ 6561a^5 - 10935a^7 + 7290a^9 - 2430a^{11} + 405a^{13}$ $- 27a^{15} = 27y^5$	
4	×	9	9		$\{ 19683a^7 - 45927a^9 + 45927a^{11} - 25515a^{13}$ $+ 8505a^{15} = 9y^7$
6	-	7	10	$\{ 27a - 819a^3 + 7371a^5 - 30888a^7$ $+ 72930a^9 - 107406a^{11}$ $+ 104652a^{13} - 69768a^{15}$	
+8	-	9			$\} = 1$
+5					

Here  $a =$  the Chord of  $\frac{1}{182}$  Part of the Circle.

Proceeding on in this Method of continually trisecting the Arch of every new Chord, and still connecting the produc'd  $\mathcal{A}$ equations into one, as in the two last Trisections, it will not be difficult to obtain the Chord of any assign'd Arch, how small soever it be.

Now in order to facilitate the Work of raising these  $\mathcal{A}$ equations to any considerable Height, it will be convenient to add some few useful Observations concerning the Nature, and of such Contractions as may be safely made in them; which, being well understood will render the Work very easy.

1. I have observ'd that every Trisection will gain or advance one Figure in the Circle's Periphery, but no more. Therefore so many Places of Figures as are at first design'd to be perfect in the Periphery, so many Trisections must be repeated to raise an  $\mathcal{A}$ equation, that will produce a Chord answerable to that Design.

2. I have also found, that all the superior Powers (of  $a$ ) whose Indices are greater than the Number of Trisections, viz. whose Indices are greater than the Number of design'd Figures, may be wholly rejected as insignificant.

3. When once the Number of Trisections, and thence the highest Power (of  $a$ ) is determin'd, the third Process, viz. the third Trisection) may be made a fix'd or a constant Canon; for by it, and Multiplication only. all the succeeding Trisections, how many soever they are) may be completed, without repeating the several Involutions.

4. In raising and collecting the Co-efficients of the several Powers, (of  $a$ ) it will be sufficient to retain only so many significant Figures



Figures ( at  $a^3$  ) as there is design'd to be Places of Figures in the Periphery; ( or at most but two more ) and every succeeding superior Power may be allow'd to decrease two Places of significant Figures. But herein great Care must be taken to supply the Places of those Figures that are omitted with Cyphers, that so the whole and exact Number of Places may be truly adjusted; otherwise all the Work will be erroneous.

Now the Number of those supplying Cyphers may be very conveniently denoted by Figures plac'd within a Parenthesis; thus,  $576(8)a^3$ , may signify  $57600000000a^3$ , as in the following Æquations. The like may be done with Decimal Parts, thus  $(,7)658$  may signify  $,0000000658$ , &c. which will be found very useful in the Solution of these, and the like Æquations.

The aforesaid Contractions may be safely made, because both the superior Powers of  $a$ , which are rejected; as also those Numbers that are omitted in the Co-efficients ( and supply'd with Cyphers ) would produce Figures so very remote from Unity, as that they would not affect the Chord design'd. That is. they would not affect the Chord in that Place wherein the design'd Periphery is concern'd, as will in Part appear in the following Example.

If these Directions be carefully minded, it will be easy to raise an Æquation that will produce the Side of a regular Polygon, whose Number of Sides shall be vastly numerous, consequently infinitely small. But I presume it will be sufficient for an Example, to find the Side of a Polygon consisting of  $258280326$  equal Sides; that is, if I find the Chord of  $\frac{1}{258280326}$  Part of the Circle's Periphery, and that requires but sixteen Trisections, which being order'd as before directed, will produce this Æquation.

$$\left. \begin{aligned} &43046721a - 332360179486968612(4)a^3 \\ &+ 769837653199714(20)a^5 - 8491218532841(35)a^7 \\ &+ 54633331143(50)a^9 - 230083348(66)a^{11} \\ &+ 6830988(79)a^{13} - 15072(94)a^{15} \end{aligned} \right\} = 1$$

Here the Value of  $a$  will have 23 Places of Figures true; that is, the Sides of the inscrib'd and circumscrib'd Polygons, will be exactly the same to 23 Places of Decimal Parts, but not farther; all which may be easily obtain'd at two Operations. And for the first, will be sufficient to take only three Terms of the Æquation; which will admit of being yet farther contracted, thus

$$\text{Let } \left\{ \begin{aligned} &43046721a - 3323601794(12)a^3 \\ &+ 76983765(27)a^5 \end{aligned} \right\} = 1$$

And

# 352 Elements of Geometry. Part III.

And let  $r + e = a$ . Then rejecting all the Powers of  $e$ , that arise by Involution above  $eee$ ,

$$\text{it will be } r^3 + 3rre + 3ree + eee = aaa$$

$$\text{And } r^5 + 5r^4e - 10r^3ee + 10rree = a^5$$

Then the first single Value of  $r$  may be thus found;

$$43046721) 1,00000000 (300000002 = r$$

This  $,00000002 = r$  being duly involv'd, and its Powers multiply'd into their respective Co-efficients, will produce

$$+86093442 + 43046721e$$

$$-02658881 - 3988322e - 199416(9)ee - 3324(18)eee \} = 1$$

$$+00024635 + 61587e + 6159(9)ee + 308(18)eee$$

$$\text{Viz. } 83459196 + 39119986e - 193257(9)ee - 3016(18)eee = 1$$

$$\text{Hence } 39119986e - 193257(9)ee - 3016(18)eee = 0,16540804$$

All the Terms of this last Æquation being divided by  $193257(9)$  the Co-efficient of  $ee$ , it will then become

$$,0000002024e - ee - 156(5)eee = ,00000000000000008558968 = D$$

$$\text{Consequently, } \sum \frac{D + 156(5)eee}{,0000002024 - e} = e$$

Operation.

$$\begin{array}{r} . . . . . \\ ,0000002024) ,00000000000000008558968 ( ,000000004 = e \\ - ,0000000043: + ,0000000000000000009984 = 156(5)eee \end{array}$$

$$1 \text{ Di. } ,000000198) ,00000000000000008568952 (000000004327$$

$$2 \text{ Di. } ,0000001981$$

$$792$$

$$6489$$

$$5943$$

$$5465$$

$$3962$$

$$\text{First } r = ,00000002$$

$$+ e = ,000000004327$$

$$\&c.$$

$$r + e = ,000000024327 = a. \text{ Or rather new } r \text{ for a 2d Operation.}$$

Now if this first Value of  $a = ,000000024327$  were not continued to more Places of Figures, by a second Operation, but only multiply'd into the Number of Chords,

$$\text{Viz. } ,000000024327 \times 258280326 = 6,28318539 \&c. \text{ the Periphery of that Circle, whose Diameter is 2, nearer than either}$$

*Archimedes*



# Chap. 6. Of the Circle's Periphery. 353

*Archimedes*, or *Mætius's* Proportion : For *Archimedes* makes it 6,285714 &c. viz. As 7 to 22. And *Mætius* makes it 6,28318584 &c. viz. as 113 to 355.

But if the whole *Æquation* before propos'd be now taken, and we proceed to a second Operation, the Value of *a* may be increas'd with twelve Places of Figures more, and those may be obtain'd by plain Division only.

Thus, let  $r + e = a$ , as before, and let all the Powers of *e* be now rejected as insignificant;

Then will  $\left\{ \begin{array}{l} r + e = a \\ r^3 + 3r^2e = a^3 \\ r^5 + 5r^4e = a^5 \\ r^7 + 7r^6e = a^7 \end{array} \right\}$  and  $\left\{ \begin{array}{l} r^9 + 9r^8e = a^9 \\ r^{11} + 11r^{10}e = a^{11} \\ r^{13} + 13r^{12}e = a^{13} \\ r^{15} + 15r^{14}e = a^{15} \end{array} \right\}$

The several Powers of  $r = ,000000024327$  being rais'd and multiply'd into their respective Co-efficients, will produce these following Numbers.

$$\begin{array}{r} +1,047197581767 \\ - ,047849196598394865 \\ + ,000655906484595355 \\ - ,000004281440413375 \\ + ,000000016302517863 \\ - ,000000000040631167 \\ + ,000000000000071388 \\ - ,000000000000000093 \\ + 43046721e \\ 5900751e \\ 134810e \\ 1232e \\ 6e \\ 0e \\ 0e \\ 0e \end{array} \Bigg\} = 1$$

Viz.  $1,0000000026474745106 + 37279554e = 1$

Hence  $37279554e = 1,0000000026474745106 - 1 = D$

Or rather  $37279554e = ,0000000026474745106 = D$

Consequently,  $\frac{D}{37279554} = -e$

Operation.

$37279554) ,000000026474745106 ( (5^{15})710167967 = -e$   
 $260956878$

$37905730$   
 $37279554$

$62617660$   
 $37279554$

&c.  
 Z z

Laſt

Last  $r = ,0000000024327$

—  $e = .00000000000000000710167967$

$r - e = .0000000024326999289832033 = a$  the Chord or Side of the Polygon requir'd.

The next Work will be to examine how many Places of these Figures will hold true to the Circle's Periphery : In order to that let  $a$  be represented by the Chord  $Bb$ ,

in the annexed Scheme; and let  $Bx = x b$ . Then will  $Bx = \frac{1}{2} a = (.57)$

$121634996440160165$  And  $\square BC -$

$\square Bx = \square Cx$ . Let the Radius  $BC$  be

as before. Then will  $\sqrt{\square BC - \square Bx}$

$= Cx = ,9999999999999999, \&c.$

But  $Cx : xB :: CA : AD$  { PerFig.

Or  $Cx : Bb :: CA : Dd$  {

Ergo  $Dd = (.57)243269992898320354$  the

Side of the circumscribing Polygon.

Then will  $a \times 258280326$  be the Perimeter of the inscrib'd Polygon,

And  $Dd \times 258280326$  will be the Perimeter of the circumscribing Polygon.

That is,  $6,2831853071795859 =$  the Perimeter of the inscrib'd Polygon.

And,  $6,2831853071795865 =$  the Perimeter of the circumscrib'd Polygon.

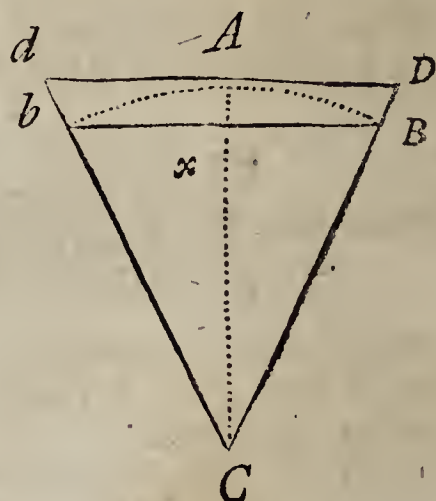
Hence 'tis evident, that the Circle's Periphery, whose Diameter is 2, may be concluded  $6,2831853071795864$  true, because the Perimeters of the inscrib'd and circumscrib'd Polygons are so far very near being co-incident, or the same.

'Tis possible there may be some who will think this is tedious and troublesom Work; but if those please to consider, that if this Periphery were to be found by the aforesaid Method of Bisection, it would-require these following Extractions.

$$\text{viz. } \left\{ \begin{array}{l} \sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} \\ + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} \\ + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} \\ + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3} \end{array} \right. \text{Multi-} \\ \text{ply'd into } 402809984.$$

Here the first Root (viz.  $\sqrt{3}$ .) must be extracted at least to one hundred and two Places of Figures. The second Root (viz.

$\sqrt{2}$





$\sqrt{2} + \sqrt{3}$  ) must have 99 Places of Figures in it. The third Root (*viz.*  $\sqrt{2} + \sqrt{2} + \sqrt{3}$  ) must have 96 Places in it, &c. every Extraction being allow'd to decrease three Places, that so the last Root (*viz.* the Chord sought ) may consist of 24 Places of Figures, as above.

I say, whoever duly considers the Trouble of these so often repeated Extractions, will, I presume, be pleas'd with what I have done. For truly when I consider of the great Time and Care requir'd in them, I cannot but admire at the Patience of the laborious *Van Culen*, who proceeded that Way until he had found the Circle's Periphery to thirty six Places of Figures, to wit,

6.28318530717958647692528676655900576.

These Numbers are said to be engraven upon his Tomb-stone in *St. Peter's Church* in *Leyden*, for a Memorial of so great a Work.

Having thus obtain'd the Circle's Periphery, its Area may easily be found (to the same Number of Figures) by *Problem 6*.

That is, if Half the Periphery of any Circle be multiply'd into Half its Diameter, the Product will be that Circle's Area, as will appear farther on. Therefore 3.141592653589793 will be the Area of the Circle whose Diameter is 2.

Thus I have shew'd the young Geometer how to find the Circle's Periphery and Area to what Exactness he pleases to approach; for precisely true they cannot be found, notwithstanding the late Pretensions of a certain *Frenchman*, who hath publish'd to the World. (in the *Works of the Learned* ) that after twenty five Years Study he had found the Quadrature of the Circle: But if he had perus'd the 83d Chapter of *Dr. Wallis's Algebra*, he might there have seen his Error, *viz.* the Impossibility of what he pretended to; for it is as impossible to square the Circle (that is, to find its true Area) as it is to find the Root of a Surd Number.

*Note*, What I have here propos'd and done by the Trisection of an Arch, may as easily and much more speedily be perform'd by Quinquesection or Septisection, &c. But because the Scheme for Trisection is more simple, and may be easier understood by a Learner than those of the other Sections, (of which see my *Compendium of Algebra*. Pages 76 and 79) I have for that Reason made Choice of Trisection.

As to the Proportion of one Circle to another, and of the Circle to the Ellipsis, &c. those shall be fully shew'd when we come to the fifth Part.

Before I conclude this Part, I shall make some Use or Application of the above found Periphery, in finding the Quantity of Angles, which is done by the Help of Right-lines, call'd Sines and Tangents, the Length whereof is calculated to every Degree and Minute of a Quadrant, by much Labour. But I shall here shew how to find the Natural Sine (and consequently the natural Tangent) of any propos'd Arch or Angle, by two *Æquations*, without the Help of any preceding Sine as usual; which I did some Years ago communicate to the ingenious Mr. *Joseph Raphson*, and he so well approv'd of them, as to make them the 20th and 21st Problems in the second Edition of his *Analysis Æquationum Universalis*.

And because in finding the Quantity of Angles, every Circle is suppos'd to be divided into 360 equal Parts, call'd Degrees; every Degree is subdivided into 60 Parts, call'd Minutes; and every Minute into 60 Seconds, &c. (See Page 294.)

Therefore 360 ) 6 2831853 &c. (0,0174532925 &c, is an Arch of the above-found Periphery, equal to the Arch of one Degree.

And 60 ) 0,0174532925 &c. (0,0002908882 &c. = the Arch of one Minute.

Then if the given Arch (or Angle) be less than 45 Degrees, reduce it into Minutes, and multiply those Minutes into this constant Multiplier, viz. 0,0002908882 calling the Product *p*. And for the Sine sought put *a*. Then it will be

$$-aaaa + 12paaa - 195aa - 36ppaa + 240pa = 45pp.$$

Example.

Let it be requir'd to find the Sine of  $19^{\circ}. 13' = 1153'$

Here  $0,0002908882 \times 1153 = 0,3353940946 = p$

$$\text{And } -a^4 + 4,024729a^3 - 199,049611aa + 80,494583a = 5,06201394.$$

$$\text{Let } r + e = a$$

$$\text{Then } \begin{cases} rr + 2re + ee = aa \\ rrr + 3rre + 3ree = aaa \\ rrrr + 4rrre + 6rree = aaaa \end{cases}$$

Note, In this Case the first *r* may always be taken equal to the first Figure in the Product = *p*. Viz. here  $r = 0,3$  which being involv'd as its Powers direct, and those Powers multiply'd into the respective Co-efficients of the *Æquation*, it will be

$$\left. \begin{array}{l} + 24,1483 + 80,49e \\ - 17,9144 - 119,43e - 199,05ee \\ + 0,1086 + 1,08e + 3,62ee \\ - 0,0081 - 0,11e - 0,54ee \end{array} \right\} = 5,06201394$$

---


$$\text{Viz. } 6\ 3341 - 37,97e - 195,97ee = 5,06201$$

Hence



Hence  $37.97e + 195.97ee = 1.27239$

And  $0.193e + ee = 0.006492 = D$

Theorem.  $\sum \frac{D}{,193 + e} = e$

Operation.  $,193 ) 0.006492 ( 0.029 = e$   
 $+ e = ,029 ) \quad 42$

1. Divisor  $,21 \quad 2292$

2. Divisor  $,222 \quad 1998$

First  $r = 0.3$

$+ e = 0.029$

$r + e = 0.329 = r$  for a second Operation.

Which being involv'd and multiply'd, &c. as before, will produce these Numbers.

$$\begin{array}{r} + 26.48271781 + 80.49458e \\ - 21.54532894 - 130.97464e - 199.0496ee \\ + 0.14332578 + 1.30692e + 3.9724ee \\ - 0.01171611 - 0.14244e - 0.6494ee \end{array}$$

Viz.  $55.06899854 - 49.31558e - 195.7266ee - 5.06201394$

Hence  $49.31558e + 195.7266ee = .0069846$ ; which being divided by  $195.7266$  the Co-efficient of  $ee$  will become

$$,25196e + ee = .0000356854 = D$$

Then  $\sum \frac{D}{,25196 + e} = e$

Operation.  $0.25196 ) .0000356854 ( 0.0001415 = e$   
 $+ e = 0.00014 \quad 2520$

1. Divisor  $0.2520 \quad 104854$

2. Divisor  $0.25210 \quad 100840$

40140

25210

Last  $r = 0.329$

$+ e = 0.0001415$

&c.

$r + e = a = 0.3291415$  being the natural Sine of  $19^\circ . 15'$   
 As was requir'd.

Thus you may find the right Sine of any Arch or Angle less than 45 Degrees.

But

# 358 Elements of Geometry. Part III.

But if the given Arch be greater than 45 Degrees, you must take its Compliment to 90°. viz. subtract it from 90 Degrees, and reduce the Remainder into Minutes, As before.

Then multiply the Square of those Minutes into this constant Multiplier, 0,000000084616 calling their Product  $p$ , and putting  $a$  = the Sine sought, As before. Then will

$$a^4 + 28a^3 + 195aa + 36pa + 108pa - 28a = 196 - 81p$$

*Example.*

Suppose it were requir'd to find the Sine of 75°. 32'. Or (which is the same thing) to find the Co-sine of 14°. 28'. = 868' whose Square 753424  $\times$  0,000000084616 = 0,06375172518 =  $p$ . Hence the Equation in Numbers will be

$$aaaa + 28aaa + 197,295062aa - 21,114814a = 190,8361102588$$

$$\text{Let } r - e = a$$

$$\text{And } r = 1$$

$$\text{Then } \begin{cases} rr - 2re + ee = aa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$$

*Note.* I here take  $r = 1$  because the Arch is so near to 90°. and therefore I make it  $r - e = a$ .

$$\text{Then } \left\{ \begin{array}{l} - 21,1148 + 21,11e \\ + 197,2956 - 394,59e + 197,29ee \\ + 28,0000 - 84,00e + 84,00ee \\ + 1,0000 - 4,00e + 6,00ee \end{array} \right\} = 190,8361$$

$$\text{Viz. } 205,1808 - 461,48e + 287,29ee = 190,8361$$

$$\text{Hence } 461,48e - 287,29ee = 14,3447$$

$$\text{And } 1,606e - ee = ,049930 = D$$

$$\text{Theorem. } \left\{ \frac{D}{1,606 - e} = e \right.$$

$$\text{Operation. } \begin{array}{r} 1,606 \ ) \ 0,049930 \ ( 0,031 = e \\ \underline{- e = 0,031} \end{array}$$

$$\begin{array}{r} 1. \text{ Divisor } \quad 1,57 \quad 2830 \\ \hline 2. \text{ Divisor } \quad 1,575 \quad 1575 \\ \hline \quad \quad \quad \&c. \end{array}$$

$$\text{First } r = 1,000$$

$$\underline{- e = 0,031}$$

$$r - e = 0,969 = r$$

for a Second Operation; which being involv'd as before, will produce these following Numbers.



$$\begin{aligned}
 & - 20,460254766 + 21,11481e \\
 & + 185,252368710 - 382,35783e + 197,2951ee \\
 & + 25,475889852 - 78,87272e + 81,5960ee \\
 & + 0,881647759 - 3,63941e + 5,6337ee
 \end{aligned}$$

$$\begin{aligned}
 \text{Viz. } 191,149651515 - 443,75515e + 284,5248ee = \\
 190836110559
 \end{aligned}$$

Hence it will be  $44375515e - 284,5248ee = 0,313541256$

$$\text{And } 1,55963e - ee = ,0011019821 = D$$

$$\text{Then } \left\{ \frac{D}{1,55963 - e} = e \right.$$

$$\begin{aligned}
 \text{Operation. } 1,55963 ) 0,0011009821 \quad (0,0007068 = e \\
 - e = 0,00070
 \end{aligned}$$

$$\text{1. Divisor } 15589$$

$$\text{2. Divisor } 1,55893$$

$$109123$$

$$1075210$$

$$935358$$

$$1398520$$

$$1247144 \text{ \&c.}$$

$$\text{Last } r = 0,969$$

$$- e = 0,0007068$$

$$r - e = a = 0,9682932 \quad \text{the Sine of } 75^\circ. 32'. \text{ As was requir'd.}$$

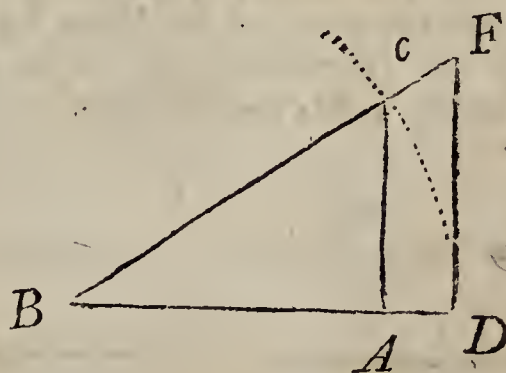
Having found the Sine and Co-sine of any Arch, the Tangent is usually found by this Proportion;

Viz.  $\left\{ \begin{array}{l} \text{As the Co-sine of any Arch: is to the Sine of that} \\ \text{Arch: : so is the Radius: to the Tangent of} \\ \text{the same Arch.} \end{array} \right.$

For supposing  $BC = BD$  Radius,  $AC$  the Sine of the Arch  $CD$ . Then  $BA$  is the Co-sine, and  $FD$  the Tangent of the same Arch. But  $BA : CA :: BD : FD$ , &c.

Now by this Proportion, there is requir'd to be given, both the Sine and Co-sine of the same Arch, to find the Tangent.

'Tis true, if the Radius, and either the Sine or the Co-sine be given, the other may be found



$$\text{Thus, } \sqrt{\square BC - \square CA} = BA. \quad \text{Or } \sqrt{\square BC - \square BA} = CA.$$

But if either the Sine or Co-sine be given, the Tangent may (I presume) be more easily found by the following Theorems.

Let

# 360 Elements of Geometry. Part III.

Let  $BC=1$ .  $CA=s$ .  $BA=x$ . and  $FD=r$ . Then if  $s$  be given,  $r$  may be found by this

$$\text{Theorem } \left\{ \sqrt{\frac{ss}{1-ss}} = r \right.$$

Or if  $x$  be given,  $r$  may be found by this

$$\text{Theorem } \left\{ \sqrt{\frac{xx}{1-xx}} = r \right.$$

Let the Sine of  $19^\circ. 13'$ . ( before found, be given, viz.  $0,3291415 = s$ . To find  $r$  the Tangent of the same Arch.

First,  $0,3291415 \times 0,3291415 = 0,108334127 = ss$ .

Again  $1 - 0,108334127 = 0,891665873 = 1 - ss$ .

Then  $0,891665873 \div 0,108334127 = 0,1214963253$

And  $\sqrt{0,1214963253} = 0,3485632 = r$  the Tangent of  $19^\circ. 13'$ . As was requir'd.

And so you may proceed to find  $r$  the Tangent, when  $x$  = the Co-sine is given.

Perhaps it may be here expected, that I should have shew'd and demonstrated (or at least have inserted) the Proportions from whence the foregoing Equations for making Sines were produc'd; but I have omitted them, as also their Use in computing the Sides and Angles of plane Triangles by the Pen only, (viz. without the Help of Tables) for the Subject of another Discourse hereafter, if Health and Time permit.

In the mean time, what is here done may suffice to shew, that the making of Sines by such a laborious and operose Way, as was formerly used, is in a great Measure overcome, which, I think, I may justly claim as my own.



A N

# INTRODUCTION

T O T H E

# Mathematicks.

## P A R T IV.

*Thomas* C H A P. I. *Dimin*

### Definitions of a Cone and its Sections.

**T**Here are several Definitions given of a Cone: The learned Dr. Barrow upon *Euclid* hath it thus:

“ A Cone (*saitk be*) is a Figure made when one Side  
 “ of a Rectangled Triangle, (*viz.* one of those Sides  
 “ that contain the Right-angle ) remaining fix’d, the  
 “ Triangle is turn’d round about, till it return to the  
 “ Place from whence it first mov’d. And if the fix’d  
 “ Right-line be equal to the other, which containeth the  
 “ Right-angle, then the Cone is a rectangled Cone;  
 “ but if it be less, it is an obtuse-angled Cone; if great-  
 “ er, an acute-angled Cone. The Axis of a Cone, is  
 “ that fix’d Line about which the Triangle is mov’d.  
 “ The Base of a Cone is the Circle, which is describ’d  
 “ by the Right-line mov’d about, (Defin. 18, 19, 20.  
 “ *Euclid* II.

Sir Jonas Moor. in his *Treatise of Conical Sections*, (taken out of the Works of Mydorgius ) defines it thus:

“ If a Line of such a Length as shall be needful, shall  
 “ upon a Point fix’d above the Plane of a Circle, so  
 “ move about the Circle, until it return to the Point  
 “ from whence the Motion begun, the Superficies that  
 “ is made by such a Line, is call’d a conical Superficies,  
 “ and

“and the solid Figure contain'd within that Superficies  
 “and the Circle, is call'd a Cone. The Point remain-  
 “ing still is the Vertex of the Cone, &c.

Altho' both these Definitions are equally true, and, with a little Consideration, may be pretty easily understood; yet I shall here propose one, very different from either of them, and, as I presume, more plain and intelligible, especially to a Learner.

If a Circle describ'd upon stiff Paper (or any other pliable Matter) of what Bigness you please, be cut into two, three, or more Sectors, either equal, or unequal, and one of those Sectors be so roll'd up as that the Radius's may exactly meet each other, it will form a Conical Superficies.

That is, if the Sector  $HVG$  be cut out of the Circle, and so roll'd up as that the Radius's  $VH$  and  $VG$  may just meet each other in all their Parts, it will form a Cone, and the Center  $V$  will become a solid Point, call'd the *Vertex* of the Cone; the Radius  $VH$  being every where equal, will be the Side of the Cone, and the Arch  $HG$  will become a Circle, whose Area is call'd the Cone's Base.

A Right-line being suppos'd to pass from the Vertex or Point  $V$ , to the Center of the Cone's Base, as at  $C$ , that Line (*viz.*  $VC$ ) will be the *Axis*, or perpendicular Height of the Cone.

If a Solid be exactly made in such a Form, it will be a complete or perfect Cone; which I shall all along call a right Cone, because its Axis  $VC$  stands at Right-Angles with the Plane of its Base  $HG$ , and its Sides are every-where equal.

Any Cone whose Axis is not at Right-Angles with the Plane of its Base, may be properly call'd an imperfect Cone, because its Sides are not every where equal (as in the annexed Figure.) Now, such an imperfect Cone is usually call'd a Scalene, or oblique Cone.

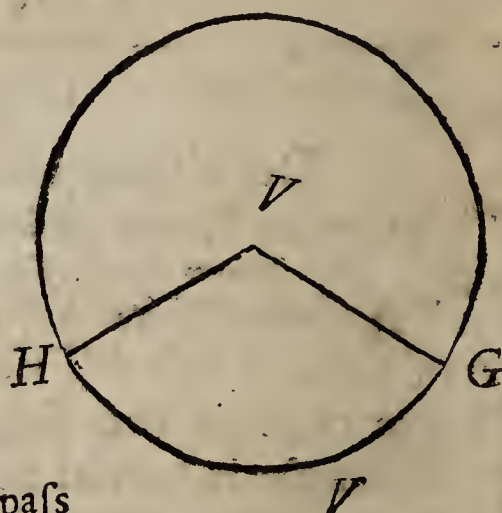
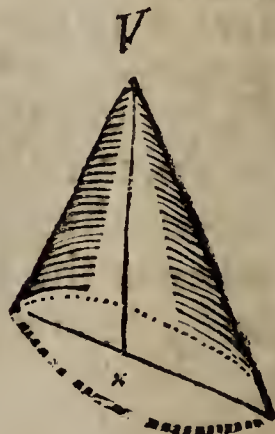
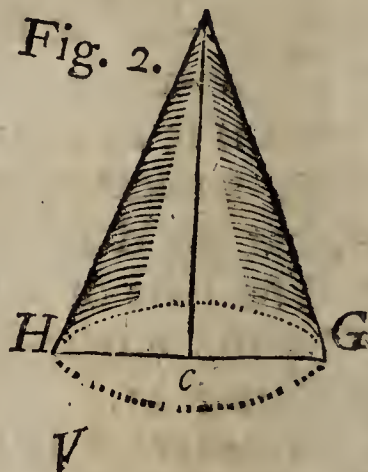


Fig. 2.



Any



Any Solid Cone may be cut by Planes, ( which I shall all along hereafter call Right-lines ) into five *Sections*.

## S E C T. I.

If a Right-Cone be cut directly through its Axis, the Plane or Superficies of that Section will be a plane Iſosceles Triangle, as *HVG* Figure 2. viz. the Sides (*HV* and *VG*) of the Cone will be the Sides of the Triangle, the Diameter (*HG*) of the Cone's Base will be the Base of the Triangle, and (*VC*) its Axis will be the perpendicular Height of the Triangle.

## S E C T. II.

If a Right-Cone be cut (any where) off by a Right-line parallel to its Base, as *hg* (it will be easy to conceive, that) the Plane of that Section will be a Circle, because the Cone's Base is such; wherein one thing ought to be clearly understood, which may be laid down as a *Lemma* to demonstrate the Properties of the following Sections.

*Lemma.*

If any two Right-lines inscrib'd within a Circle, do cut or cross each other (as *hg* doth *ba* in the annexed Figure) the Rectangle made of the Segments of one of the Lines will be equal to the Rectangle made of the Segments of the other Line, (See *Theorem 15*, Page 315.)

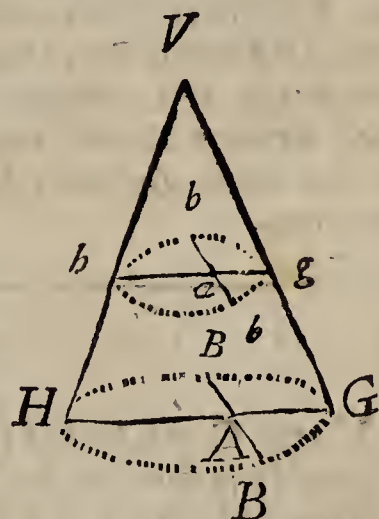
That is,  $ba \times ga = ba \times ab$  } &c.  
And  $HA \times GA = BA \times AB$  }

Consequently, if  $ba = ab$

And if  $BA = AB$ .

Then it will be,  $ba \times ga = \square ba$ .

And in the Cone's Base  $HA \times GA = \square BA$ .



## S E C T. III.

If a Right Cone be (any where) cut off by a Right-line, that cuts both its Sides, but not parallel to its Base (as *TS* in the following Figure) the Plane of that Section will be an Ellipsis (vulgarly call'd an Oval, viz. an oblong or imperfect Circle, which hath several Diameters, and two particular Centers. That is,

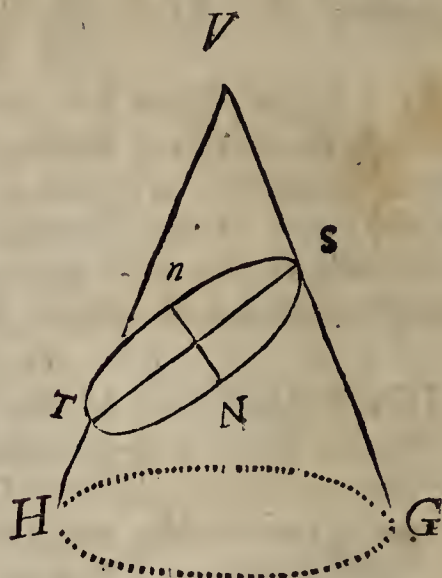
1. Any Right-line that divides an Ellipsis into two equal Parts, is call'd a Diameter; amongst which, the longest and the shortest are particularly distinguish'd from the rest, as being of most general Use; the other are only applicable to particular Cases.

A a a 2

2. The

2. The longest Diameter (as  $TS$ ) is called the **Transverse-Diameter**, or **Transverse-Axis**; being that Right line which is drawn through the Middle of the Ellipsis, and doth shew or limit its Length.

3. The shortest Diameter, call'd the **Conjugate-Diameter**, is a Right-line that doth intersect or cross the **Transverse-Diameter** at Right-angles, in the middle or common Center of the Ellipsis, (as  $Nn$ ) and doth limit the Ellipsis Breadth.



4. The two Points, which I call particular Centers of an Ellipsis, (for a Reason which shall be shew'd farther on) are two Points in the **Transverse-Diameter**, at an equal Distance each Way from the **Conjugate-Diameter**, and are usually call'd *Nodes*, *Focus's*, or *burning Points*.

5. All Right-lines within the Ellipsis that are parallel to one another, and can be divided into two equal Parts, are call'd *Ordinates* with respect to that Diameter which divides them. And if they are parallel to the **Conjugate**, viz. at Right-angles with the **Transverse-Diameter**, then they are call'd *Ordinates rightly apply'd*. And those two that pass through the *Focus's*, are remarkable above the rest, which being equal and situated alike, are call'd both by one Name, viz. *Latus Rectum* or *Right Parameter*, by which all the other *Ordinates* are regulated and valued; as will appear farther on.

## S E C T. IV.

If any Cone be cut into Two Parts, by a Right-line parallel to one of its Sides, (as  $SA$  in the following Scheme) the Plane of that Section (viz.  $SbBAbs$ ) is call'd a *Parabola*.

1. A Right-line being drawn through the Middle of any Parabola, (as  $SA$ ) is call'd its **Axis** or intercepted Diameter.

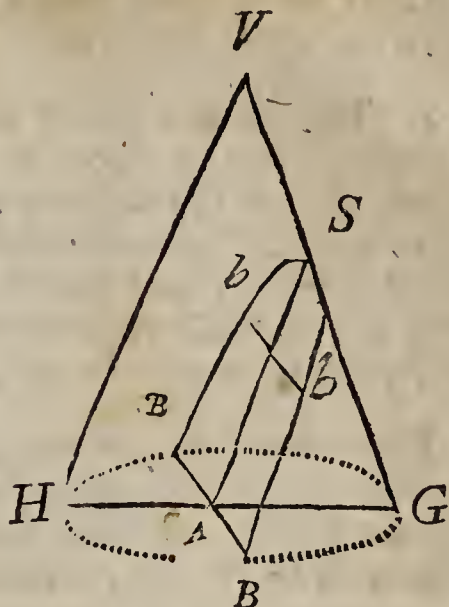
2. All Right lines that intersect or cut the Axis at Right-angles, (as  $BB$ , and  $bb$  are suppos'd to cut or cross  $SA$ ) are call'd *Ordinates rightly apply'd*. (as in the Ellipsis) and the greatest Ordinate as  $BB$ , which limits the Length of the Parabola's Axis ( $SA$ ) is usually call'd the **Base** of the Parabola.

3. That



3. That Ordinate which passes through the Focus, or Burning-Point of the Parabola, is call'd the *Latus Rectum*, or Right-Parameter (as in the Ellipsis) because by it all the other Ordinates are proportion'd, and may be found.

4. The Node, Focus, or Burning-Point of the Parabola, is a Point in its Axis (but not a Center as in the Ellipsis) distant from the Vertex or Top of the Section, (*viz.* from *S*) just one fourth Part of the *Latus Rectum*; as shall be shew'd farther on.

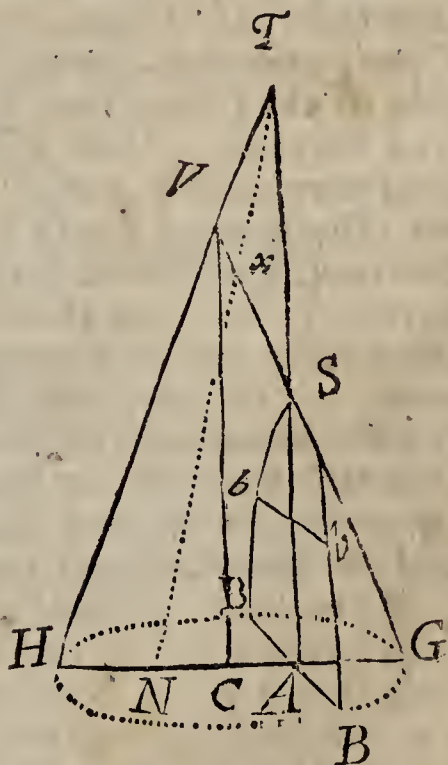


5. All Right-lines drawn within a Parabola parallel to its Axis are call'd Diameters; and every Right-line that any of those Diameters doth bisect or cut into two equal Parts, is said to be an Ordinate to that Diameter which bisects it.

## S E C T. V.

If a Cone be any where cut by a Right-line, either parallel to its Axis, (as *SA*, or otherwise as *xN*) so as the cutting Line being continu'd through one Side of the Cone, (as at *S*, or *x*) will meet with the other Side of the Cone, if it be continu'd or produc'd beyond the Vertex *V*, as at *T*. Then the Plane of that Section, (*viz.* the Figure *SbBBbS*) is call'd an *Hyperbola*.

1. A Right-line being drawn through the Middle of any Hyperbola, *viz.* within the Section, (as *sa* or *xN*) is call'd the Axis or intercepted Diameter, (as in the Parabola) and that Part of it which is continu'd or produc'd out of the Section, until it meet with the other Side of the Cone continu'd, *viz.* *Ts*, or *Tx* &c. is call'd the Transverse-Diameter, or Transverse-Axis of the Hyperbola.



2. All Right-lines that are drawn within an Hyperbola, at Right-Angles to its Axis, are call'd Ordinates rightly apply'd; as in the Ellipsis and Parabola.

3. That

3. That Ordinate which passes through the Focus of the Hyperbola, is call'd *Latus Rectum*, or *right Parameter*, for the same Reason as in the other Sections.

4. The middle Point of the Transverse-Diameter is call'd the Center of the Hyperbola, from whence may be drawn two Right-lines (out of the Section) call'd *Assymptotes*, because they will always incline (that is, come nearer and nearer) to both Sides of the Hyperbola, but never meet with (or touch) them, altho' both they and the Sides of the Hyperbola were infinitely extended, as will plainly appear in its proper Place.

These five Sections, *viz.* the Triangle, Circle, Ellipsis, Parabola, and Hyperbola, are all the Planes that can possibly be produc'd from a Cone. But of them the Three last are only call'd *Conic Sections*, both by the Antient and Modern Geometers.

## S C H O L I U M.

Besides the foregoing Definitions, it may not be amiss to add, by Way of Observation, how one Section may (or rather doth) change or degenerate into another.

An Ellipsis being that Plane of any Section of the Cone, which is between the Circle and Parabola, it will be easy to conceive that there may be great Variety of Ellipses produc'd from the same Cone; and when the Section comes to be exactly parallel to one Side of the Cone, then doth the Ellipsis change or degenerate into a Parabola. Now a Parabola being that Section whose Plane is always exactly parallel to the Side of the Cone, cannot vary as the Ellipsis may: For so soon as ever it begins to move out of that Position, (*viz.* from being parallel to the Cone's Side) it degenerates either into an Ellipsis, or into an Hyperbola. That is, if the Section inclines towards the Plane of the Cone's Base, it becomes an Ellipsis; but if it incline towards the Cone's Vertex, it then becomes an Hyperbola, which is the Plane of any Section that falls between the Parabola and the Triangle. And therefore there may be as many Varieties of Hyperbola's produc'd from one and the same Cone, as there may be Ellipses.

To be brief, a Circle may change into an Ellipsis, the Ellipsis into a Parabola, the Parabola into an Hyperbola, and the Hyperbola into a plane Isocles Triangle. And the Center of the Circle, which is its Focus or Burning-Point, doth, as it were, part or divide it self into two Focus's so soon as ever the Circle begins to degenerate into an Ellipsis; but when the Ellipsis changes into a Parabola, one End of it flies open, and one of its Focus's vanishes,



vanishes, and the remaining Focus goes along with the Parabola when it degenerates into an Hyperbola. And when the Hyperbola degenerates into a plane Iſoſceles Triangle, this Focus becomes the Vertical Point of the Triangle, (*viz.* the Vertex of the Cone.) So that the Center of the Cone's Base may be truly ſaid to paſs gradually through all the Sections, until it arrives at the Vertex of the Cone, ſtill carrying its *Latus Rectum* along with it. For the Diameter of a Circle being that Right-line which paſſes through its Center or Focus, and by which all other Right-lines drawn within the Circle are regulated and valued, may, (I preſume) be properly call'd the Circle's *Latus Rectum*; and altho' it loſes the Name of Diameter when the Circle degenerates into an Ellipſis, yet it retains the Name of *Latus Rectum*, with its firſt Properties in all the Sections, gradually ſhortening as the Focus carries it along from one Section to another, until at laſt it and the Focus becomes coincident, and terminate in the Vertex of the Cone.

I have been more particular, and full in theſe Definitions, than is uſual in Books of this Subject, which I hope is no Fault, but will prove of Uſe, eſpecially to a Learner; and altho' they may perhaps ſeem a little ſtrange, and at firſt hard to be underſtood, yet when they are well conſider'd and compar'd with a Cone, cut into ſuch Sections as have been defin'd, they will not only be found true, but will alſo help to form a true and clear Idea of each Section.

## C H A P. II.

### *Concerning the chief Properties of an Ellipſis.*

*Note.* If the Tranſverſe-Diameter of an Ellipſis, as *T S* in the following Figure, be interſected or divided into any two Parts by an Ordinate rightly apply'd, as at the Points *A, C, a*, &c. Then are thoſe Parts, *T A, T C, T a*, and *S A, S C, S a*, &c. uſually call'd Abſciſſa's, (which ſignifies Lines or Parts cut off) and by the Rectangle of any two Abſciſſa's, is meant the Rectangle of ſuch two Parts as, being added together, will be equal to the Tranſverſe-Diameter.

$$\text{As } TA + SA = TS. \text{ And } TC + SC = TS. \\ \text{Or } TA + SA = TS, \text{ \&c.}$$

S E C T.

## S E C T. I.

Every Ellipsis is proportion'd, and all such Lines as relate to it, are regulated by the Help of one general Theorem.

*Theorem.* { *As the Rectangle of any two Abscissa's : is to the Square of Half the Ordinate which divides them :: So is the Rectangle of any other two Abscissa's : To the Square of half that Ordinate which divides them.*

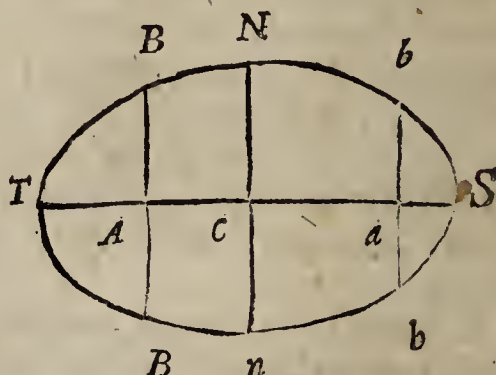
That is,

$$TA \times SA : \square BA :: Ta \times sa : \square ba$$

$$TA \times SA : \square BA :: TC \times SC : \square NC$$

$$TC \times SC : \square NC :: Ta \times Sa : \square ba$$

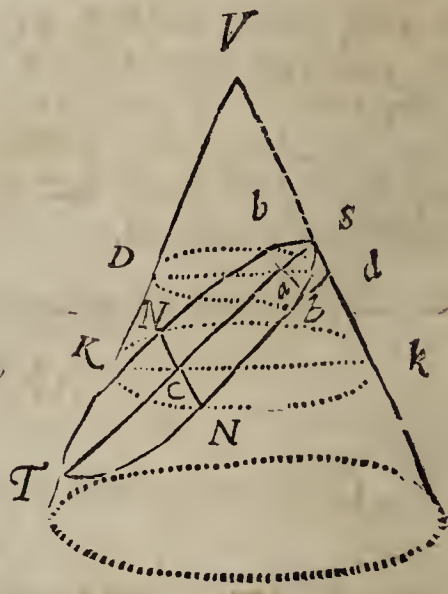
&c.



## Demonstration.

Let the annex'd Figure represent a right Cone, cut through both its Sides by the Right-line TS; then will the Plane of that Section be an Ellipsis, (by Sect. 3. Chap. 1.) TS will be the Transverse-Diameter N C N and *b a b* will be the Ordinates rightly apply'd. As before.

Again, if the Line *D d* and *K k* be parallel to the Cone's Base, they will be Diameters of Circles (by Sect. 2. Chap. 1.) Then will  $\triangle TCK$  and  $\triangle TaD$  be alike. Also  $\triangle sad$  and  $\triangle sck$  will be alike.



Ergo	1	$Sa : ad :: SC : Ck$	} Per Theorem 13.
And	2	$TC : CK :: Ta : aD$	
I	3	$Sa \times Ck = ad \times SC$	
2	4	$Ta \times CK = TC \times aD$	
2 x 3	5	$Sa \times Ck \times Ta \times CK = ad \times SC \times TC \times aD$	Per Axiom 3.
But	6	$CK \times Ck = \square NC$	} Per Lemma Sect. 2.
And	7	$aD \times ad = \square ba$	
Then		for $CK \times Ck$ , and $aD \times ad$ , take $\square NC$ , and $\square ba$	
5, 6, 7.	8	$Sa \times Ta \times \square NC = TC \times SC \times \square ba$	Per Axiom 5.
Hence	9	$Sa \times Ta : \square ba :: TC \times SC : \square NC$	See Page 194.

Q. E. D.  
-Or



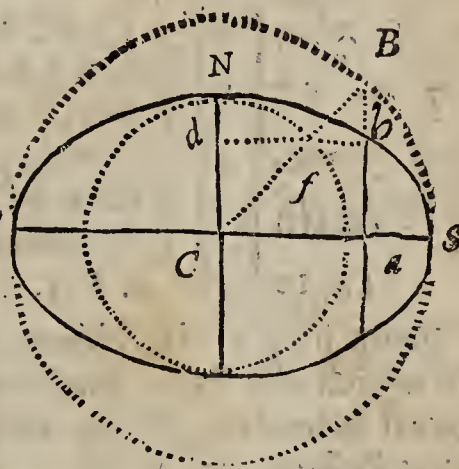
Or the Truth of these Proportions may be otherwise prov'd by a Circle, without the Help of the Cone. Thus;

Let any Ellipsis be circumscrib'd and inscrib'd with Circles, as in the following Figure. Then from any Point in the circumscribed Circle's Periphery, as at *B*, draw the Right-line *B a* parallel to the semi-conjugate Diameter *NC*, then will *b a* be a semi-ordinate rightly apply'd to the Transverse Diameter *TS*, as before.

Again, from the Point *b* (in the Ellipsis's Periphery) draw the Right line *b d*, parallel to the Transverse *TS*; and draw the Radius *BC*.

Then will  $\triangle BCa$  and  $\triangle Cfd$  be alike.

Therefore	1	$\{ BC : Ba :: Cf : dC$
		$\{ \text{Per Theorem 13.}$
But	2	$\{ TC = BC \quad NC = Cf \quad T$
		$\{ \text{and } ba = dC$
Consequ.	3	$TC : Ba :: NC : ba$
Or	4	$TC : NC :: Ba : ba$
4 in $\square$ 's	5	$\square TC \square NC :: \square Ba : \square ba$
But	6	$\{ Ta \times Sa = \square Ba$
		$\{ \text{Per Lem. Sect. 2. C. I.}$
Therefore	7	$\{ Ta \times Sa :: \square ba :: TC \times SC = \square TC : \square NC.$
		$\{ \text{As before.}$



And so for any other Abscissa's, and their Semi-ordinates.

These Proportions being found to be the true and common Properties of every Ellipsis, all that is farther requir'd in ( or about ) that Section, may be easily deduc'd from them.

## Sect. 2. To find the Latus Rectum, Or Right Parameter of any Ellipsis.

There are several Ways of finding the *Latus Rectum*; but I think none so easy. and shews it so plainly to be the third principal Line in the Ellipsis, as the following Theorem.

**Theorem.**  $\left\{ \begin{array}{l} \text{As the Transverse Diameter : is in Proporti-} \\ \text{on to the Conjugate : : So is the Conjugate : To} \\ \text{the Latus Rectum.} \end{array} \right.$

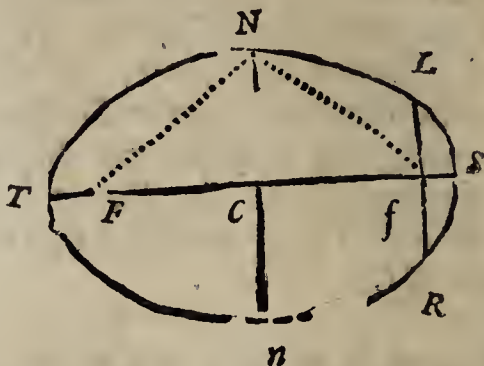
Viz. ( in the following Figure )  $TS : Nn :: Nn : LR$  the *Latus Rectum*.

### Demonstration.

From the last Proportions take either of the Antecedents, and its Consequent, viz. either  $TC \times SC : \square NC.$  Or  $Ta \times Sa : \square ba$   
*B b b* *and*

and make  $TS$  the third Term, to which find a fourth Proportional, and it will be  $= LR$ :

Thus	1	$TC \times SC : \square NC :: TS : LR$
But	2	$\{ TC = SC$ And $NC = Cn$
Therefore	3	$TC \times SC = \frac{1}{4} \square TS$
And	4	$\square NC = \frac{1}{4} Nn$
1, 3, 4	5	$\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$
5	6	$\frac{1}{4} \square TS \times LR = \frac{1}{4} Nn \times TS$
6 $\times$ 4	7	$\square TS \times LR = \square Nn \times TS$
7 $\div$ $ST$	8	$\{ TS \times LR = \square Nn$ Which gives the following Analogy.
viz.	9	$TS : Nn :: Nn : LR$
Again	10	$\{ TC \times SC : \square NC :: Ta \times Sa : \square ba.$ By common Properties.
1, 10	11	$TS : LR :: Ta \times Sa : \square ba$



From hence 'tis evident, that  $LR$  thus found, is that Ordinate by which the other Ordinates may be regulated and found. Therefore (according to its Definition, *Sect. 3. Chap. I.*) it is the true *Latus Rectum*. Q. E. D.

### Conseſtary.

Hence it follows, that if the Transverse, and Conjugate Diameters of any Ellipsis are given (either in Lines or Numbers) the *Latus Rectum* may be easily found; and then any Ordinate, whose Distance from the Conjugate is given, may be found, as above.

### Sect. 3. To find the Focus of any Ellipsis.

The Focus is the Distance of the *Latus Rectum* from the Conjugate or Middle of the Ellipsis, (*vide* Definition 4. Page 358.) And that Distance is always a mean Proportional between the Half Sum and Half Difference of the Transverse and Conjugate Diameters, which gives this Theorem.

**Theorem** { From the Square of Half the Transverse, subtract the Square of Half the Conjugate; the square Root of their Difference will be the Distance of each Focus from the middle or common Center of the Ellipsis.

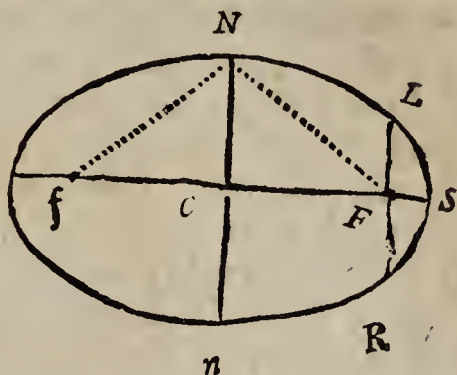
That is, supposing the Points  $f$  and  $F$  to be the two Focus's, viz.  $fC = CF$  and  $TC = \frac{1}{2} TS$ .  $NC = \frac{1}{2} Nn$ . Then  $TC + NC : fC :: FC : TC - NC$ . Ergo  $\square FC = \square TC - \square NC$ . Consequently,  $FC = \sqrt{\square TC - \square NC}$ .

Demon



Demonstration.

First	1	$TS \times LR = \square Nn$ .	By 8 Step of the last Process.
And	2	$TS:LR::TF \times SF: \square LF$ .	Per common Properties.
That is	3	$TS:LR::TC + CF \times TC - CF: \frac{1}{4} \square LR = \square LF$	
	4	$\frac{1}{4} \square LR \times TS =$	
3	4	$\square TC - \square CF \times LR$	
4	5	$\frac{1}{4} LR \times TS = \square TC - \square CF$	
1	6	$\frac{1}{4} TS \times LR = \frac{1}{4} \square Nn = \square$	
		$NC$	
5,	7	$\square NC = \square TC - \square CF$	
7,	8	$\square CF = \square TC - \square NC$	
8	9	$CF = \sqrt{\square TC - \square NC}$	



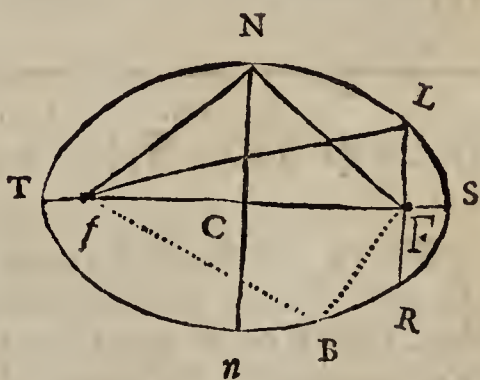
Now from hence is deduc'd that notable Proposition, upon which is grounded the usual Method of describing an Ellipsis, and drawing of Tangents, &c.

**Proposition.** { If from the two Focus's of any Ellipsis there be drawn two Right-lines, so as to meet each other in any Point of the Ellipsis's Periphery, the Sum of those Lines will be equal to the Transverse.

*viz.*  $fN + NF = TS$ .  $fL + LF = TS$ . Or  $fB + BF = TS$ , &c.

Demonstration.

First	1	$\square CF + \square NC = \square TC$	
		By 8th of the last.	
But	2	$\square CF + \square NC = \square NF$	
		By Theorem 15.	
1,	2	$\square NF = \square TC$	
		By Axiom 5.	
3	4	$NF = TC$	
Hence		$2NF = 2TC = TS$	
Again	5	$TS:LR::TF \times FS: \square LF$ .	By common Properties.
Conseq.	6	$\frac{1}{2} TS: \frac{1}{2} LR:: TF \times FS: \square LF$	
But		$\frac{1}{2} TS = TC$ . And $\frac{1}{2} LR = LF$	
Ergo	7	$TC:LF::TC + CF \times TC - CF: \square LF$	
7	8	$TC \times LF = \square TC - \square CF$	
But	9	$\square fF + \square LF = \square fL$ . By Theorem 11.	
That is.	10	$4 \square CF + \square LF = \square fL$ for $2 CF: = fF$	
8	11	$4 TC \times LF = 4 \square TC - 4 \square CF$	
10	12	$4 \square TC + \square LF = 4 TC \times LF: + \square fL$	
12	13	$4 \square TC: - 4 TC \times LF: + \square LF = \square fL$	



$$\begin{array}{l|l|l}
 13. \text{ } w \text{ } 2 & 14. & 2 \text{ TC} - \text{LF} = f \text{ L} \\
 14. + \text{LF} & 15. & 2 \text{ TC} = f \text{ L} + \text{LF.} \text{ But } 2 \text{ TC} = \text{TS} \\
 \text{Ergo} & 16. & f \text{ L} \times \text{L} f = \text{TS.}
 \end{array}$$

Q. E. D.

And this Proposition must needs hold true to every Point in the Ellipsis's Periphery, *viz.* at B, &c. as will evidently appear to any one who rightly considers, That

As a Thread just the Length of the Diameter of any Circle, having its two Ends ty'd together, and then mov'd about a Point in the Center, (*viz.* by making it a double Radius) will, by drawing another Point in its Extremity, describe the Periphery of a Circle (*vide* Definition, Page 280.) Even so,

If a Thread, just the Length of the Transverse Diameter (TS) having its two Ends so fix'd upon the two Focus's, (f and F) as that it may be mov'd about them, by drawing a Point in its Extremity, (*viz.* at its full Stretch) it will describe the true Periphery of an Ellipsis.

Now although this easy Way of describing, or, as usually phras'd, drawing an Ellipsis, be mechanical, and known even to most Joyners, Carpenters, &c. yet it gives as complete and clear an Idea of that Figure, as any other Way whatsoever; and by describing it thus about its two Focus's, as a Circle is about its Center, doth plainly shew those two Points are not improperly call'd particular Centers in Definition 4. Sect. 3. Chap. I. For each of them bears much the same Respect to the Ellipsis's Periphery, as the Circle's Center doth to its Periphery.

#### Sect. 4. To describe, or delineate an Ellipsis several Ways.

There are several (other) Ways of describing an Ellipsis, both Geometrically and Numerically, according to peculiar Occasions; but I shall only mention two or three of them, leaving the rest to the Learner's Genius. Now in order to that Work, it will be convenient to consider what Lines are requisite to limit or bound its Form, which I take to be chiefly these following.

1. If the Transverse and Conjugate are given, the Ellipsis is perfectly limited, (*vide* Confectary Page 363.) For if TS and Nn, be set at Right-angles in their Middle at C, and TC or CS, be set off from N, or n, both Ways upon the Transverse to f and F; (*viz.* make  $fN = TC = NF$ ) then will those Points f and F be the two Focus's, (by 4th Step of the last Process) and then the Ellipsis may be describ'd as above.

2. If



2. If the Transverse Diameter and *Latus Rectum* are given, the Ellipsis is truly limited, because by them the Conjugate may be found, by *Seç. 2.*

3. Or if only the Transverse, and the Proportion it hath either to the Conjugate or *Latus Rectum*, be given, the Ellipsis is thereby limited: As for Instance, suppose the given Ratio between the Transverse and Conjugate to be, As  $a$ : to  $d$ ;

Viz.  $a : d :: TS : Nn$ , then  $\frac{TS \times d}{a} = Nn$ , &c.

4. If either the Transverse, or Conjugate, and the Distance of the Focus from the Conjugate be given, the Ellipsis is limited, because by them the Conjugate or Transverse may be found.

These being premis'd, and the preceding Work a little consider'd, it must needs be easy to describe, or delineate any Ellipsis in *Plano*, either Geometrically or Numerically.

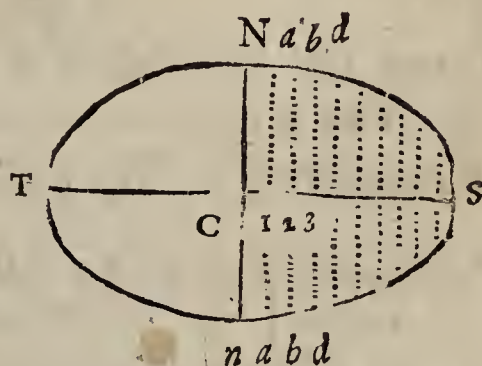
### 1. To describe an Ellipsis Numerically by Points.

Suppose the Transverse Diameter  $TS = 20$ , and the Conjugate  $Nn = 12$ , (either Inches or any other equal Parts) and let them cross each other at Right-angles in their Mid-dles, As in the Point  $C$ .

Then will  $TC = CS = 10$ .

And  $NC = Cn = 6$

And it will be  $20 : 12 :: 12 : 7,2 =$  the *Latus Rectum*.



Again  $20 : 7,2$  Or rather take their Ratio.

Thus  $\begin{cases} 1 : 0,36 :: 10 + 1 \times 10 - 1 : \square a. \parallel 1. \\ 1 : 0,36 :: 10 + 2 \times 10 - 2 : \square b. \parallel 2. \\ 1 : 0,36 :: 10 + 3 \times 10 - 3 : \square d. \parallel 3. \end{cases}$  &c.

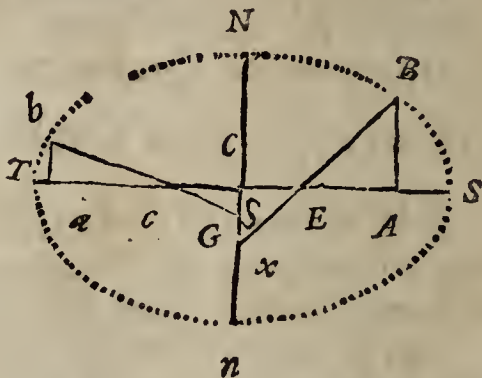
Viz.  $\begin{cases} 100 - 1 \times 0,36 = \square a.1. \text{ Hence } \sqrt{99 \times 0,36} = 5,97 \text{ } \&c. = a.1. \\ 100 - 4 \times 0,36 = \square b.2. \quad \sqrt{96 \times 0,36} = 5,88 \text{ } \&c. = b.2. \\ 100 - 9 \times 0,36 = \square d.3. \quad \sqrt{91 \times 0,36} = 5,72 \text{ } \&c. = d.3. \end{cases}$

If so many Semi-ordinates as may be thought convenient (the more the better) be found in this manner. and every one of them be set off at Right-angles from its respective Point in the Transverse Diameter each Way; viz. from 1 to  $a$ , from 2 to  $b$  from 3 to  $d$ . &c. Then if a curve Line be carefully drawn with an even Hand, through those extreme Points  $a, b, d$ , &c. it will be the Ellipsis Periphery requir'd.

2. To

## 2. To describe an Ellipsis Geometrically by Points.

Having the Transverse and Conjugate Diameters given, viz.  $TS$  and  $Nn$ , plac'd at Right-angles in their Middles, as before; then from either End of the Conjugate, viz.  $N$  (or  $n$ ) set off Half the Transverse Diameter to  $x$ . That is, make  $Nx = TC$  (continuing the Conjugate  $Nn$ , when it is shorter than  $TC$ .) Or, which is all one, make  $Cx = TC - NC$ . Then take any Point in the Line  $Cx$  at Pleasure. Suppose it at  $G$ , and from that Point at  $G$  set off the Distance  $Cx$  to the Transverse, as at  $E$ , viz. make  $GE = Cx$ , and join the Points  $GE$  with a Right-line, produc'd so far beyond  $E$ , as to make  $EB = NC$ . Consequently,  $GB = TC$ .



Then, I say, that where-ever the Point  $G$  was taken between  $C$  and  $x$ , the Point  $B$  will just touch (or fall in) the Ellipsis's Periphery.

## Demonstration.

Draw the Right-line  $BA$  perpendicular to  $TS$ . viz. let  $BA$  be a Semi-ordinate rightly apply'd to the Transverse Diameter  $TS$ . Then  $\triangle GCE$  and  $\triangle BAE$  will be alike.

Consequen.	1	$CE:AE::EG:EB$ . By Theorem 13.
1, And	2	$CE+AE:AE::EG+EB:EB$ . See Page 192.
But	3	$CE+AE=CA$ . $EG+EB=TC$ . And $EB=NC$ .
Therefore	4	$CA:AE::TC:NC$
6, in $\square$ 's	5	$\square CA:\square AE::\square TC:\square NC$
5, $\therefore$	5	$\frac{\square CA \times \square NC}{\square TC} = \square AE$
But	6	$\square NC - \square AB = \square AE$
That is	7	$\square EB - \square AB = \square AE$
6, 7	8	$\frac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$
8 $\times \square TC$	9	$\square CA \times \square NC = \square NC \times \square TC - \square AB \times \square TC$
9 $\pm$	10	$\square NC \times \square TC - \square CA \times \square NC = \square AB \times \square TC$
10. Analogy	11	$\square TC:\square CN::\square TC-\square CA:\square AB$
That is	12	$TC \times CS:\square NC::TC+CA \times TC-CA:\square AB$

Which is according to the common Properties of the Ellipsis. Therefore the Point  $B$  is truly found. Q. E. D.

Hence



Hence it follows, that if a convenient Number of such Lines as  $GEB$ , be so drawn, as above directed, from the like Number of Points taken between  $C$  and  $\infty$ , &c. their extreme Points (as at  $B$ ) will be those Points by which (with an even Hand) the Ellipsis may be truly described, as before:

But if this be well understood, it will be very easy to conceive how to describe an Ellipsis very readily, without drawing those Lines, by having a thin, strait, narrow Ruler, just the Length of  $TC$ , made somewhat sharp at both Ends, upon which, from one of its Ends, set off the Length of  $NC$ . Then if the Point upon the Ruler which represents  $E$  be gradually or easily mov'd along the Transverse  $TS$ , and at the same time the Point or End representing  $G$  be kept sliding close along the Conjugate  $Nn$ , 'tis evident from the Work above, that the End of the Ruler representing  $B$ , will, by that Motion, assign the true Periphery of the Ellipsis requir'd. For by that Motion the strait Edge of the Ruler doth supply an infinite Number of the aforesaid Lines; as will appear very plain and easy in Practice.

### SCHOLIUM.

Now from hence was deduc'd the first Invention of that well-contriv'd Instrument for drawing an Ellipsis by one Motion, commonly call'd the Elliptical Compasses, being usually made of Brass, and compos'd of three Parts, two of which represent (or rather supply) the Transverse and Conjugate Diameters set together at Right angles; and the third Part is a moveable Ruler, which performs the Office of the last mentioned thin Ruler. But because the making of it is so well-known to most Mathematical Instrument-Makers, especially to that accurate and ingenious Artist Mr. *John Rowley*, Mathematical-Instrument-Maker, under St. *Dunstan's* Church in *Fleet-street*, *London*; who for his great Skill in contriving, framing, and graduating all Kind of Mathematical-Instruments, may, I believe, be justly call'd one of the best Workmen of his Trade in *Europe*. I think it needless therefore to give a particular Description of that Instrument.

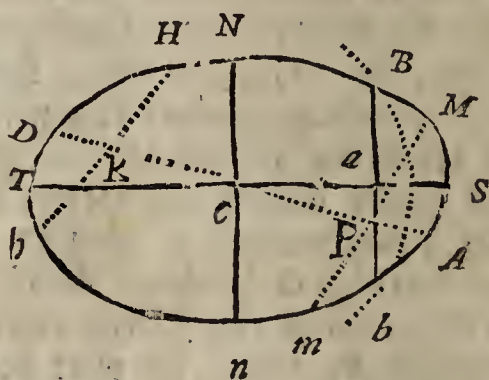
Also from hence came that ingenious Invention of making Engines for turning all Sorts of Elliptical or Oval Work, as Oval-Boxes and Picture-Frames, &c.



**Sect. 5.** *Any Ellipsis being given, To find its Transverse and Conjugate Diameters.*

Suppose the given Ellipsis to be  $TNSn$  (in the annex'd Scheme) in which let it be requir'd to find the Transverse Diameter  $TS$ , and its Conjugate  $Nn$ .

Draw within the Ellipsis any two Right-Lines parallel to each other, as  $Hh$  and  $Mm$ ; and bisect those Lines, viz. find the middle Point of each, as at  $K$  and  $P$ . Then through those Points  $K$  and  $P$ , draw a Right-line, as  $DA$ , and it will be a Diameter; for it will divide the Ellipsis into two equal Parts. (See Definit. 1. Page 357.) Consequently the Middle of  $DA$  will be the true Middle or common Center of the Ellipsis,



as at  $C$ .

For 'tis the Nature, or Property of all Diameters, howsoever they are drawn in any Ellipsis, (as it is in a Circle) to cut or cross one another in the common Center, or Middle of the Figure, as at  $C$ .

Upon the Point  $C$  describe an Arch of any Circle that will cut the Ellipsis Periphery in two Points, as at  $B$  and  $b$ ; then joyn those Points  $Bb$  with a Right-line, and it will be an Ordinate, through whose Middle, (as at  $a$ ) and the common Center  $C$ , the Transverse Diameter  $TS$  must pass.

For  $BS = Sb$ , and  $Ba$  is at Right-angles with  $TS$ ; therefore the Line  $Bb$  is an Ordinate rightly apply'd to  $TS$  the Transverse Diameter. And if through the Point  $C$  there be drawn the Right-line  $Nn$  parallel to  $Bb$ , it will become the Conjugate; as was requir'd.

**Sect. 6.** *To draw a Tangent, or Right-line that may touch the Ellipsis's Periphery in any assign'd Point.*

The drawing of Tangents to or from any assign'd Point in the Ellipsis's Periphery, admits of three Cases.

**CASE I.** If it be requir'd to draw a Tangent that may touch the Ellipsis in either of the extreme Points of its Transverse Diameter, as at  $T$  or  $S$ , it is plain the Tangent must be drawn parallel to the Conjugate Diameter  $Nn$ , as  $HK$  in the following Figure is suppos'd to be.

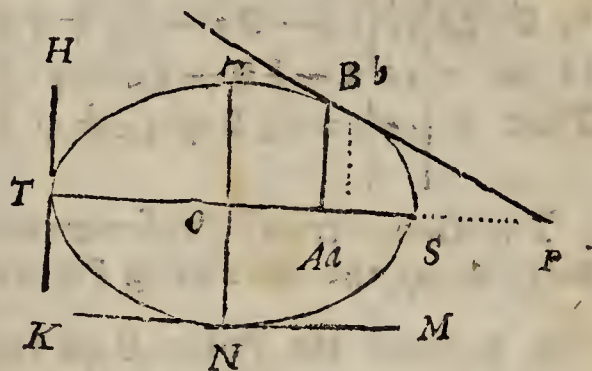
**CASE**



CASE 2. Or if the Tangent must be drawn to touch the Ellipsis in either of the extreme Points of its Conjugate Diameter, as at *N* or *n*, 'tis as evident that it must be drawn parallel to the Transverse Diameter *TS*, as *K M*. Consequently, if that Tangent, and the Transverse were both infinitely continu'd, they would never meet.

CASE 3. But if it be requir'd to draw a Tangent that may touch the Ellipsis in any other Point, as at *B*, &c.

Then if the Tangent and the Transverse-Diameter be both continu'd, they will meet in some Point, as at *P*; and those two Points (*viz.* *B* and *P*) do so mutually depend upon each other, that one of them must be assign'd in order to find the other, that so the Tangent may by them be truly drawn.



Let  $D = TS$ .  $y = AS$ : and  $z = AP$ . Then if  $y$  be given,  $z$

may be found by this Theorem 
$$\sqrt{\frac{Dy - yy}{\frac{1}{2}D - y}} = z$$

Or if  $z$  be given,  $y$  may be found by this Theorem:

Theorem. 
$$\sqrt{\frac{D + z}{z} \pm \sqrt{\frac{DD + zz}{4}}} = y$$

## Demonstration.

Draw the Semi-Ordinate *ba*, as in the Figure; then will  $\triangle BAP$  and  $\triangle bap$  be alike. Put  $x = Aa$  the Distance between the two Semi-ordinates (*viz.* between *BA* and *ba*) which we suppose infinitely small.

Then	1	$z : z - x :: BA : ba$ . By Theorem 13.
But	2	$D - y \times y : D - y + x \times y - x :: \square BA : \square ba$
That is,	3	$Dy - yy : Dy - yy + 2yx - Dx - xx :: \square BA : \square ba$
in $\square$ 's.	4	$zz : zz - 2zx + xx :: \square BA : \square ba$
Suppose	5	$x = 0$ That so $x$ may be every where rejected.
3, Then	6	$Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$
4, And	7	$zz : zz - 2z :: \square BA : \square ba$
6, 7	8	$Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$
8	9	$2yz - Dz = 2yy - Dy$
9	10	$yz - \frac{1}{2}Dz = yy - Dy$
10	11	$\frac{1}{2}Dz - yz = Dy - yy$

C c c

II	÷	I2	$z = \frac{Dy - yy}{\frac{1}{2}D - y}$	} } Which is the 1st Theorem, and gives the following <i>Analogy</i> .
Analogy.	I3	$\frac{1}{2}D - y : y :: D - y : z.$	Viz, $CA : SA :: TA : AP$	
I0	$-yz$	I4	$yy - Dy - yz = -\frac{1}{2}Dz.$	
I4	C □	I5	$yy - Dy - yz + \frac{1}{4}DD - \frac{1}{2}Dz + \frac{1}{4}zz = \frac{1}{4}DD + \frac{1}{4}zz$	
I5	w 2	I6	$y - \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	
That is.	I7	$y = \frac{1}{2}D + \frac{1}{2}z \pm \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$		Which is the 2d. Theor. Q.E.D.

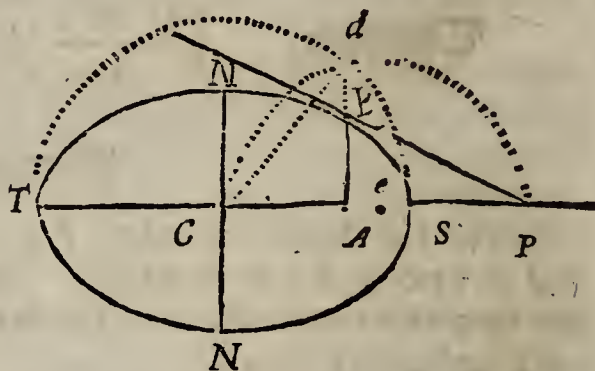
The Geometrical Performance of these two Theorems is very easy, as by the following Figure.

1. Suppose the Point  $B$  in the Ellipsis's Periphery were given; and it were requir'd to find the Point  $P$ , &c.

Make  $TC$  Radius, and upon the common Center  $C$  describe the Semi-circle  $TdS$ , and joyn the Points  $C$  and  $d$  with a Right-line; then bisect that Line, ( by *Prob. 2. Page 287.* ) and mark the Point where the bisecting Line would cross the Transverse, as at  $\parallel e$ . Upon that Point  $\parallel e$ , with the Radius  $Ce$ , ( or  $ed$  ) describe another Semi-circle. producing the Transverse-Diameter to its Periphery, and it will assign the Point  $P$ .

For if  $D = TS$ .  $y = AS$ .  $z = AP$ . as before.

Then	I	$D - y \times y = \square dA$
And	2	$\frac{1}{2}D - y \times z = \square dA$
For	3	$TA : dA :: dA : SA$
And	4	$CA : dA :: dA : AP$
But		$CA = \frac{1}{2}D - y$ , &c.
I, 2	5	$\begin{cases} \frac{1}{2}Dz - yz = Dy - yy \\ \text{As at II Step before.} \end{cases}$



Therefore the Point  $P$  is truly found. Consequently, if a Right-line be drawn through those Points  $B$  and  $P$ , it will be the Tangent requir'd, according to the first Theorem.

2. The Converse of this is as easy; to wit, if the Point  $P$  be given, thence to find the Point  $B$  in the Ellipsis's Periphery. Thus, circumscribe Half the Ellipsis with the Semi-circle  $TdS$ , as before; and bisect the Distance between the Points  $C$  and  $P$ , as at  $\parallel e$ , viz. let  $Ce = eP$ . Then making  $Ce$  Radius, upon the Point  $\parallel e$  describe the Semi-circle  $CdP$ ; and from the Point where the two Semi-circles intersect or cross each other, as at  $\parallel d$ , draw the Right-line  $dA$  perpendicular to the Transverse  $TS$ , and

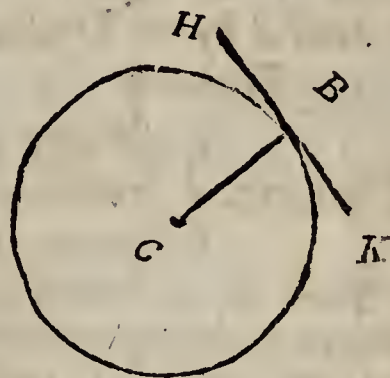


and it will assign the Point of Contact  $B$  in the Ellipsis's Periphery through which the Tangent must pass.

But a Practical Method of drawing Tangents to any assign'd Point in the Ellipsis's Periphery, may (without finding the afore-said Point  $P$ ) be easily deduc'd from the following Property of Tangents drawn to a Circle, which is this.

If to any Radius of a Circle, as  $CB$  there be drawn a Tangent-Line (as  $HK$ ) to touch the Radius at the Point  $B$ , the two Angles which the Tangent makes with the Radius, will always be two Right-angles, ( 16, 17, 18, 19 *Euclid* 3.)

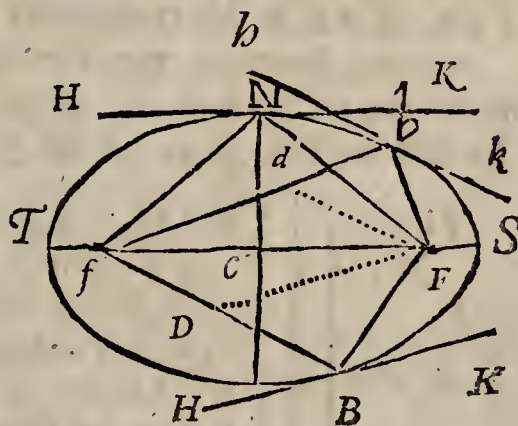
That is,  $\angle HBC = \angle CBK = 90^\circ$



In like manner the two Angles made between the Tangent and the two Lines drawn from the Focus's of an Ellipsis to the Point of Contact, will always be equal, but not Right-angles, save only at the two Ends of the Transverse-Diameter.

These being well consider'd, and compar'd with what hath been said in *Page* 366, it must needs be easy to understand the following Way of drawing Tangents to any assign'd Point in the Ellipsis's Periphery; which is thus:

Having by the Transverse and Conjugate-Diameters found the two Focus's  $f$ , and  $F$ , by *sect.* 3. from them draw two Right-lines to meet each other in the assign'd Point of Contact, as  $fb$  and  $Fb$  (or  $fB$  and  $FB$ ) in the annex'd Figure. Next set off (viz. make)  $bd = bF$ , (or  $BD = BF$ ) and join the Points  $Fd$  (or  $F.D$ ) with a Right-line.



Then, I say, if a Right-line be drawn through the Point of Contact  $b$  (or  $B$ ) parallel to  $dF$ , or ( $DF$ ) it will be the Tangent requir'd. For it is plain, that as the  $\angle fNH = \angle FNK$  when the Tangent is parallel to the Transverse-Diameter, even so is the  $\angle fbb = \angle Fbk$ , (and  $\angle fBH = \angle FBK$ ) and will be every where so, as the Point of Contact  $b$  (or  $B$ ) and its Tangent, is carry'd about the Ellipsis's Periphery with the Lines  $fbF$  (or  $fBF$ )

## C H A P. III.

Concerning the chief Properties of every Parabola.

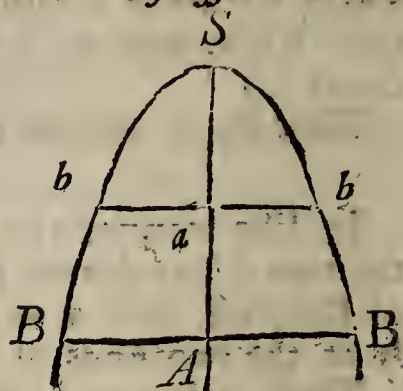
**N**ote, In every Parabola, the intercepted Diameter, or that Part of its Axis which is between the Vertex and that Ordinate which limits its Length, as  $sa$  or  $SA$ , &c. is call'd Abscissa.

Sect. 1. The Plane or Figure of every Parabola, is proportion'd by its Ordinates and Abscissa's, as in the following Theorem.

Theorem.  $\left\{ \begin{array}{l} \text{As any one Abscissa: is to the Square of its} \\ \text{Semi-ordinate:} :: \text{so is any other Abscissa: to the} \\ \text{Square of its Semi-ordinate.} \end{array} \right.$

That is, if we suppose the annex'd Figure to be a Parabola, wherein  $sa$ , and  $SA$  are Abscissa's, and  $ba$ ,  $BAB$ , Ordinates rightly apply'd, it will

be  $sa : \square ba :: SA : \square BA$  } wheresoever  
Or  $sa : SA :: \square ba : \square BA$  } the Points  $a$ ,  
And so for any other Abscissa, &c.

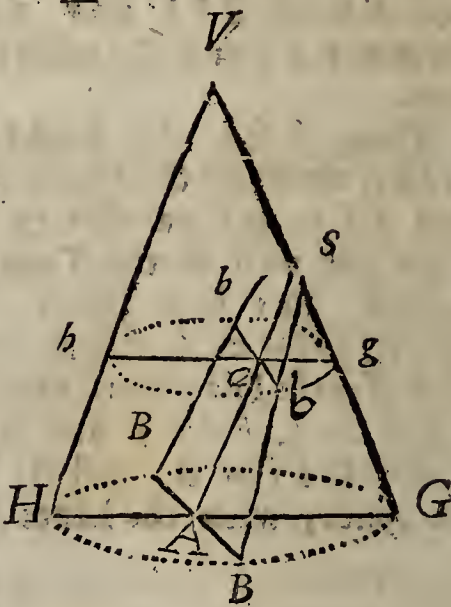


## Demonstration.

Let the following Figure  $HVG$  represent a Right Cone, cut into two Parts by the Right-line  $SA$  parallel to its Side  $VH$ . Then the Plane of that Section, viz.  $BbsbB$  will be a Parabola by Sect. 4. Page 358. wherein let us suppose  $SA$  to be its Axis, and  $ba$ ,  $BAB$  to be Ordinates rightly apply'd to that Axis.

Again, imagine the Cone to be cut by the Right-line  $bg$  parallel to its Base  $HG$ . Then will  $bg$  be the Diameter of a Circle, by Sect. 2. Page 357. And  $\triangle sag$  like to  $\triangle SAG$ .

Therefore	1	$\{ sa : ag :: SA : AG$
	2	$\{ \text{By Theorem 13.}$
I		$sa \times AG = SA \times ag$
$2 \times ba$	3	$\{ sa \times AG \times ba = SA \times ag \times ba$
		$\{ \text{By Axiom 3.}$
But	4	$\{ HA = ba \text{ because } SA$
		$\{ \text{is parallel to } VH$
And	5	$\{ \square BA = AG \times HA \}$ By Lem.
		$\{ \square ba = ag \times ba \}$ P. 357.
3, 4, 5	6	$\{ SA \times \square BA = SA \times \square ba$
		$\{ \text{By Axiom 5.}$
6, Analogy	7	$\{ sa : \square ba :: SA : \square BA$
		$\{ \text{Vide Page 194.}$



Q. E. D.

These



# Chap. 3. Concerning the Parabola. 381

These Proportions being prov'd to be the common Property of every Parabola, all that is farther requir'd about that Section, or Figure, may from thence be easily deduc'd.

## Sect. 2. To find the Latus Rectum, or Right Parameter of any Parabola.

The *Latus Rectum* of a Parabola hath the same Ratio or Proportion to any Abscissa, and its Semi-ordinate, as the *Latus Rectum* of an Ellipsis hath to its Transverse and Conjugate Diameters, and may be found by this Theorem.

**Theorem.**  $\left\{ \begin{array}{l} \text{As any Abscissa : is in Proportion to its Semi-ordinate : : so is that Semi-ordinate to the} \\ \text{Latus Rectum.} \end{array} \right.$

Let  $L =$  the *Latus Rectum*.

Then  
And  
1  
2  
3  
4  
5  
6  
7

1	$Sa : ba :: ba : L$	$\left\{ \begin{array}{l} \text{wherever the Points } a, \text{ and } A, \\ \text{are taken in the Axis.} \end{array} \right.$
2	$Sa : BA :: BA : L$	
3	$\frac{\square ba}{Sa} = L$	Or $Sa \times L = \square ba$
4	$\frac{\square BA}{SA} = L$	Or $SA \times L = \square BA$
5	$\frac{\square BA}{SA} = \frac{\square Sa}{ba}$	Per Axiom 5.
6	$Sa \times \square BA = Sa \times \square ba$	Which gives this
7	$Sa : \square ba :: SA : \square BA$	The same as the 7th Step of the last Process; therefore $L$ (thus found) is the true <i>Latus Rectum</i> , by which all the Ordinates may be regulated and found, according to its Definition in Section 4. Page 358.

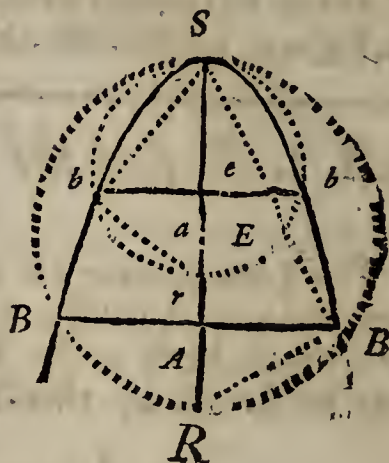
For by the 3d Step  $Sa \times L = \square ba$ . And by the 4th Step  $SA \times L = \square BA$ . Consequently,  $\sqrt{Sa \times L} = ba$  And  $\sqrt{SA \times L} = \square BA$ ; and so for any other Ordinate.

Or if the Ordinates are given, to find their Abscissa's; then it will be,  $L : ba :: ba : Sa$ . And  $L : BA :: BA : SA$ , &c.

Consequently,  $\frac{\square ba}{L} = Sa$ . And  $\frac{\square BA}{L} = SA$ , &c.

From the Consideration of these Proportions, it will be easy to conceive how to find the *Latus Rectum* Geometrically, thus:  
Join

Join the Vertical Point  $S$  of the Axis, and either extreme Point of any Ordinate, as  $B$ , (or  $b$ ) with a Right-line, viz.  $SB$  (or  $sb$ ) and bisect that Line, (by *Problem 2. Page 287.*) marking the Point where the bisecting Line doth intersect or cross the Axis, as at  $E$ , (or  $e$ ) and with the Radius  $SE$  (or  $se$ ) upon the Point  $E$  (or  $e$ ) describe a Circle; (as in the annex'd Figure) then will the Distance between the Ordinate and that Point where the Circle's Periphery cuts the Axis, viz.  $AR$  (or  $ar$ ) be the true *Latus Rectum* requir'd.



For  $SA : BA :: BA : AR$ . And  $sa : ba :: ba : ar$ . By *Theor. 13* Therefore  $AR = L$ . And  $ar = L$  by the 1st and 2d Steps above.

### CONSECTARY.

From these Proportions of finding the *Latus Rectum*, it will be easy to deduce and demonstrate this following Theorem.

**Theorem.** *As the Latus Rectum: Is to the Sum of any two Semi-ordinates :: so is the Difference of those two Semi-ordinates: to the Difference of their Abscissa's.*

Suppose any Right-line drawn within the Parabola, as  $bD$ , parallel to its Axis  $SA$ ; then will that Line (viz.  $bD$ ) be a Diameter (by *Def. 5. Page 359.*) which will make  $ED = AB + ab$ ,  $DE = AB - ab$ , and  $bD = SA - sa$ . Then it will be

$L : ED :: DB : bD$  according to the Theorem,

**Demonstration.**

First 1  $\left\{ \begin{array}{l} SA = \frac{\square BA}{L} \text{ By Step 2.} \\ \text{of the last Process.} \end{array} \right.$

And 2  $\left\{ \begin{array}{l} sa = \frac{\square ba}{L} \text{ By Step 1.} \\ \text{of the last Process.} \end{array} \right.$

1 — 2 3  $SA - sa = \frac{\square BA - \square ba}{L}$

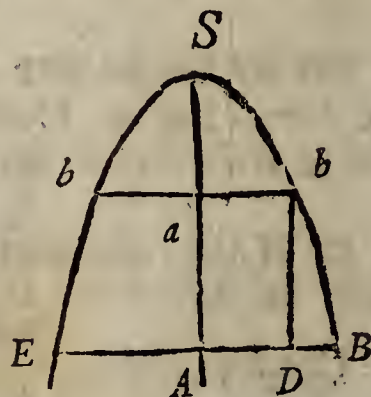
3  $\times$  L 4  $SA - sa \times L = \square BA - \square ba$

But 5  $\square BA - \square ba = BA + ba \times BA - ba$

4 = 5. 6  $SA - sa \times L = BA + ba \times BA - ba$

6, Analogy 7  $L : BA \times ba :: BA - ba : SA - sa$

Or 8  $L : ED :: DB : bD$



$\left\{ \begin{array}{l} \text{Which gives} \\ \text{the following} \\ \text{Analogy.} \end{array} \right.$

This

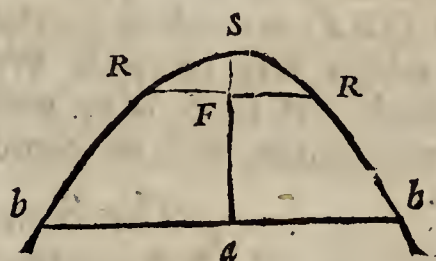


This peculiar Property of the Parabola was first publish'd Anno 1684. by one Mr. Thomas Baker, Rector of Bishop Nympton in Devonshire, in a Treatise intitl'd, *The Geometrical Key: or, The Gate of Equations unlock'd*; wherein he hath shew'd the Geometrical Construction and Solution of all Cubic and Biquadratic adfect'd Equations by one general Method, which he calls a *Central Rule*, deduc'd from this peculiar Property of the Parabola.

## Sect. 3. To find the Focus of any Parabola.

The Focus of every Parabola is that Point in its Axis through which the *Latus Rectum* doth pass. (See *Definition 3. Sect. 4. Page 359.*) Therefore its Distance from the Vertex of the Parabola may be easily found, either by the *Latus Rectum* it self, or by any other Ordinate, and its Abscissa.

Thus, suppose the Point at *F* to be the Focus, *S* the Vertex, the Ordinate *RFR = L* the *Latus Rectum*, and *b a b* any other Ordinate.



Then will  $SF = \frac{1}{4}L$ . Or  $SF = \frac{\square ba}{4Sa}$

## Demonstration.

First	1	$SF \times L = \square FR$	By <i>Sect. 2. Page 375.</i>
And	2	$FR = \frac{1}{2}L$	For the Ordinate $RFR = L$ as above.
2	3	$\square FR = \frac{1}{4} \square L = \frac{1}{2}L \times \frac{1}{2}L$	
1, =	4	$SF \times L = \frac{1}{4} \square L$	
4 ÷ L	5	$SF = \frac{1}{4}L$	As by <i>Definition 4. Sect. 4. Page 359.</i>
Again	6	$\frac{\square b a}{S a} = L$	by the third Step in <i>Page 375.</i>
Conseq.	7	$\frac{\square b a}{4 S a} = \frac{1}{4}L$	&c. As above.

Q. E. D.

## Sect. 4. To Describe, or Draw a Parabola several Ways.

Note, There are two or three Ways of drawing a Parabola Instrumentally at one Motion; but because those Instruments or Machines are not only too perplex'd for a Learner to manage, but also a little subject to Error, I have therefore chosen to shew how that Figure may be (the best) drawn by a convenient Number of Points, viz. Ordinates found, either Numerically or Geometrically, according to the Data; which, if the Work of the three last Sections be well consider'd, must needs be very easy.

I. If



1. If any Ordinate and its Abscissa are given, there may by them be found as many Ordinates as you please to assign or take Points in the Parabola's Axis, (by Sect. 1. Page 374.) and the Curve of the Parabola may be drawn by the extreme Points of those Ordinates, as the Ellipsis was Page 367.

2. If the *Latus Rectum*, and either any Ordinate, or its Abscissa, are given, then any assign'd Number of Ordinates may by them be found, (by Sect. 2. Page 375,) either Numerically or Geometrically, &c.

3. If only the Distance of the Focus from the Vertex of the Parabola be given, any assign'd Number of Ordinates may be found by it. For  $SF = \frac{1}{4}L$  the *Latus Rectum*, and  $\frac{1}{2}L = FR$  as in the last Section; and it will be, as  $SF$  is to  $\square FR$  :: so is any other Abscissa (viz.  $SA$ , or  $SA$ , &c.) : to the Square of its Semi-ordinate, (viz.  $\square ba$ , or  $\square BA$ ) according to the common Property of the Parabola.

Altho' any of these Ways of finding the Ordinates are easy enough, yet that Way which may be deduc'd from the following Proposition, will be found much more easy, and ready in Practice.

**Proposition.** *The Sum of any Abscissa and focal Distance from the Vertex, will be equal to the Distance from the Focus to the extreme Point of the Ordinate which cuts off that Abscissa.*

For Instance, Suppose  $S$  to be the Vertex of any Parabola, the Point  $F$  to be its Focus, and  $AB$  any Semi-ordinate rightly apply'd to its Axis  $SA$ . Then I say, where-ever the Point  $A$  is taken in the Axis, it will be  $SA + SF = FB$ . Consequently, if  $Sf = SF$ , it will be  $fA = FB$ .



### Demonstration.

First	1	$SF = \frac{1}{4}L$ by the 7th Step, Sect. 3.
Ergo	2	$fA = FA + \frac{1}{2}L$ by Construction above.
2	3	$\square fA = \square FA + FA \times L : \frac{1}{4}LL$
Again	4	$SA = FA + \frac{1}{4}L$ by the Supposition and Figure.
$4 \times L$	5	$SA \times L = FA \times L : + \frac{1}{4}LL$ But $SA \times L = \square AB$
Ergo	6	$\square AB = FA \times L : + \frac{1}{4}LL$
3 — 6	7	$\square fA - \square AB = \square FA$ Conse. $\square fA = \square FA + \square AB$
But	8	$\square FA + \square AB = \square FB$ By Theorem II.
Ergo	9	$\square fA = \square FB$
9 un 2	10	$fA = FB$ Q. E. D.

This



## Chap. 3. Concerning the Parabola. 385

This Proposition being well understood, it will be very easily apply'd to Practice, supposing the Focal Distance given, or any other Data by which it may be found.

Thus draw any Right Line to represent the Parabola's Axis, and from its Vertical Point, as at  $S$ , set off the Focal Distance both upwards and downwards, *viz.* make  $Sf = SF$  the Distance of the given Focus from the Vertex, as in the last Scheme. Then by the Proposition it is evident, that if never so many Lines be drawn ordinately at Right Angles to the Axis the true Distance between the Point  $f$  out of the Parabola, and any of those Lines (or Ordinates) being measur'd or set off from the Focus  $F$  to the same Line (or Ordinate) it will assign the true Point in that Line through which the Curve must pass. That is, it will shew the true Limits or Length of that Ordinate, as at  $B$  in the last Scheme.

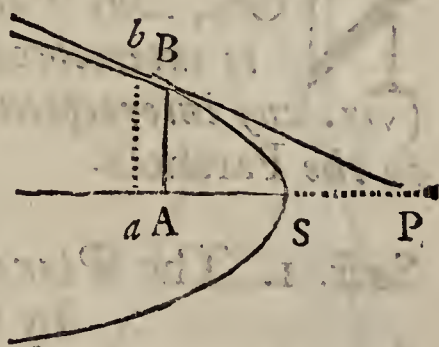
Proceeding on in the very same Manner, from Ordinate to Ordinate, you may with great Expedition and Exactness find as many Ordinates (or rather their Points only, like  $B$ ) as may be thought convenient; which being all joyn'd together with an even Hand, will form the Parabola requir'd.

*N.B.* The more Ordinates (or their Points) there are found, and the nearer they are to one another, the easier and exacter may the Curve of the Parabola be drawn. The same is to be observ'd when any other Curve is requir'd to be drawn by Points.

### Sect. 5. To draw a Tangent to any given Point in the Curve of a Parabola.

*Tangents* are very easily drawn to the Curve of any Parabola.

For, Supposing  $S$  to be its Vertex,  $B$  the Point of Contact, *viz.* the Point where the Tangent must touch the Curve. And  $P$  the Point where the Tangent will intersect, (or meet with) the Parabola's Axis produc'd. Then if from the Point of Contact  $B$ , there be drawn the Semi-ordinate  $BA$  at Right Angles to the Axis  $SA$ , where-soever the Point  $A$  falls in the Axis, it will be  $SP = SA$ .



### Demonstration.

Draw the Semi-ordinate  $ba$  (as in the Figure) then will the  $\triangle BAP$ , and  $\triangle bAP$  be alike. Let  $y = AS$  the Abcissa, and  $D d d$   $z = SP$ ;

$z = SP$ ; put  $x = Aa$  the Distance between the two Semi-ordinates; which we suppose to be infinitely near each other, as in the Ellipsis, *Page 371*.

Then	1	$y + z : BA :: y + z + x : ba$ . Per Theorem 13.
1, Or	2	$y + z : y + z + x :: BA : ba$ . See Page 192.
Again	3	$y : \square BA :: y + x : \square ba$ . Per Theorem Page 374.
3, Or	4	$y : y + x :: \square BA : \square ba$
2 in $\square$ 's	5	$\begin{cases} yy + 2yz + zz : yy + 2yz + 2yx + zz + 2zx + \\ xx :: \square BA : \square ba \end{cases}$
4, 5	6	$\begin{cases} y : y + x :: yy + 2yz + zz : yy + 2yz + 2yx + zz + \\ 2zx + xx \end{cases}$
6, ∴	7	$\begin{cases} yy + 2yz + yx + zz + 2zx + \frac{zzx}{y} = \\ yy + 2yz + 2yx + zz + 2zx + xx \end{cases}$
That is,	8	$\frac{zzx}{y} = yx + xx$ . Consequently $\frac{zz}{y} = y + x$
Suppose	9	$x = 0$ And rejected, as in the Ellipsis, <i>Page 371</i> .
Then	10	$\frac{zz}{y} = y$ Consequently $zz = yy$
10 w 2	11	$z = y$ That is, $SP = SA$ . <span style="float: right;">Q.E.D</span>

#### C H A P. IV.

*Concerning the chief Properties of the Hyperbola.*

**N**Ote, Any Part of the Axis of an Hyperbola, which is intercepted between its Vertex and any Ordinate, (*viz.* any intercepted Diameter) is call'd an Abscissa, As in the Parabola.

Sect. I. The Plane of every Hyperbola is proportion'd (by this General Theorem.

Theorem. { *As the Sum of the Transverse and any Abscissa multiply'd into that Abscissa: is to the Square of its Semi-ordinate :: so is the Sum of the Transverse and any other Abscissa multiply'd into that Abscissa : To the Square of its Semi-ordinate.*

That



# Chap. 4. Concerning the Hyperbola. 387

That is, if  $TS$  be the Transverse Diameter,

And  $\begin{cases} Sa, SA, \text{ Abscissa's.} \\ ba, BA, \text{ Semi-ordinates.} \end{cases}$

Then is  $\begin{cases} Ta = TS + Sa. \\ TA = TS + SA. \end{cases}$

And it will be

$$Ta \times Sa : \square ba :: TA \times SA : \square BA.$$

That is,

$$TS + Sa \times Sa : \square ba :: TS + SA \times SA : \square BA$$

&c.

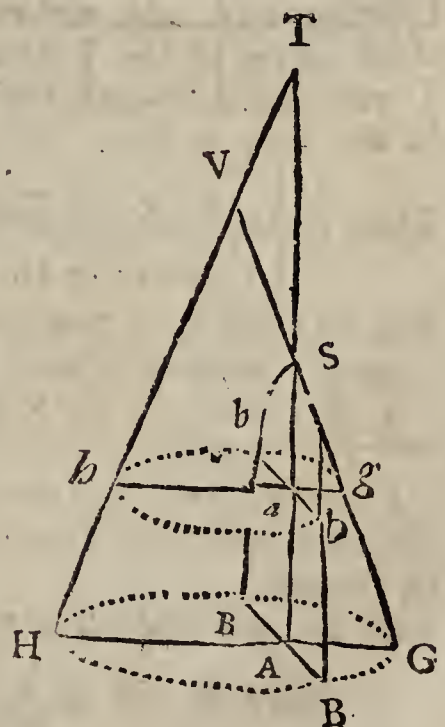
## Demonstration.

Let the following Figure  $HVG$  represent a Right Cone cut into two Parts by the Right Line  $SA$ ; then will the Plane of that Section be an Hyperbola (by *Sect. 5. Chap. 1.*) in which let  $SA$  be its Axis, or intercepted Diameter,  $bab$ , and  $BAB$ , Ordinates Rightly apply'd (as before in the Parabola) and  $TS$  its Transverse Diameter.

Again, if the Cone is suppos'd to be cut by  $hg$ , parallel to its Base  $HG$ , it will also be the Diameter of a Circle, &c. as in the Ellipsis and Parabola.

Then will the  $\triangle Sga$  and  $\triangle SGA$  be alike, also the  $\triangle Tab$  and  $\triangle TAH$  will be alike; therefore it

will be	1	$Sa : ag :: SA : AG$
And	2	$Ta : ab :: TA : AH$
1 $\therefore$	3	$Sa \times AG = SA \times ag$
2 $\therefore$	4	$Ta \times AH = TA \times ab$
3 $\times$ 4	5	$\begin{cases} Sa \times Ta \times AG \times AH = \\ = SA \times TA \times ag \times ab \end{cases}$
But	6	$ag \times ab = \square ab$
And	7	$\begin{cases} AG \times AH = \square AB \\ \text{Per Lemma Page 357.} \end{cases}$
5, 6, 7	8	$\begin{cases} Sa \times Ta \times \square AB = \\ SA \times TA \times \square ab \end{cases}$
8, Anal.	9	$a \times Ta : \square ab :: SA \times TA : \square AB, \&c.$



Q. E. D.

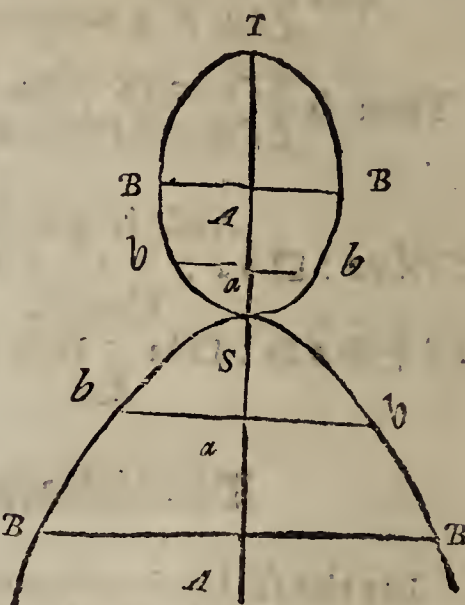
These

These Proportions are the common Property of every Hyperbola, and do only differ from those of the Ellipsis in the Signs  $+$  And  $-$  As plainly appears in all the following Proportions.

That is, if we suppose  $TS$ , the Transverse Diameter, common to both Sections, (*viz.* both the Ellipsis and Hyperbola) As in the annex'd Scheme.

Then in the Ellipsis it will be  
 $TS - Sa \times Sa : \square ab :: TS - SA \times SA : \square AB$   
 (As by Sect. 1. Chap. 2.)

And in the Hyperbola it is  
 $TS + Sa \times Sa : \square ab :: TS + SA \times SA : \square AB$   
 As above.



And therefore all that is farther requir'd in the Hyperbola, may (in a manner) be found as in the Ellipsis, due Regard being had to changing of the Signs.

### Sect. 2. To find the Latus Rectum, or Right Parameter of any Hyperbola.

From the last Proportion take either of the Antecedents and its Consequent, *viz.* either  $Ta \times Sa : \square ab$ . Or  $TA \times SA : \square AB$ . To them bring in the Transverse  $TS$  for a third Term. and by those Three find a Fourth Proportional (as in the Ellipsis) and that will be the *Latus Rectum*.

Thus 1.  $Ta \times Sa : \square ab :: TS : \frac{\square ab \times TS}{Ta \times Sa} = \text{the Latus Rectum, which call } L \text{ (as in the Parabola.)}$

Then 2.  $TS : L :: Ta \times Sa : \square ab$ .

But 3.  $Ta \times Sa : \square ab :: TA \times SA : \square AB$ . Therefore

2. 3. 4.  $TS : L :: TA \times SA : \square AB, \&c.$

Consequently  $L$  is the true *Latus Rectum*, or Right Parameter, by which all the Ordinates may be found, according to its Definition in Chapter 1.

And because  $TS + Sa = Ta$ . Let it be  $TS + Sa$ , instead of  $Ta$ .

Then it will be  $\frac{\square ab \times TS}{TS \times Sa : + \square Sa} = L$

And in the Ellipsis it would be  $\frac{\square ab \times TS}{TS \times Sa : - \square Sa} = LR = L$ .

Sect.



# Chap 4. Concerning the Hyperbola. 389

## Sect. 3. To find the Focus of any Hyperbola.

The Focus being that Point in the Hyperbola's Axis, through which the *Latus Rectum* must pass; (As in the Ellipsis and Parabola) it may be found by this *Theorem*.

**Theorem.** { To the Rectangle made of half the Transverse into half the *Latus Rectum*, add the Square of half the Transverse; the Square Root of that Sum will be the Distance of the Focus from the Center of the Hyperbola.

### Demonstration.

Suppose the Point at *F*, in the annex'd Scheme, to be the Focus sought. Then will  $FR = \frac{1}{2}L$ .

Let  $TC = CS$  be half the Transverse; then is the Point *C* call'd the Center of the Hyperbola (for a Reason that shall be hereafter shew'd.) and  $SF = a$ .

Again, Let  $CS = d$ . And  $SF = a$

Then 1  $2d : L :: 2d + a \times a : \frac{1}{4}LL$   
That is, 2  $TS : L :: TS + SF \times FS : \square FR$

3  $\frac{1}{2}dL = 2da + aa$

4  $dd + \frac{1}{2}dL = dd + 2da + aa$

5  $\sqrt{dd + \frac{1}{2}dL} = d + a = FC$

Or, 6  $\sqrt{dd + \frac{1}{2}dL} - d = a = SF$

In the Ellipsis it is,  $2d : L :: 2d - a \times a : \frac{1}{4}LL$ .

That is,  $\frac{1}{2}dL = 2da - aa$ . &c.

The Geometrical Effecton of the last *Theorem* is very easily perform'd, thus;

Make  $Sx = \frac{1}{2}L$ , viz. half the *Latus Rectum*; and let  $CS = d$ , as above.

Upon *Cx* (as a Diameter) describe a Circle; and at *S* the Vertex of the Hyperbola draw the Right Line  $nSN$  at Right Angles to *Cx*; then join the Points *CN* with a Right Line, and 'twill be  $CN = d + a = FC$ .

For 1  $CS : SN :: SN : Sx$  per Fig.

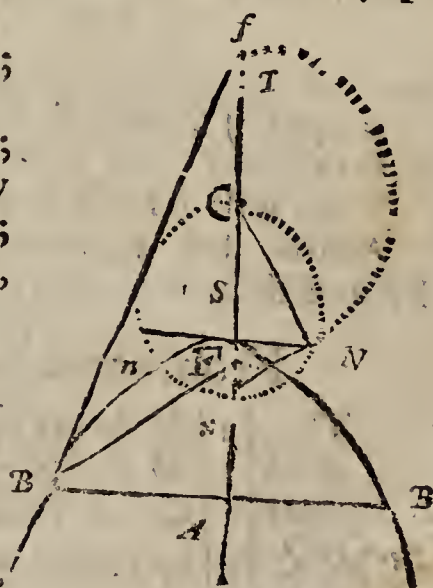
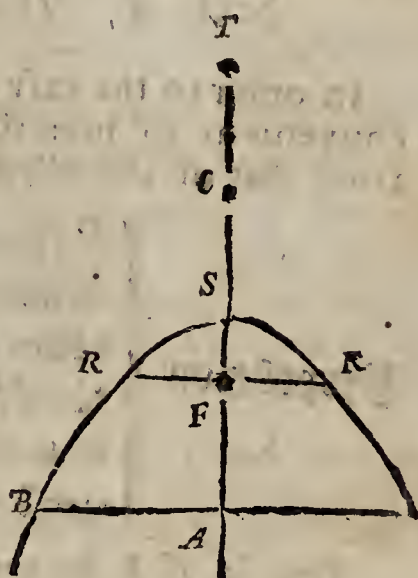
That is, 2  $d : SN :: SN : \frac{1}{2}L$

3  $\frac{1}{2}dL = \square SN$

But 4  $dd + \square SN = \square CN$

5  $dd + \frac{1}{2}dL = \square CN$

6  $\sqrt{dd + \frac{1}{2}dL} = CN = d + a$ , &c.



Now

Now here is not only found the Distance of the Hyperbola's Focus, either from its Center C, or Vertex S; but here is also found that Right Line usually call'd its Conjugate Diameter, viz. the Line  $nSN$ , which bears the same Proportion to the Transverse and *Latus Rectum* of the Hyperbola, as the Conjugate Diameter of the Ellipsis doth to its Transverse, and *Latus Rectum*.

For in the Ellipsis  $TS : Nn :: Nn : LR$ . per sect. 2. Page 363.

Consequently  $\frac{1}{2} TS : \frac{1}{2} Nn :: \frac{1}{2} Nn : \frac{1}{2} LR$ .

But  $\frac{1}{2} TS = d$ .  $\frac{1}{2} Nn = SN$ . And  $\frac{1}{2} LR = \frac{1}{2} L$ .

Therefore  $d : SN :: SN : \frac{1}{2} L$ . As at the second Step above.

What Use the aforesaid Line  $nSN$  is of, in Relation to the Hyperbola, will appear farther on.

#### Sect. 4. To Describe an Hyperbola in Plano.

In order to the easy describing of any Hyperbola in Plano, it will be convenient to premise the following Proposition, which differs from that of the Ellipsis in Sect. 3. Chap. 2. only in the Signs.

**Proposition.** { If from the Focus's of any Hyperbola, there be drawn two Right Lines, so as to meet each other in any Point of the Hyperbola's Curve, the Difference of those Lines (in the Ellipsis it is their Sum) will be equal to the Transverse Diameter.

That is, if F be the Focus, and it be made  $Cf = CF$ , (as in the last Scheme) then the Point  $f$  is said to be a Focus out of the Section (or rather of the opposite Section) and it will be  $fB - FB = TS$ .

#### Demonstration.

Suppose  $fC$ , or  $CF = z$ , and  $SA = x$ . Let  $CS$ , or  $TC = d$ , as before.

Then will  $fA = d + x + z$ , And  $FA = d + x - z$ .

Again, Let  $FB = b$ , and  $fB = b$ . Then  $2d = b - b$ , by the Proposition.

From these substituted Letters it follows,

That	1	$dd + 2dx + 2dz + xx + 2zx + 2z = \square fA$
And	2	$dd + 2dx - 2dz + xx - 2zx + 2z = \square FA$
But		$\square fA + \square AB = \square fB$ . And $\square fA + \square AB = \square FB$
Per 4th of last	3	$dd + \frac{1}{2}LL = dd + 2da + aa = \square FC = 2z$ .

$$3 = dd$$



# Chap. 4. Concerning the Hyperbola. 391

3	—	dd	4	$zz - dd = \frac{1}{2}dL$
4	÷	$\frac{1}{2}d$	5	$\frac{zz - dd}{\frac{1}{2}d} = L$
Again			6	$2d : L :: 2d + xxx : \square AB$ By com. Properties.
5,		6	7	$2d : \frac{zz - dd}{\frac{1}{2}d} :: 2dx + xx : \square AB$
7	∴		8	$\frac{2dzzx + zxxx - 2dddx - ddx}{dd} = \square AB$
I	+	8	9	$\frac{\begin{cases} dd + 2dx + 2dz + xx + 2zx + zz + \\ 2dzzx + zxxx - 2d^3x - ddx \end{cases}}{dd} = \square fA + \square AB = bb$
2	+	8	10	$\frac{\begin{cases} dd + 2dx - 2dz + xx - 2zx + zz + \\ 2dzzx + zxxx - 2d^3x - ddx \end{cases}}{dd} = \square fA + \square AB = bb$
9	×	dd	11	$d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zxxx = ddbb$
10	×	dd	12	$d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zxxx = ddbb$
11	w	2	13	$dd + dz + zx = db$
12	w	2	14	$dd - dz - zx = db$
13	÷	d	15	$d + z + \frac{zx}{d} = b$
14	÷	d	16	$d - z - \frac{zx}{d} = b$
16,	or		17	$z + \frac{zx}{d} - d = b$
15	—	17	18	$2d = b - b$

Altho' the Equation at the 16th Step be in it self impossible, because  $z$  is greater than  $d$ ; (by the 4th Step) yet from thence it will be easy to conclude, that the Difference between  $d$  and  $z + \frac{zx}{d}$  will give the true Value of  $b$ , as in the 17th Step.

But because I would leave no Room for the Learner to doubt about changing the Equation,  $d - z - \frac{zx}{d} = b$  into that of

$z + \frac{zx}{d} - d = b$ , it may be convenient to illustrate the whole Process in Numbers, whereby (I presume) it will plainly appear that  $b - b = TS$ .

In order to that, Let the Transverse  $TS = 2d = 12$ . Then  $d = 6$  Suppose the Abscissa  $SA = x = 4$ . And the Semi-ordinate  $AB = 3$

First	I	$TS + SA \times SA : \square AB :: TS : L$ per Sect. 2.
I, Viz.	2	$12 + 4 \times 4 = 64 : 9 :: 12 : 16875 = L$
Again	3	$\sqrt{dd + \frac{1}{2}dL} = d + a = CF$ per Sect. 3.
3, Viz.	4	$\sqrt{36 + 5.0625} = 6.408 = CF = z$
Then	5	$d + x + z = 6 + 4 + 6.408 = 16.408 = fA$
And	6	$d + x - z = 6 + 4 - 6.408 = 3.592 = FA$

5	⊙	2	7	$269,2224 = \square f A$
6	⊙	2	8	$12,9024 = \square F A$
	But	9		$9 = \square A B$ For $AB = 3$ by Supposition.
7	+	9	10	$278,2224 = \square f A + \square A B = \square f B$
8	+	9	11	$21,9024 = \square F A + \square A B = \square F B$
10	w	2	12	$16,68 = f B$
11	w	2	13	$4,68 = F B$
12	—	13	14	$12.00 = f B - F B = TS.$ Which was to be prov'd

If this Proposition be truly understood, it must needs be easy to conceive how to describe the Curve of any Hyperbola very readily by Points, when the Transverse Diameter and the Focus are given, (or any other Data by which they may be found, As in the preceding Rules.) Thus,

Draw any strait Line at Pleasure, in it set off the Length of the given Transverse  $TS$ , and from its extreme Points or Limits, viz.  $T, S$ , set off  $Tf = SF$ , the Distance of the given Focus; (viz. the Point  $f$  without, and  $F$  within the Section, as before) that done, upon the Point  $f$  (as a Center) with any assum'd Radius, greater than  $TS$ , describe an Arch of a Circle; then from that Radius take the Transverse  $TS$ , making their Difference a second Radius, with which upon the Point  $F$  within the Section, describe another Arch to Cut or Cross the first Arch, as at  $B$ . Then will that Point  $B$  be in the Curve of the Hyperbola, by the last Proposition. And therefore 'Tis plain, that proceeding on in this Manner, you may find as many Points (like  $B$ ) as may be thought convenient; (the more there are, and nearer they are together, the better) which being all joyn'd together with an even Hand (as in the Parabola) will form the Hyperbola requir'd.



There are several other Ways of delineating an Hyperbola in Plano; one Way is by finding a competent Number of Ordinates, as by Sect. I. &c. But I think none so easy and expeditious as this Mechanical Way; I shall therefore, for Brevity's Sake, pass over the rest, and leave them to the Learner's Practice, as being easily deduc'd from what hath been already said.



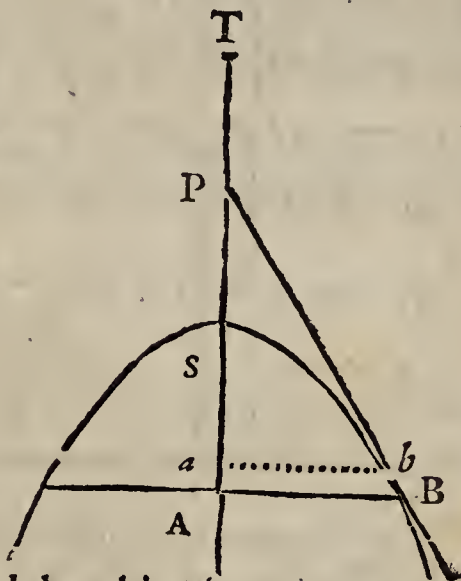
## Sect. 5. To draw a Tangent to any given Point in the Curve of an Hyperbola.

The drawing of a Tangent that will touch any given Point in the Curve of an Hyperbola, may be easily perform'd by Help of a Theorem, as in the Ellipsis, Sect 6. Chap. 2.

Let  $\begin{cases} D=TS \text{ the Transverse Diameter} \\ L=\text{the Latus Rectum} \\ y=SA \text{ the Abscissa.} \end{cases}$   
 And  $z=AP$   $\begin{cases} \text{the Distance between the} \\ \text{Ordinate, and that Point} \\ \text{in the Transverse cut by} \\ \text{the Tangent.} \end{cases}$

Then if  $y$  be given,  $z$  may be found by this Theorem.  $\begin{cases} Dy+yy \\ \frac{1}{2}D+y \end{cases} = z$

Which differs from that in the Ellipsis only in the Signs, vide Page 371.



Or if  $z$  be given, then  $y$  may be found by this Theorem.

Theorem.  $\sqrt{\frac{DD+zz}{4}} : +\frac{1}{2}D+z-\frac{1}{2}D=y$

### Demonstration.

Draw the Semi-Ordinate  $ba$ , as in the Figure. and an infinite small Space between the two Semi-Ordinates. As before in the Ellipsis, &c,

put $x=Aa$	1	$D:L::Dy+yy:\square AB$
Then	2	$TS:L::TS+SA\times SA:\square AB$
That is,	3	$\frac{DyL+yyL}{D}=\square AB$
1	4	$D:L::Dy+yy-2yx-Dx+xx:\square ab$
Again	5	$TS:L::TS+Sa\times Sa:\square ab$
That is,	6	$\frac{DyL+yyL-2yxL-DxL+xxL}{D}=\square ab$
4	7	$z:AB::z-x:ab \text{ Viz. } PA:AB::Pa:ab$
per Figure	8	$zz:\square AB::zz-2zx+xx:\square ab$
7 in $\square$ 's	9	$x=0$ and every where rejected, as in the Ellipsis.
Suppose	10	$zz:\frac{DyL+yyL}{D}::zz-2z:\square ab$
Then 3, 9	11	$\frac{DyLzz+yyLzz-2DyLz-2yyLz}{Dzz}=\square ab$
10		

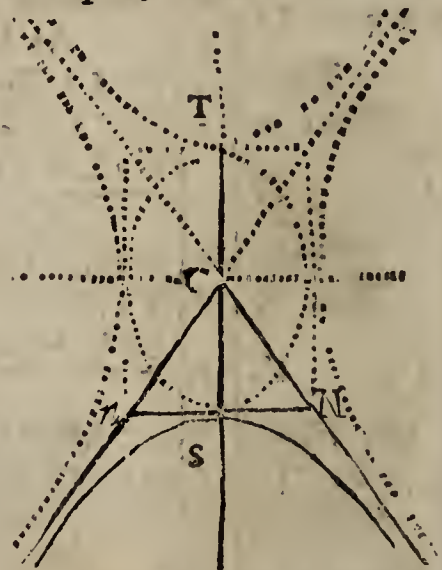
6,	11	12	$\left\{ \begin{array}{l} \frac{DyL + yyL - zyL - DL}{D} = \\ \frac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} \end{array} \right.$	
12 Reduced	13		$\frac{1}{2}Dz + zy = Dy + yy$	
13 Analogy	14		$\frac{1}{2}D + y:yy::D + y:z.$	<i>Viz.</i> CA:SA::TA:AP
13 $\div \frac{1}{2}D - y$	15		$z = \frac{Dy + yy}{\frac{1}{2}D \times y}$	Which is the first Theorem.
13 $- zy$	16		$yy + Dy - zy = \frac{1}{2}Dz$	
16 $\square C$	17		$yy + Dy - zy + \frac{DD - 2Dz + zz}{4} = \frac{DD + zz}{4}$	
17 $w 2$	18		$y + \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{DD + zz}{4}}$	
18 $\pm$	19		$y = \frac{\sqrt{D + zz}}{4} : + \frac{1}{2}z - \frac{1}{2}D$	Which is the second Theorem.

Q. E. D.

The Geometrical Effect of the first of these Theorems is very easy; for by the 14th Step it is evident, that there are three Lines given to find a fourth proportional Line. By *Problem 3. Page 308.*

## S C H O L I U M.

From the Comparisons which have been all along made in this Chapter, between the Hyperbola and the Ellipsis, it will be easy (even for a Learner) to perceive the Coherence that is in (or between) those two Figures. But for the better understanding of what is meant by the Center and Asymptotes of an Hyperbola, consider the annex'd Scheme, wherein it is evident (even by Inspection) that the opposite Hyperbola's will always be alike, because they will always have the same Transverse Diameter common to both, &c. See *Sect. 1. of this Chapter.* Also, that the middle Point, or common Center of the Ellipsis, is the common Center to all the four Conjugal Hyperbola's.



And the two Diagonals of the Right-angled Parallelogram, which circumscribes the Ellipsis, or is inscrib'd to the four Hyperbola's, being continu'd, will be such Asymptotes to those Hyperbola's as are defin'd, *Chap. 1. Sect. 5. Defin. 4.*

Sect.



Sect. 6. To draw the Asymptotes of an Hyperbola, &c.

Having found the *Latus Rectum*, (by *Seçt.* 2.) and the Conjugate Diameter  $n$  SN, in its true Position, by *Seçt.* 3. Then through the Center C of the Hyperbola, and the Extreme Points  $n$ , N, of its Conjugate Diameter, draw two Right Lines, as CN, and Cn, infinitely continued, (as in the following Figure) and they will be the Asymptotes requir'd.

That is, they are two such Right Lines as, being infinitely extended, will continually incline to the Sides of the Hyperbola, but never touch them.

### Demonstration.

Suppose the Semi-Ordinates  $ab$  and  $AB$  to be rightly apply'd to the Axis  $TA$ , and produced both Ways to the Asymptotes, as at  $fg$  and  $FG$ . Then will the  $\triangle CSN$ ,  $\triangle Cag$ ,  $\triangle CAG$ , be alike,

Let  $d = CS = TC$ . And  $L =$  the *Latus Rectum*. As before.

Put  $\begin{cases} e = Sa \\ y = SA \end{cases}$  the Abscissa's. Then  $\begin{cases} d + e = Ca. \\ d + y = CA. \end{cases}$

Then  $| \text{I} | d : \text{SN} :: d + e : ag.$  *Viz.*  $\text{CS} : \text{SN} :: \text{Ca} : ag$

r in □'s | 2 | dd : □ SN :: dd + 2de + ee : □ ag

But  $3 \frac{1}{2} d L = \square \text{ SN.}$  Per Sect. 3.

$$2, 3 \therefore 4 \frac{ddL + 2deL + eeL}{2d} = \square ag$$

Again 5 2d: L :: 2d<sup>e</sup> + ee:  $\square$  ab. Per Sect. 2.

$$5 \therefore 6 \frac{2 \text{ de } L + ee \bar{L}}{2d} = \square ab$$
$$4-6 \quad 7 \quad \frac{dL}{2} = \square ag - \square ab$$

But  $\left. \begin{array}{l} 8 \quad ag + ab = bf \\ 9 \quad ag - ab = bg \end{array} \right\} \text{per Fig.}$

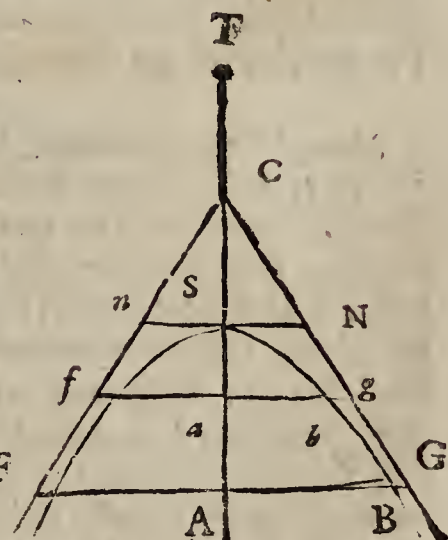
$$8 \times 9 \mid 10 \mid \square ag - \square ab = bf \times bg$$
$$7, \quad 10 \mid 11 \quad bf \times bg = \frac{1}{2} dL$$

Again  $12 \quad dd : \square SN :: dd + 2dy + yy : \square AG$

That is,  $\square CS : \square SN :: \square CA ; \square AG$

$$3, 12 \therefore 13 \quad \frac{ddL + 2dyL + yyL}{2d} = \square AG$$

But  $I4 \quad 2d : L :: 2dy + yy : \square AB. \text{ Per Sect. 2.}$

$$\therefore \frac{2 dy L + vyL}{2d} = \square AB$$


13 — 15	15	$\frac{dL}{2} = \square AG - \square AB$
Also	17	$AG + AB = BF$
	18	$AG - AB = BG$
17 X 18	19	$\square AG - \square AB = BF \times BG$
16,	19	$BF \times BG = \frac{1}{2} dL$
11, & 20 ÷	21	$bg = \frac{\frac{1}{2} dL}{bf}$ And $BG = \frac{\frac{1}{2} dL}{BF}$

From the last Step it is evident, that the *Asymptotes* are nearer the *Hyperbola* at *G*, than at *g*, and consequently will continually approach its *Curve*. For  $BF) \frac{1}{2} dL (= BG$  is less than  $bf) \frac{1}{2} dL (= bg$ , because the *Divisor*  $BF$  is greater than the *Divisor*  $bf$ ; and it must needs be so, where-ever the *Ordinates* are produced to the *Asymptotes*, from the Nature of the *Triangles*.

Again, from the 7th and 16th Steps it is evident, that the *Asymptotes* can never really meet and be coincident with the *Curve* of the *Hyperbola*, although both were infinitely extended; because  $\frac{1}{2} dL$  will always be the Difference between the Square of any Semi-ordinate, and the Square of that Semi-ordinate when it is produced to the *Asymptote*.

## CONSECTARY.

From hence it follows, that every Right Line which passes thro' the Center, and falls within the *Asymptotes*, will cut the *Hyperbola*; and all such Lines are call'd *Diameters*, as in the *Ellipsis*, because the Properties of the *Hyperbola* and *Ellipsis* are the same.

*Note*, Every *Diameter*, both in the *Ellipsis*, *Parabola* and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which should they be distinctly handled, and the Effect of all such Lines as relate to them; as also, the Nature and Properties of such Figures as may be inscrib'd and circumscrib'd to all the Sections, with the various Habitude or Proportions of one *Hyperbola* to another, &c. would afford Matter sufficient to fill a large Volum. But thus much may suffice for an Introduction; I shall therefore desist pursuing them any farther, being fully satisfy'd, that if what I have already done be well understood the rest must needs be very easy to any one that intends to proceed farther on that Subject.



A N

# INTRODUCTION TO THE Mathematicks.

## P A R T V.

**T**HE Method of finding out any particular Quantity, (viz. either any Line, Superficies or Solid) by a regular Progression or Series of Quantities continually approaching to it, which being infinitely continu'd, would then become perfectly equal to it, is what is commonly call'd *Arithmetick of Infinites*; which I shall briefly deliver in the following Lemma's, and apply them to Practice in finding the *Superficial* and *Solid Contents* of *Geometrical Figures* farther on.

### L E M M A I.

*In any Series of equal Numbers, representing Lines or other Quantities) As 1. 1. 1. 1. &c. Or 2. 2. 2. 2. &c. Or 3. 3. 3. 3. &c. If one of the Terms be multiply'd into the Number of Terms, the Product will be the Sum of all the Terms in the Series.*

This is so very plain and easy to be understood, that it needs no Example.

### L E M M A II.

*If the Series of Numbers in Arithmetical Progression begin with a Cypher, and the common Difference be 1. As 0. 1. 2. 3. 4. &c. (representing a Series of Lines or Roots beginning with a Point) If the last Term be multiply'd into the Number of the Terms, the Product will be double the Sum of the Series.*

That is, putting  $L$  = the last Term,  $N$  = the Number of Terms, and  $S$  = the Sum of all the Series.

Then

Then will  $N L = 2S$ . Consequently,  $\frac{1}{2} N L = S$ .  
*viz.* one half of so many Times the greatest Term, as there are Number of Terms in the Series.

Thus  $\sum \frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10}{20} = \frac{1}{2} N L$ .  
the Sum of the Series  $= \frac{1}{2} N L$ .

And this will always be so, how many Terms soever there are,  
by *Confess.* I. Page 185.

### L E M M A III.

*If a Series of Squares whose Sides or Roots are in Arithmetical Progression, beginning with a Cypher, &c. (as in the last Lemma) be infinitely continued, the last Term being multiply'd into the Number of Terms, will be Triple to the Sum of all the Series. viz.  $NLL = 3S$ , or  $\frac{1}{3} NLL = S$ .*

That is, the Sum of such a Series will be one Third of the last or greatest Term, so many Times repeated as is the Number of Terms in the Series.

#### *Instances in Square Numbers.*

1.  $\sum \frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$
2.  $\sum \frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18}$
3.  $\sum \frac{0+1+4+9+16}{16+16+16+16+16} = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24}$  &c.

From these Instances it's evident, that as the Number of Terms in the Series does increase, the Fraction or Excess above  $\frac{1}{3}$  does decrease, the said Excess always being  $\frac{1}{6N-6}$  which, if we sup-

pose the Series to be infinitely continued, will then become infinitely small, *viz.* in Effect nothing at all.

Consequently,  $\frac{1}{3} NLL$  may be taken for the true or perfect Sum of such an infinite Series of Squares.

### L E M M A IV.

*If a Series of Cubes, whose Roots are in Arithmetical Progression, beginning with a Cypher, &c. (as above) be infinitely continued, the Sum of all the Series will be  $\frac{1}{4} NLLL = S$ .*

That is, is one Fourth of the last or greatest Term, so many Times repeated as is the Number of Terms.

*Instances*



*Instances in Cube Numbers.*

If 0. 1. 2. 3. &c. be the Roots of the Cubes.

Then 1.  $\left\{ \frac{0+1+8+27}{27+27+27+27} = \frac{36}{108} = \frac{4}{12} = \frac{1}{4} + \frac{1}{12} \right.$

2.  $\left\{ \frac{0+1+8+27+64}{64+64+64+64+64} = \frac{100}{320} = \frac{10}{32} = \frac{5}{16} = \frac{1}{4} + \frac{1}{16} \right.$

3.  $\left\{ \frac{0+1+8+27+64+125}{125+125+125+125+125+125} = \frac{225}{750} = \frac{45}{150} = \frac{3}{10} \right.$   
 $\left. = \frac{6}{20} = \frac{1}{4} + \frac{1}{20} \right.$

From all these Examples it plainly appears, that as the Number of Terms in the Series increases, the Fraction or Excess above  $\frac{1}{4}$  decreases; the Excess being always  $\frac{1}{4N-4}$  which, if we suppose the Series to be infinitely continued, will become infinitely small, or rather nothing. As in the last Lemma.

Consequently,  $\frac{1}{4}$  NLLL may be taken for the true and perfect Sum of all the Terms in such an infinite Series of Cubes.

L E M M A V.

*If a Series of Biquadrates, whose Roots are in Arithmetical Progression, beginning with a Cypher, &c. as before, be infinitely continued, the Sum of all the Terms in such a Series will be  $\frac{1}{5}$  NL<sup>4</sup>.*

The Truth of this may be manifested by the like Process, as in the foregoing Lemma's, and so on for higher Powers. But if any one desires a farther Demonstration of these Series, he may (I presume) meet with ample Satisfaction in Dr. Wallis's History of Algebra, Chap. 78 and 79, wherein the Doctor concludes with these Words:

“ Thus having shew'd, that in a Progression of Laterals (or Arithmetical Proportionals) beginning at 0. the Sum of 2, 3, 4, 5, 6 Terms, is always equal to  $\frac{1}{2}$  of so many times the greatest, and there being no Pretence of Reason why we should then doubt it, in a Progression of 7, 8, 9, 10, &c. we conclude it so to be, tho' such a Number of Terms be supposed infinite.

“ Again, in a Progression of their Squares, having shew'd, that in 2, 3, 4, 5, 6 Terms, the Aggregate is always more than  $\frac{1}{3}$  of so many times the greatest; and the Excess always such aliquot Part of

“ of the greatest, as is denominated by six times the Number of  
“ Terms wanting 1. (As if the Terms be 2, it is  $\frac{1}{3} + \frac{1}{6}$ ; if 3, it  
“ is  $\frac{1}{3} + \frac{1}{12}$ ; if 4, it is  $\frac{1}{3} + \frac{1}{18}$ ; if 5, it is  $\frac{1}{3} + \frac{1}{24}$  of so many times the  
“ greatest Term, and so onward) we may well conclude, (there  
“ being no Pretence of Reason why to doubt it in the rest) that  
“ it will be so. how many soever be such Number of Terms.  
“ And because such Excess as the Number of Terms do increase,  
“ will become infinitely small (or less than any assignable) we con-  
“ clude (from the Method of Exhaustions) that, if the Number of  
“ Terms be supposed infinite, such Excess must be supposed to va-  
“ nish, and the Aggregate of such infinite Progression supposed  
“ equal to  $\frac{1}{3}$  of so many times the greatest.

“ In like manner, having prov'd that such Progression of Cubes  
“ doth (as the Number of Terms encreaseth) approach infinitely  
“ near to  $\frac{1}{4}$  of so many times the greatest) and of Biquadrates to  $\frac{1}{5}$ ,  
“ and so of Surfsolids to  $\frac{1}{6}$  of so many times the greatest, and so  
“ onwards as we please to try, and there being no Pretence of Rea-  
“ son why to doubt it as to the rest) we may take it as a sufficient  
“ Discovery, that (universally) the Aggregate of such infinite Pro-  
“ gression is equal (or doth approach infinitely near to such a Part  
“ of so many times the greatest, as is denominated by the Exponent  
“ or Number of Dimensions) of such Power (as is that according  
“ to which the Progression is made) increased by 1. namely, of La-  
“ terals  $\frac{1}{2}$ ; of Squares  $\frac{1}{3}$ ; of Cubes  $\frac{1}{4}$ ; of Biquadrates  $\frac{1}{5}$ ; (of so ma-  
“ times the greatest) and so onwards infinitely.

This Discourse of the Doctor's I thought convenient to insert  
supposing it may give some Satisfaction to the Learner, to hear so  
great a Man as Dr. *Wallis's* Arguments about the Truth of these  
Series, which I have briefly deliver'd in the foregoing Lemma's.

### L E M M A VI.

*If any two Series or Ranks of Proportionals, have the  
same Number of Terms, (whether Finite or Infinite)  
it will always*

*As the first Term of one Series : is to the first Term of  
be { the other Series :: so is the Sum of all the Terms in  
the one Series : to the Sum of all the Terms in the  
other Series.* (12. c. 5.)



As in these Numbers,	1	3	Or these Numbers	4	5
	2	6		12	15
	3	9		36	45
	4	12		108	135
	5	15		324	405
	6	18		972	1215

That is,  $1 : 3 :: 21 : 63$       And  $4 : 5 :: 1456 : 1820$  &c.

The Application of these Lemma's to Geometrical Quantities, viz. to Lines, Superficies and Solids, wholly depends upon granting the following Hypotheses.

## The Hypotheses.

1. That every Line is suppos'd to consist or be compos'd of an infinite Series of equidistant Points.

2. A Surface (viz. the Area of any Figure) to consist of an infinite Series of Lines, either strait or crooked, according as the Figure requires.

3. A Solid to consist of an infinite Series of Planes, or Superficies, according as its Figure requires.

Not that we suppose Lines, which have really no Breadth, can fill a Space or Superficies; or that Planes, which have not any Thickness, can constitute a Solid: But by what we here call Lines, are to be understood small Parallelograms (or other Superficies) infinitely narrow, yet so, as that their Breadths, being all taken and put together, must be equal to the Figure they are suppos'd to fill up.

And those Planes or Superficies, which are here said to constitute a Solid, are to be understood infinitely thin; yet so, as that their Depths or Thicknesses (which are hereafter also call'd Lines) being all taken together, must be equal to the Height of the propos'd Solid.

Now, in order to render these Hypotheses as easy for a Learner to understand as I can, I shall here propose a very plain and familiar Example.

Viz. Let us suppose any Book to be compos'd or made up of 100, 200, 300 (more or less) Leaves of fine Paper; such a Book, being close put together, will have Length, Breadth and Depth or Thickness, and therefore may not (improperly) be call'd a Solid; and each of its Edges (being evenly cut) will be a Superficies compos'd of a Series of small Parallelograms, every one of their Breadths being only the Edge of a single Leaf of Paper; and if we conceive the Thickness of every one of those Leaves

F f f

to

to be divided into 10, or 100, or 1000, &c. they will then become such a Series of infinitely small Lines, as are (by the Hypothesis) said to compose, or fill up a Superficies.

And all the Superficies of those infinitely thin, or divided Leaves of Paper, will become such a Series of Planes, or Superficies, as are said to constitute a Solid, *viz.* such a Solid as the Bigness and Figure of that Book.

Now according to this Idea of Lines, Superficies, and Solids, one may, without the least Prejudice to any Demonstration, admit of the following Definitions, and Theorems.

### Definitions.

I. The Area's of Squares, and all other Parallelograms, are compos'd or fill'd up with an infinite Series of equal Right-lines.

II. The Area of every plane Triangle, is compos'd of an infinite Series of Right-lines parallel to its Base, and equally decreasing, until they terminate in a Point at the Vertical Angle.

III. The Area of a Circle may be compos'd either of an infinite Series of concentric or parallel Circles; or of an infinite Series of Chord Lines parallel to its Diameter; or of an innumerable Multitude of Sectors.

IV. The Area of an Ellipsis may be compos'd, either of an Infinite Series of Ordinates rightly apply'd. or of an infinite Series of Right-lines parallel to its Transverse-Diameter.

V. The Area's of the Parabola, and Hyperbola, are compos'd of an infinite Series of Ordinates, or may also be compos'd of Right-lines parallel to their Axis, &c.

VI. A Prism is a solid Body, contain'd or included within several equal Parallelograms, having its Bases or Ends equal and alike; and 'tis generally named according to the Figure of its Base. That is,

VII. A Cube (or Solid like a Dye) is a Prism bounded or included within six equal Square-Planes.

VIII. A Parallelopipedon is a Prism that hath its Sides bounded or included within four equal Parallelograms, and two Square-Bases or Ends.

IX. A Cylinder (or Solid like the Rolling-Stone in a Garden) is only a round Prism, having its Bases or Ends a perfect Circle.

X. The



X. The Solidity of every Prism is compos'd of an infinite Series of equal Planes, parallel and alike to that of its Base.

XI. A Pyramid is a Solid bounded or included within several plane Triangles, set upon any Polygonous Base, having their Vertical-Angles all meeting together in a Point, call'd the Vertex, and takes its Name from the Figure of its Base, *viz.* if it have a Square-Base, 'tis called a Square-Pyramid, if a Triangle-Base, 'tis call'd a Triangular-Pyramid, &c.

XII. A Cone is always a round Pyramid, which hath been already defin'd in Page 361, &c.

XIII. The Solidity of every Pyramid is compos'd or constituted of an infinite Series of Planes, parallel and alike to that of its Base, equally decreas'g until they terminate in a Point at the Vertex.

XIV. A Sphere or Globe (*viz.* a Ball) is a Solid bounded or included within one regular Superficies, being form'd or generated by the Rotation of a Semi-circle about its Diameter; (call'd the Axis of the Sphere) and its Solidity is compos'd or constituted of an infinite Series of concentric Circles, whose Diameters are the Chords of that Circle by which it was form'd.

XV. A Spheroid (or Egg-like Figure) is a Solid bounded with one regular Superficies, form'd by the Rotation of a Semi-Ellipsis about its Transverse-Diameter (call'd the Axis of the Spheroid) and its Solidity is constituted of an infinite Series of concentric Circles, whose Diameters are the Ordinates of that Ellipsis by which it was form'd.

XVI. There is another Sort of a Solid call'd an Oblate-Spheroid; being form'd by the Rotation of an Ellipsis about its Conjugate-Diameter; and 'tis like to a flat Turnip.

XVII. If a Semi-parabola be turn'd about its Axis, it will form a Solid call'd a Parabolic-Conoid, being compos'd or constituted of an infinite Series of Circles, whose Diameters are the Ordinates of the Parabola.

XVIII. If a Parabola be turn'd about its Base or greatest Ordinate, it will form a Solid call'd a Pyramidoid, but most commonly a Parabolic Spindle, which will be constituted of an infinite Series of Circles, whose Diameters are Right-lines parallel to the Parabola's Axis.

XIX. If an Hyperbola be turn'd about its Axis, it will form a Solid call'd an Hyperbolic-Conoid, being constituted of an infinite Series of Circles, whose Diameters are the Ordinates of the Hyperbola.

F f f 2

XX. The

XX. The curve Superficies of all circular Solids (*viz.* Cylinders, Cones, Spheres, &c.) are compos'd of an infinite Series of the Peripheries of those Circles which constitute their Solidities.

Upon these Definitions are grounded all the following Theorems; and therefore, if they were diligently compar'd with their respective Figures, it must needs be of great Help to the Learner, and would render all that follows very easy; wherein I shall begin with what hath been already demonstrated, by way of introducing the rest.

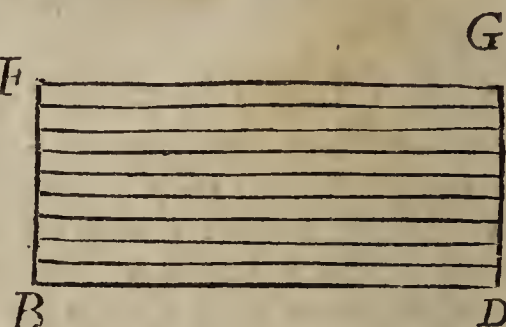
### THEOREM I.

*The Area of every Right-lin'd Parallelogram, is obtain'd by multiplying the Length into its Breadth.*

That is,  $BD \times FB =$  the Area of the Parallelogram,  $BD FG$ . By Lemma I. compar'd with Definition I.

*Example,* Suppose  $BD = 26$ . And  $FB = 9$ .

Then  $26 \times 9 = 234$  the Area. See Problem I. Page 339.



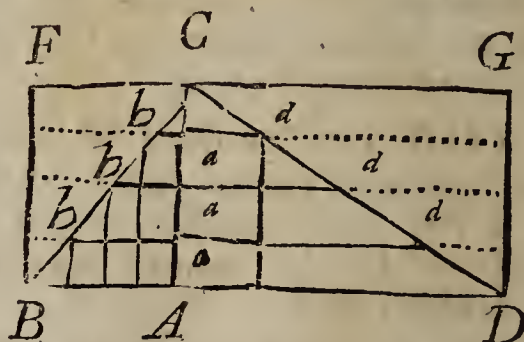
### THEOREM II.

*The Area of every plane-Triangle is equal to Half the Area of its circumscribing Parallelogram.*

That is,  $\frac{BD \times CA}{2} =$  the Area of  $\triangle BCD$ , in the following Figure.

#### Demonstration.

Suppose the Perpendicular  $CA$  to be divided into an infinite Number of equal Parts, as at the Points  $a, a, a$ , &c. and through those Points there were drawn Right-lines parallel to the Base  $BD$ , *viz.*  $b, a, d, b, a, d, b, a, d$ , &c. Then will those Lines be a Series of Terms in Arithmetical Progression, beginning at the Point  $C$ , *viz.*  $c, bd, 2, bd, 3, bd$ , &c. as is evident by the Figure, wherein  $BD$  is the greatest Term  $= L$ , and  $CA$  the Number of Terms  $= N$ .



But



But  $\frac{1}{2} NL = S$ , by Lemma 2. And  $S =$  the Triangle's Area, by Definition 2. Q. E. D.

Example, Let  $BD = 26$ , and  $CA = 9$ . As above.

Then  $\frac{26 \times 9}{2} = 117$ . Or  $\frac{26}{2} \times 9 = 117$ . Or thus  $26 \times \frac{9}{2} = 117$ .

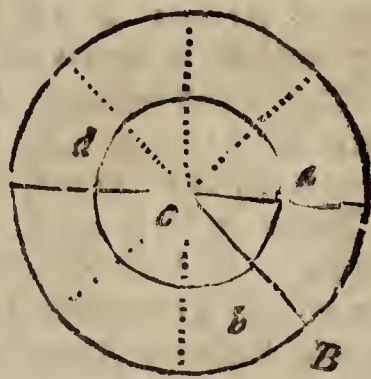
the Area requir'd. See Problem 3. Page 330.

### THEOREM III.

*The Peripheries of Circles are in Proportion one to another, as their Diameters are.*

#### Demonstration

Let the Periphery of a Circle be divided into any Number of equal Arches, by Right-lines drawn from the Center, viz. Radius's ) suppose them 8, as in the annex'd Figure, wherein  $AB$  is one of them. Then if through any Point in the Radius there be drawn a concentric or parallel Circle, its Periphery will also be divided into 8 equal Arches by those Radius's; one whereof will be  $ab$ , and the  $\triangle Cab$  will be like to  $\triangle CAB$ .



Therefore  $Ca : ab :: CA : AB$ . Or  $Ca : CA :: ab : AB$

Consequently  $2Ca : 2CA :: 8ab : 8AB$ .

But  $2Ca = da$  the Diameter of the Circle, whose Periphery is  $8ab$ .

And  $2CA = DA$  the Diameter of the Circle, whose Periphery is  $8AB$ . Therefore, &c. as by Theorem. Q. E. D.

Example. In Chapter 6. Part III. it was found, that if the Diameter of a Circle be 2. its Periphery will be 6.2831853071795864

Etgo  $2 : 6.2831853, \&c. :: 1 : 3,14159265, \&c.$  the Periphery of the Circle whose Diameter is 1.

#### Corollary.

Hence it follows, that because Unity or 1. may be made the first Term in the Proportion, therefore 3,14159265, &c. may be made a constant or settled Factor; which being multiply'd into any propos'd Diameter, will produce the Periphery of that Circle.

Note. Instead of 3,14159265, &c. it may be sufficient to take only 3,1416.

Or

Or in whole Numbers the Proportion may be,  
 As 7: 22:: Diam:: Periphery } these Numb. may serve, and  
 Or 113: 355:: Diam:: Periphery } are often us'd in com. Practice.

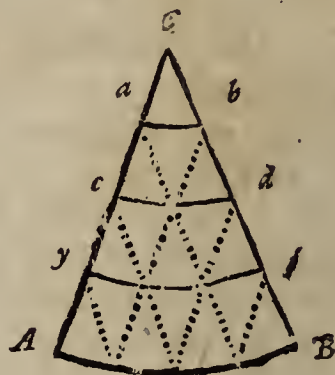
### THEOREM IV.

*The Area of any Sector of a Circle, is equal to Half the Rectangle of the Radius into its Arch.*

That is,  $\frac{CA \times AB}{2} =$  the Area of A C B.

#### Demonstration.

Suppose the Radius CA to be divided into an infinite Series of equidistant Points, as *a, e, y, &c.* and through those Points there were drawn concentric or parallel Arches, as *ab, ed, yf &c.* Then they will be a Series of Arches in Arithmetical Progression, beginning at the Point C, (*viz.* 0, 1, 2, 3, &c.) as plainly appears by the Figure, wherein the greatest Term is  $AB = L$ , and Number of Terms is  $CA = N$ . But  $\frac{1}{2} NL = S$  the Sum of all the Series, by Lemma 2. And  $S =$  the Sector's Area, by Definition 3.



Q. E. D.

*Example.* Let the Radius  $CA = 12$ . And the Arch  $AB = 8$ .

Then  $\frac{12 \times 8}{2} = 48$ . Or  $\frac{1}{2} \times 8 = 48$ . Or  $\frac{3}{2} \times 12 = 48$ . the Area of the Sector A C B.

### THEOREM V.

*The Area of every Circle, is equal to Half the Rectangle of the Radius into its Periphery.*

*That is, according to Archimedes, a Circle is equal to a right-angled Triangle, whose Sides containing the Right-angle, are equal, one to the Radius, and the other to the Perimeter of that Circle. Pro 1. de dimensione circuli.*

The Truth of this Theorem may be easily deduc'd from the last, thus, if we suppose the last Sector to be one eighth Part of a Circle; then it follows, that  $\frac{8 AB \times CA}{2} = 4 AB \times CA$  will be the Area of the whole Circle.

But  $4 AB =$  Half the Circle's Periphery; and  $CA =$  Half its Diameter. Therefore, &c. As per Theorem.

Q. E. D.

*Example*



*Example,* If the Diameter be Unity, or 1. the Periphery will be 3,14159265, &c. by Theorem 3.

Then  $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816$ , &c. ( Or 0,7854 for common Use ) will be the Area of that Circle.

## SCHOLIUM.

From hence naturally flows the following Proportion between the Square, and its inscrib'd Circle.

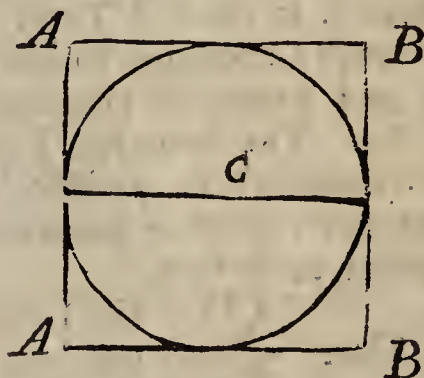
*Proportion.*  $\left\{ \begin{array}{l} \text{As the Perimeter ( viz. the Sum of the } \\ \text{four Sides ) of any Square : is to its Area ::} \\ \text{So is the Periphery of the inscrib'd Circle :} \\ \text{To its Area.} \end{array} \right.$

That is, supposing  $AB = D =$  the Side of the Square, and the Diameter of its inscrib'd Circle.

Then  $4D =$  the Perimeter ;  $DD =$  the Area of the Square ; and  $3,1416D =$  the Periphery of the Circle. By Theorem 3.

But  $4D : DD :: 3,1416D : 0,7844DD =$  the Circle's Area.

And if  $D = 1$ . Then  $4D = 4$  and  $DD = 1 \times 1 = 1$ . And the Periphery will be 3,1416,



Then  $4 : 1 :: 1 : 0,7854$  &c. As in the Example above.

And from hence may be easily deduc'd the following Theorem.

## THEOREM VI.

*The Area's of all Circles are in Proportion one to another, as the Squares of their Diameters. ( 2. e. 12. )*

For if  $D =$  the Diameter of one Circle, and  $d =$  the Diameter of another Circle.

Then will  $0,7854DD$  be the Area of one Circle, and  $0,7854dd$  will be the Area of the other Circle, as above.

But  $0,7854DD : 0,7854dd :: DD : dd$ . Or thus.

Let  $D =$  the Diameter, and  $P =$  the Periphery of one Circle :  $d =$  the Diameter, and  $p =$  the Periphery of another Circle.

Then  $1 \left| \frac{1}{2} D \times \frac{1}{2} P = \frac{1}{4} DP = A \text{ the Area of one Circle.} \right.$

And  $2 \left| \frac{1}{2} d \times \frac{1}{2} p = \frac{1}{4} dp = a \text{ the Area of the other Circle.} \right.$

$1 \times 4 \quad 3 \quad DP = 4A$

$2 \times 4 \quad 4 \quad dp = 4a$

$3 \div D \quad 5 \quad P = \frac{4A}{D}$

(per last Theorem.)

$4 \div d$

$$4 \div d \quad 6 \quad p = \frac{4a}{d}$$

But 7  $P : p :: D : d$ . Per Theorem 3.

$$5, \quad 6, \quad 7 \quad 8 \quad D : d :: \frac{4A}{D} : \frac{4a}{d}$$

$$8 \quad \therefore 9 \quad 4DDa = 4dda. \quad \text{That is, } DDa = d d A$$

$$9, \text{ Anal. } 10 \quad DD : A :: dd : a. \quad \text{Or } A : a :: DD : dd.$$

Q.E.D.

## Corollary.

Hence it follows that because the Square of 1 is 1. (*viz.*  $1 \times 1 = 1$ ) and 0,78539816, &c. Or 0,7854 is the Area of the Circle whose Diameter is 1. (as before) therefore it will be  $1 : 0,7854 ::$  So is the Square of any Circle's Diameter: To its Area. And because 1. is the first Term in the Proportion, therefore 0,7854 may be made a constant Factor; which being multiply'd into the Square of any propos'd Diameter, will produce the Area of that Circle.

Note, The four last Theorems do plainly shew the Reason of all the common or practical Problems about a Circle; which for the Learner's further Satisfaction, I have here inserted together; supposing as before,

That  $\left\{ \begin{array}{l} D = \text{the Diameter} \\ P = \text{the Periphery} \\ A = \text{the Area} \end{array} \right\}$  of any propos'd Circle.

Then

Prob. 1. D being given, to find P.

$$1 \quad 1 : 3,1416 :: D : P. \quad \text{Per Theorem 3.}$$

I  $\therefore$ 

$$2 \quad 3,1416 D = P$$

Example.

$$\left\{ \begin{array}{l} \text{Suppose } D = 32. \quad \text{Then } 3,1416 \times 32 = 100,5312 \\ \text{the Periphery.} \end{array} \right.$$

Prob. 2. D, being given, To find A.

$$3 \quad 1 : 0,7854 :: DD : A. \quad \text{Per Theorem 6.}$$

3  $\therefore$ 

$$4 \quad 0,7854 DD = A$$

Example.

Suppose  $D = 32$ . As before.

Then

$$DD = 32 \times 32 = 1024.$$

And

$$0,7854 \times 1024 = 804,2496 \text{ the Area requir'd.}$$

Prob. 3. P, being given, to find D.

$$5 \quad D = \frac{P}{3,1416} \quad \left\{ \begin{array}{l} \text{Or } \left\{ \begin{array}{l} \text{because } \frac{1}{3,1416} = 0,3183 \\ \text{therefore } 0,3183 P = D. \end{array} \right. \end{array} \right.$$

This being only converse to the first needs no Exam.



			Prob. 4. P, being given, To find A.	
2	⊙ 2	6	9,86965	DD = PP
	6 ÷	7	DD = $\frac{PP}{9,86965}$	Or 0,10132 PP = DD
	4 ÷	8	DD = $\frac{A}{0,7854}$	Or 1,2732 A = DD
	For		0,7854 = 1,2732	
	7, 8	9	$\frac{PP}{9,86965} = \frac{A}{0,7854}$	Or 0,10132 PP = 1,2732 A
9 × &c.		10	$\frac{PP}{12,5664} = A$	Or 0,07957 PP = A
			Prob. 5. A, being given, To find D.	
8 w 2		11	D = $\sqrt{\frac{A}{0,7854}}$	Or D = $\sqrt{1,2732 A}$
			Prob. 6. A, being given, To find P.	
10 × &c.		12	PP = 12,5664 A	Or PP = $\frac{A}{0,07957}$
12 w 2		13	P = $\sqrt{12,5664 A}$	Or P = $\sqrt{\frac{A}{0,07957}}$

These six Problems contain all the Variety that can be proposed about finding the Periphery, Diameter, and Area of any Circle.

But if it be requir'd to find the Area of any Segment, or Part of a Circle cut off by a Chord, that Work will require a farther Consideration.

First, As to the Data, there must always be given the Diameter; or, either the Periphery, or Area of the Circle, in order to find the Diameter.

Secondly, There must also be given, either the Chord, which is the Base of the Segment, or the Versed Sine, which is the Height of the Segment.

That is, either BG, or AF, in the following Scheme, must be given, that so the Area of the  $\triangle BCG$  may be found.

Then it's evident, (by the Figure) that if the Area of the  $\triangle BCG$  be taken from the Area of the Sector CBAG, the Remainder will be the Area of the Segment BAG.

And if the Area of the Segment BAG be taken from the whole Area of the Circle, the Remainder will be the Area of the other Segment DBG.

*Example in Numbers.*

Let there be given  $DA = 32$ . as *Prob. 1.*

And the Versed Sine  $AF = 6$ .

Then  $\frac{1}{2}DA = BC = CA = 16$ .

And  $CA - AF = CF = 10$ .

But  $\square BC - \square CF = \square BF$ .

Consequently  $\sqrt{\square BC - \square CF} = BF$ .

*Viz.*  $\sqrt{156} = 12,49 = BF$ .

Then by the Doctrine of plane Triangles, the Arch  $BA = \angle BCA$  may be found in Degrees and Decimal Parts;

Thus  $BC : Radius :: BF : \text{Sine } \angle BCF = 51^{\circ}, 31'$

And then it will always hold in this Proportion;

*Viz.*  $\left\{ \begin{array}{l} \text{As the Circle's Periphery in Degrees : is to its Pe-} \\ \text{riphery in equal Parts. (according to the Dimen-} \\ \text{sions taken) :: So is the Arch in Degrees (viz.} \\ \angle BCA) : \text{To the same Arch in equal Parts.} \end{array} \right.$

That is,  $360^{\circ} : 100,5312 :: 51^{\circ}, 31' : 14,3284 = BA$

Then  $14,3284 \times 16 = 229,2544$  the Area of the Sector  $BCGA$ .

And  $12,49 \times 10 = 124,9$  the Area of the  $\triangle BCG$ .

Their Difference  $104,3544 =$  the Area of the Segm.  $BAG$ .

Or the Area of any Segment may be otherwise found (as most usually it is) by a Table of the Segments of a Circle, whose Area is Unity, or 1. The Construction or making of such a Table is very well laid down in Mr. *Daric's* Book of Gaging, *Chap. 9.* which he performs in this *Problem*.

P R O B L E M

*In a Circle whose Area is Unity, and its Diameter cut by Chord Lines into 1000 equal Parts, to find the Segment to any Versed Sine propos'd, not exceeding 500 of those equal Parts.*

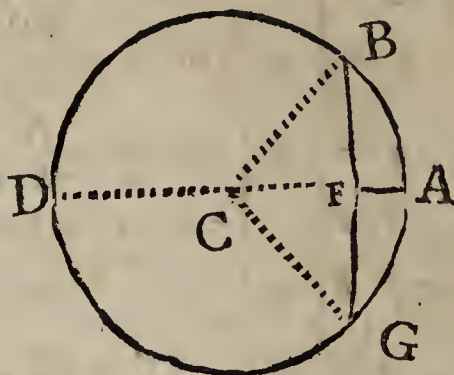
1. Multiply the Versed Sine propos'd by 0,002, and subtract the Product from an Unit or 1.

2. This Remainder you shall seek in the Common Table of Natural Sines, (the Arch being divided into Degrees and Centesima's) which being found, let its Co-arch be doubled, and call'd A.

3. You must find the Correspondent Sine to A; which Sine being found, you may call S, and then it holds

$6,2831853) 0,0174532925 \text{ A} - \text{S} (= \text{the Segment requir'd.}$

Now





Now this Segment being thus found, if you subduct it from an Unit, you have the Co-segment, &c.

*Note,* Notwithstanding what hath been said in the second Precept of this Problem, it very often falls out, that the Remainder there spoken of, cannot be truly found in the Table of Natural Sines; therefore in this Case my Advice is, that you make two Operations, one with a Sine the next Greater, and one with a Sine the next Less; and in so doing you will be sure to have the Segment requir'd bounded between the Results of those two Operations.

**Example,** *Let it be propos'd to find the Correspondent Segment to the Versed Sine 263.*

First,  $263 \times 0.002 = 0.526$ , and  $1 - 0.526 = 0.474$ , its Arch is  $28^\circ.29'$  being less than just; its Complement is  $61.71$  which being doubled, is  $123.42 = A$ ,

Then  $0.0174533 A = 2.154086286$   
 $- 0.8346556 = S$  The Sine of A.

---

6,2831853) 1,319430686 (0,209993 The Segment.

*Now I make a second Work.*

263 being multiply'd with 0,002 is 526. and  $1 - 526 = 0.474$  its Arch is  $28^\circ.30'$  being greater than just; and its Complement is  $61.70$ , which being doubled is  $123.4 = A$ .

Then  $0.0174533 A = 2.1537372$   
 $- 0.8348478 = S$  The Sine of A.

---

6,2831853) 1,3188894 (0,209907 The Segment.

So you see by these two Operations, that the Segment is bounded, and 'tis very probable it may be 0,20995.

But to abbreviate this large Factor, and this large Divisor. I shall here insert two Tablets of them, which will be ready for Use, and exact enough too.

Divisor.		Factor.	
6,2832	1	0,0174533	1
12,5664	2	0,0349066	2
18,8495	3	0,0523599	3
25,1327	4	0,0698132	4
31,4159	5	0,0872665	5
37,6991	6	0,1047197	6
43,9823	7	0,1221730	7
50,2655	8	0,1396263	8
56,5487	9	0,1570796	9

Thus far Mr. Darie, which I have here inserted to shew the Learner how, by the Help of these two Tablets, and a Table of Natural Sines, he may easily make a Table of Segments, whose Use shall be shew'd farther on, viz. when I come to Practical Gaging. In the mean Time I shall here lay down another Method, to find the Area of any Segment of a Circle (very near) by a

G g g 2

new

new *Theorem*, without the Help either of a Table of Sines or Segments, having the ~~same~~ Data as before in *Page* 404.

*viz.* Let  $\begin{cases} R = \text{the Radius, or } \frac{1}{2} \text{ Diameter of the given Circle.} \\ d = \text{the Difference between the Versed Sine and Radius.} \\ C = \text{half the Chord of the Segment's Base.} \end{cases}$

**Theorem**  $\left\{ \frac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{3}R + d} \times C = S, \text{ the Area of the Segm.} \right.$

*Example*, Suppose  $R = BC = 16$ .  $d = FC = 10$ . and  $C = BF = 12,49$ .  
As before.

Then  $2\frac{1}{3}RR = 597,3333$ .  $1\frac{1}{3}Rd = 213,3333$ .  $dd = 100$   
 $- 313,3333 = 1\frac{1}{3}Rd + dd$

$1\frac{1}{2}R + d = 34$   $284,0000$   $(8,3529$

Lastly,  $8,3529 \times 12,49 = 104,3276$  the Area of the Segment *BAG*,  
As before.

## THEOREM VII.

*As Squares are to the Area's of their inscrib'd Circles, so are Parallelograms to the Area's of their inscrib'd Ellipses.*

That is,  $\left\{ \begin{array}{l} \text{As the Square of the Diameter of any Circle : is} \\ \text{to its Area :: so is the Rectangle of the Transverse} \\ \text{and conjugate Diameters of any Ellipsis to its Area.} \end{array} \right.$

### Demonstration.

Circumscribe any Ellipsis with a Circle; and suppose an infinite Number of Chord-Lines drawn therein, all parallel to the Conjugate Diameter; as those in the annex'd Figure; then it will

be  $\left\{ \begin{array}{l} \text{As (DA) the Diameter of the Circle : is to (Nn) the conjugate Diameter of} \\ \text{the Ellipsis :: So is (BaB) any Chord in the Circle : to (bab) its respective} \\ \text{Ordinate in the Ellipsis.} \end{array} \right.$

For according to the Property of the Circle

it is 1  $TS - Ta \times Ta = \square Ba$

And by the Property of the Ellipsis

it is 2  $\square TC : \square NC :: TS - Ta \times Ta : \square ba$

1, 2 3  $\square TC : \square NC :: \square Ba : \square ba$

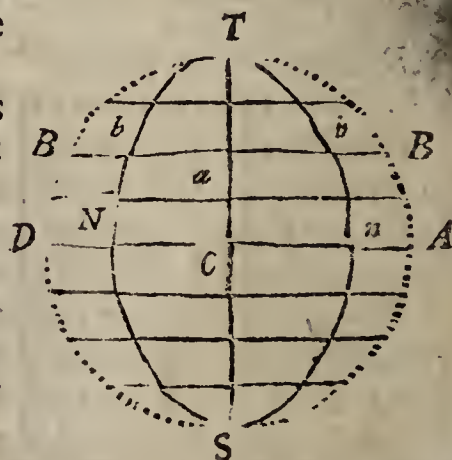
3, hence 4  $TC : NC :: Ba : ba$

Conseq. 5  $2TC : 2NC :: 2Ba : 2ba$

That is, 6  $DA : Nn :: BaB : bab$

Put 7  $D = 2TC$ . and  $d = 2NC$

Then 8  $D : d :: \text{Chord } BaB : \text{Ordinate } bab, \&c.$



But



But the Sum of an Infinite Series of such Chords, as  $BaB$ , do constitute the Area of the Circle, by Definition 3.

And the Sum of the like Series of their respective Ordinates, as  $bab$ , do constitute the Ellipsis's Area, by Definition 4.

Therefore  $D : d :: \text{Circle's Area} : \text{Ellipsis's Area}$ , by Lemma 6.

But  $D : d :: DD : Dd$ . Whence it follows,

That  $DD : \text{Circle's Area} :: Dd : \text{Ellipsis's Area}$ . Q. E. D.

Consequently, As 1 : is to 0,7854 :: so is the Rectangle, or Product of the Transverse, and Conjugate Diameters of any Ellipsis : To its Area.

Example, Suppose  $TS = 36$ . and  $Nn = 16$ . Then  $36 \times 16 = 576$ . And  $576 \times 0,7854 = 452,3904$  the Area of the Ellipsis.

### Corollaries.

1. Hence it is easy to conceive, that the Square Root of the Rectangle or Product of the Transverse, and Conjugate Diameters, will be the Diameter of a Circle, whose Area will be equal to the Ellipsis's Area.

Viz.  $\sqrt{576} = 24$  the Diameter of a Circle = to the Ellipsis.

2. All Segments of an Ellipsis and its circumscribing Circle, (whose Bases are parallel to the conjugate Diameter, and of the same Height) are in Proportion one to another, as their Bases are. That is,  $BaB : bab :: \text{Area Segment } BTB : \text{Area Segment } bTb$ ; Or  $TS : Nn :: \text{Area Segment } BTB : \text{Area Segment } bTb$ .

## THEOREM VIII.

*The Area of every Ellipsis, is a mean Proportional between the Area's of its Circumscribing, and inscrib'd Circles.*

The Truth of this Theorem may be easily deduc'd from the Last; for supposing  $D = TS$ . and  $d = Nn$ , as before. Then it is already prov'd, that  $DD : Dd :: \text{Circumscribing Circle's Area} : \text{Ellipsis's Area}$ .

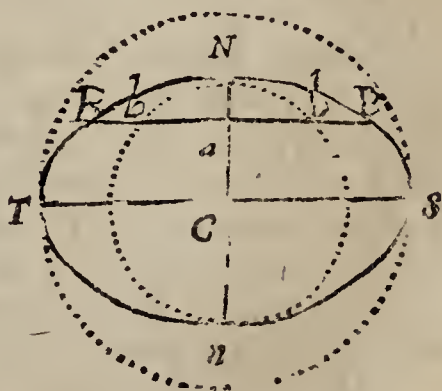
But  $DD : Dd :: Dd : dd$ .

Therefore, Ellipsis Area : Inscrib'd Circle's Area ::  $Dd : dd$ .

By Theorem 6.

Example, Let  $TS = D = 36$ . and  $Nn = d = 16$ . As before.

Then  $DD = 1296$ . and  $dd = 256$ .



Then

Then will  $\begin{cases} 1296 \times 0,7854 = 1017,8784 \text{ the great Circle's Area} \\ 256 \times 0,7854 = 201,0624 \text{ the lesser Circle's Area.} \end{cases}$

Suppose  $A =$  the Ellipsis Area. Then, according to the *Theorem*, it will be,  $1017,8784 : A :: A : 201,0624$ .

*Ergo*  $AA = 1017,8784 \times 201,0624 = 204657,07401216$ .

Consequently,  $\sqrt{204657,07401216} = 452,3904 = A$ , the Area of the Ellipsis, As before in the last *Example*.

### Corollary.

From hence it follows, that all Segments of an Ellipsis and its inscrib'd Circle, whose Bases are parallel to the Transverse Diameter, and have the same Height, are in Proportion one to another as the Area's of the Ellipsis and Circle are.

That is, Area of Circle : Area of Ellipsis :: Segment  $bNb$  : Segment  $BNB$ .

Or,  $Nn : TS ::$  Area Segment  $bNb$  : Area Segment  $BNB$ .

## THEOREM IX.

*The Solid Content of any Prism (what Figure soever its Base is of) is obtain'd by multiplying the Area of its Base into its Height.*

For Instance, a Parallelopipedon (or square Prism) is constituted of an infinite Series of equal Squares; that of its Base  $BAb$  being one of the Terms, and its Height  $DB$ , or  $GA$  the Number of all the Terms.

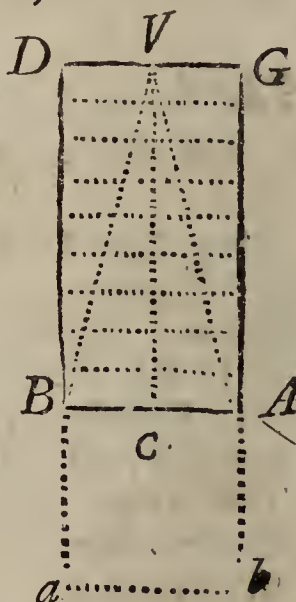
Consequently, the Area of  $BAb \times DB =$  the Sum of all the Series (by *Lemma 1.*) which is the Solidity of the Parallelopipedon  $DBG$ , by Definition 10.

*Example*, Suppose the Side of the Base  $BA = 16$  and the Height  $DB = 42$ .

Then will  $16 \times 16 = 256$  be the Area of the Base. And  $256 \times 42 = 10752$  the solid Content of the Parallelopipedon  $DBG$ .

In this Manner you may find the Solidity of all regular Polygonous Prisms, whose Bases (or Ends) are parallel and alike, what Form soever they are of.

That is, whether their Bases are Triangles, Pentagons, Hexagons, or Octagons, &c.



THEOREM



**THEOREM X.**

*Every Pyramid is the third Part of the Prism, that hath the same Base and Height with it. (7.e. 12.)*

That is, the *solid Content* of the Pyramid  $BVA$ , (in the last Figure) is one Third of its circumscribing Prism  $DBG A$ .

**Demonstration.**

For every Pyramid that hath a square Base (as  $BAb a$ , in the last Figure) is constituted of an infinite Series of Squares, whose Sides or Roots are continually increasing in Arithmetical Progression, beginning at the Vertex or Point  $V$  (See *Theor. 2.*) its Base  $BAb a$ , being the greatest Term, ( $= LL$ ) and its perpendicular Height  $VC$ , or  $DB$ , is the Number of all the Terms ( $= N$ ); but  $\frac{NLL}{3} = S$  the Sum of all the Series, by *Lemma 3.* and  $S =$  the solid Content of the Pyramid  $BVA$ , by Definition 13.

*Example.* Suppose the Side of a Pyramid's Base be  $BA = 16$ . and its Height be  $VC = 42$ . Then  $16 \times 16 = 256$  the Area of its Base  $BAb a$ . And  $\frac{256 \times 42}{3} = 3584$ . Or  $\frac{256}{3} \times 42 = 3584$ . Or thus,  $256 \times \frac{42}{3} = 3584$ , is the Solidity of that Pyramid  $BVA$ .

**Corollary.**

From hence it will be easy to conceive, that every Pyramid is  $\frac{1}{3}$  of its circumscribing Prism, what Form soever its Base is of, viz. whether it be a Square, Triangle, or Pentagon, &c.

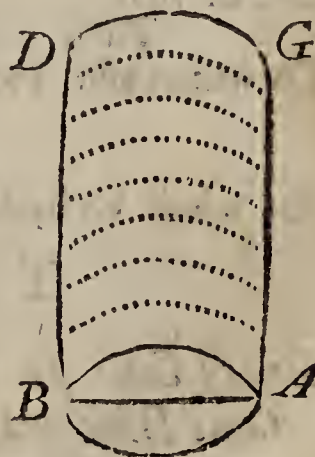
**THEOREM XI.**

*The solid Content of every Cylinder, is obtain'd by multiplying the Area of its Base into its Height.*

For every Right Cylinder is only a round Prism, being constituted of an infinite Series of equal Circles; that of its Base or End being one of the Terms, and its Height  $DB$  is the Number of all the Terms. Therefore the Area of its Base  $BA$  being multiply'd into  $DB$ , will be its Solidity, by *Lemma 1.*

Viz. Let  $D = BA$  and  $H = GA$ .

Then  $0,7854 DD \times H =$  its Solidity.



*Example;*

# 416 Arithmetick of Infinites. Part V.

*Example,* Let the Diameter of its Base be  $D=16$ , and its Height  $H=42$ .

Then  $1:0,7854::16 \times 16=256:201,0624$  the Area of its Base.

And  $201,0624 \times 42=8444,6208$  the solid Content of that Cylinder  $DBGA$ .

## Corollary.

Hence it is evident, that every Square Parallelopipedon is to its inscrib'd Cylinder, As 1 : is to 0,7854. Or in whole Numbers, as 452 : to 355, very near.

And that all Prisms are in Proportion to their inscrib'd Cylinders, as the Area's of their Bases are.

## THEOREM XII.

*The Curve Superficies of every Right Cylinder, is equal to the Rectangle made of its Height into the Periphery of its Base.*

That is,  $DB$  multiply'd into the Periphery of the Diameter  $BA$ , will produce the Curve Superficies of the last Cylinder  $DGBA$ .

For the Cylinder is constituted of an infinite Series of equal Circles; (according to the last *Theor.*) therefore its Curve Superficies is compos'd of the Peripheries of those Circles, by Definition 20. But the Periphery of its Base  $BA$  is one of the Terms, and its Height  $DB$  is the Number of Terms. Therefore, &c. As by *Lemma 1*.

To which, if there be added the Area's of both its Ends, (or Bases) the Sum will be the Superficies of the whole Cylinder.

*Example,* Suppose the Diameter of its Base to be  $BA=16$ , and its Height  $DB=42$ . As before.

Then  $1:3,1416::16:50,2656$  the Periphery of its Base.

Again,  $1:0,7854::16 \times 16=256:201,0624$  the Area of each End or Base.

Then  $50,2656 \times 42=2111,1552$  the Curve Superficies. To which add  $201,0624 \times 2=402,1248$  the Area's of both Ends.

The Sum  $=2513,2800$  is the Superficies of the whole Cylinder.

## THEOREM XIII.

*Every Cone is the third Part of a Cylinder, having the same Base with it, and their Altitudes equal. (10. e. 12.)*

Demon-



**Demonstration.**

The Truth of this Theorem may be easily conceiv'd by only considering, that a Cone is but a round Pyramid, and therefore it must needs have the same Ratio to its circumscribing Cylinder, as the Square Pyramid hath to its circumscribing Parallelopipedon; *viz.* as 1 to 3. However, to make it yet clearer, let it be farther consider'd, That

Every Right Cone is constituted of an infinite Series of Circles, whose Diameters do continually encrease in Arithmetical Progression, beginning at the Vertex or Point *V*, the Area of its Base *BA* being the greatest Term, and its Perpendicular Height *VC* the Number of all the Terms; therefore the Area of the Circle *BA*  $\times \frac{1}{3} VC$  will be the Sum of all the Series, by Lemma 3. which is the Cone's Solidity.



*Example.* Let the Diameter of its Base be  $BA = 16$ , and its Height  $VC = 42$ .

Then  $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$  the Area of the Base.

And  $\frac{201,0624 \times 42}{3} = 2814,8736$ , the Solidity of the Cone *BVA*.

Or thus,  $201,0624 \times 42^2 = 2814,8736$ , &c.

*Corollary.*

Hence it follows, that every Square Pyramid is to its inscrib'd Cone, as  $1 : 0,7854$ . (Or, as  $452 : 355$ .) Consequently, that all Pyramids have the same Ratio to their inscrib'd Cones, as the Areas of their Bases have.

**THEOREM IV.**

*The Curve Superficies of every Right Cone, is equal to half the Rectangle of the Periphery of its Base into the Length of its Side.*

The Truth of this Theorem is self-evident from the Definition of a Cone Chap. I. Part 4. where it appears, that the Curve Superficies of every Right Cone (as *BVA*) is equal to the Area of a Sector of that Circle, whose Radius is the Side of the Cone (*VB*) and its Arch equal to the Periphery of the Cone's Base (*B.A.*) But the Area of any Sector is equal to half the Rectangle of the Radius into its Arch, by Theorem 4. Therefore, &c.

H h h

*Example,*

# 418 Arithmetick of Infinites Part V.

*Example,* Suppose the Length of the Cone's Side to be  $VB$ , or  $VA = 42,7551$ .

And the Diameter of its Base,  $BA = 16$ . As before.

Then will 50,2656 be the Periphery of its Base.

And  $\frac{50,2656 \times 42,7551}{2} = 1074,5553$  &c. the Curve of the Superficies.

To which, if there be added the Area of its Base, the Sum will be the Superficies of the whole *viz.* all the Cone.

That is 1074,5553

+ 201,0614 the Area of the Base.

Sum 1275,6177 is the total Superficies. &c.

Note, The Truth of this Theorem may be prov'd from the Consideration of the last Theorem, and Definition 20.

## SCHOLIUM.

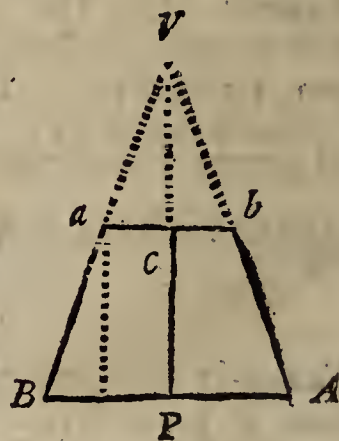
From the 10th and 13th Theorems may be easily deduced several Theorems for finding the solid Content of any Frustum or Part, either of a Pyramid or Cone, by a plane Parallel to its Base.

Suppose a square Pyramid, as  $BVA$ , to be cut by a Plane at  $ab$ , parallel to its Base  $BA$  and it were requir'd to find the Solidity of the Frustum or Part  $abAB$ . Let there be given.

$D = BA$  the Side of the greater Base.

$d = ba$  the Side of the lesser Base.

$H = CP$  the Perpendicular Height,



First 1  $-d : H :: d : \frac{dH}{D-d} = VC$  by the Figure.

Then 2  $\int DD \times \frac{H+VC}{3} =$  the whole Pyramid  $BVA$ .

By Theorem 10.

And 3  $dd \times \frac{1}{3} VC =$  the Pyramid  $aVb$  cut off.

*Viz.* 1, 2 4  $\frac{DDD H}{3D - 3d} =$  the whole Pyramid  $BVA$ .

And 1, 3 5  $\frac{ddd H}{3D - 3d} =$  the Pyramid  $aVb$ .

4 — 5 6  $\frac{DDD H - ddd H}{3D - 3d} =$  the Frustum  $abAB$ .

6, Reduc. 7  $DD + Dd + dd : \times \frac{1}{3} H =$  the Frustum  $abAB$ .

Which in Words gives this following Theorem.

THEOREM



THEOREM XV.

To the Rectangle of the Sides of the two Bases, add the Sum of their Squares; that Sum being multiply'd into one Third of the Frustum's Height, will give its Solidity.

Example, Suppose the Side of the greater Base  $BA = 16$ .

And the Side of the lesser Base (or Top)  $ab = 12$ .

The Height  $CP = 9$ .

Then  $16 \times 12 = 192$ .  $16 \times 16 = 256$ . and  $12 \times 12 = 144$ .

Next  $192 + 256 + 144 = 592$ . and  $\frac{592 \times 9}{3} = 1776$ .

Or  $592 \times \frac{9}{3} = 1776$ , the Content of the Frustum of a square Pyramid.

And if it were the like Frustum of a right Cone, it may be found by the same Theorem. Supposing  $D$  equal the Diameter of the greater Base,  $d =$  the Diameter of the lesser, and  $H =$  the Height of the Frustum.

Then seeing the Sum of all the Squares which constitute the Frustum of a square Pyramid, are to the Sum of all the Circles which constitute the like Frustum of a right Cone, in the Ratio of 1: to 0.7854. (or of 452: to 355.) Therefore,

it will be  $1 : 0.7854 :: DD + Dd + dd \times \frac{1}{3} H : 0.7854 DD + 0.7854 Dd + 0.7854 dd \times \frac{1}{3} H =$  the Cone's Frustum.

That is, in the last Example,  $1 : 0.7854 :: 1776 : 1394.8704$  the like Frustum of a right Cone.

Or, because  $0.7854 = 1.273236$ , &c. Therefore it may be made  $1.273236 DD + Dd + dd \times \frac{1}{3} H (=$  the same Frustum.

That is,  $1.273236 1776 (1394.87$ . &c. As before.

And if you take the Triple of this Divisor, viz.  $1.273236 \times 3$  it will be  $3.8197 DD + Dd + dd : \times H (=$  the Frustum, &c.

Again.

Suppose  $x = D - d$ . And  $F =$  the Frustum.

Then  $DD + Dd + dd = \frac{3}{H} F$ , by 7th Step of the last.

$1 \oplus 2 \quad 3 \quad xx = DD - 2 Dd + dd$

$2 - 3 \quad 4 \quad 3Dd = \frac{3}{H} F - xx$

$4 \div 3 \quad 5 \quad Dd = \frac{F}{H} - \frac{1}{3} xx$ . Or  $DD + \frac{1}{3} xx = \frac{F}{H}$

$5 \times H \quad 6 \quad Dd + \frac{1}{3} xx : \times H = F$  the Frustum  $abAB$ .

Hence we have another easy Theorem for finding the same Frustum.

Hhh 2

THEOREM

THEOREM XVI.

*To the Rectangle of the Sides of the two Bases, add one Third Part of the Square of their Difference; that Sum being multiply'd into the Height, will produce the Solidity.*

*Example, Let  $D = 16$ .  $d = 12$ . and  $H = 9$ . As before.*

*Then  $Dd = 192$ .  $D - d = 4 = x$ .  $\frac{1}{3}xx = \frac{4 \times 4}{3} = 5,3333$ .*

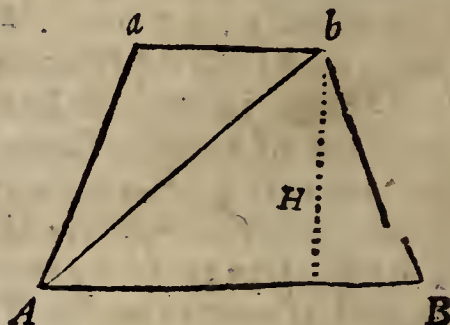
*And  $192 + 5,3333 = 197,3333$ .*

*Lastly,  $197,3333 \times 9 = 1775,9997$  the Solidity of the Frustum of the square Pyramid. As before.*

*And 3,81968) 1775,9997 (1394,87, &c. the like Frustum of a right Cone. As before.*

Either of the two last Theorems (being rightly apply'd) will produce the true solid Content of all Frustums of any kind of Pyramids that are intercepted between two parallel and alike Planes or Bases. As above.

But if such Frustums are cut thro' the Extremities of both Bases by a Diagonal Plane (as  $Ab$  in the annex'd Figure) into two Parts,  $Aab$ , and  $ABb$ , call'd Hoofs; then the Solidity of those Hoofs is usually found by dividing the middle Term  $Dd$  of the Equation  $DD + Dd + dd$  into two Parts, and adding one of those Parts to the Square of each Base.



Thus,  $DD + \frac{1}{2}Dd : \times \frac{1}{3}H =$  the great Hoof  $ABb$ .

And  $dd + \frac{1}{2}Dd : \times \frac{1}{3}H =$  the lesser Hoof  $Aab$  of the Frustum of any square Pyramid.

Then 3,8197)  $DD + \frac{1}{2}Dd : \times H (=$  the greater Hoof of a Cone.

And 3,8197)  $dd + \frac{1}{2}Dd : \times H (=$  the lesser Hoof, &c.

These are the Theorems made use of by Mr. Dary in his Book of Gaging, and are pretty near the Truth, but not exactly so; for they give the Solidity of the upper Hoof  $Aab$  a small matter too big, and the lower Hoof  $ABb$  as much too little.

Now in order to rectify that small Error, I shall here propose the two following Theorems, which come very near the Truth, and are more easily perform'd than those proposed in the first Impression of this Book.

First,



First,  $DD + \frac{1}{2}Dd + D - d : \times \frac{1}{3}H$  will be the Solidity of the greater Hoof  $ABb$ .

Secondly,  $dd + \frac{1}{2}Dd + d - D : \times \frac{1}{3}H$  will give the Solidity of the lesser Hoof  $Aab$ , of the Frustum of any Square Pyramid.

And for the like Hoofs of the Frustum of any right Cone, it will be

Thus, 38197)  $DD + \frac{1}{2}Dd - D - d : \times \frac{1}{3}H (= \text{the greater Hoof.})$

And 3,8197)  $dd + \frac{1}{2}Dd + d - D : \times \frac{1}{3}H = \text{the lesser Hoof.}$

Note, In order to avoid many Words in the following Demonstrations. let  $\odot$  signify any Circle in general; and if any two Letters be joyn'd to it, thus,  $\odot BA$  &c. it then denotes the Area of such a Circle as those two Letters represent the Radius of.

## THEOREM XVII.

*The Superficies of every Sphere (or Globe) is equal to four times the Area of its greatest Circle.*

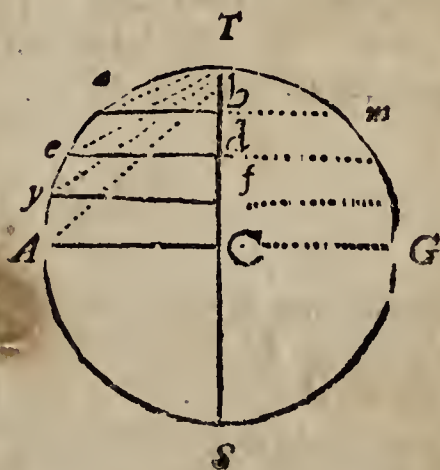
That is, of a Circle whose Diameter is the Axis of the Sphere.

### Demonstration.

If any Semi-circle (as  $ATGS$ ) be turn'd or moved about its Diameter ( $TS$ ) it will describe a solid Body call'd a Sphere, which will be constituted of an infinite Series of concentrick or parallel Circles, whose Diameters are Chords, viz.  $\odot ab$ ,  $\odot ed$ ,  $\odot ef$ , &c. by Definition 14.

Consequently, the Superficies of the Sphere will be composed of the Peripheries of those Circles which constitute its Solidity. By Definition 20.

Let  $D=TS$ , the Axis of any Sphere. Then, according to the Property of a Circle, it



will be	1	$D - Tb \times Tb = \square ab$
That is	2	$D \times Tb - \square Tb = \square ab$
Therefore	3	$D \times Tb = \square aT. \text{ For } \square ab + \square Tb = \square aT$
And	4	$D \times Td = \square eT$
	5	$D \times Tf = \square yT, \text{ \&c.}$

Hence

Hence it is evident, that the Series  $\square aT$ .  $\square eT$ .  $\square yT$  &c. are in the same *Ratio* with  $Tb$ .  $Td$ .  $Tf$ . &c. *viz.* in Arithmetical Progression. Whence it follows, that the  $\odot aT =$  the Sum of all the Circle's Peripheries between  $T$  and  $b$ ,

And the  $\odot eT =$  the Sum of all the Circle's Peripheries between  $T$  and  $d$ . &c.

Consequently, that the  $\odot AT =$  the Sum of all the Circle's Peripheries included between  $T$  and  $C$  That is,  $\odot AT =$  the Superficies of the Semi-sphere.

And because the  $\square AC + \square TC = \square AT$ , and  $\square AC = \square TC$ . Therefore  $\odot AT = 2\odot AC$  is the Superficies of the Semi-sphere.

Consequently,  $4\odot AC$  will be the Superficies of the whole Sphere. Q. E. D.

*Example.* Suppose the Axis  $TS = D = 16$ . Then  $DD = 256$ . And  $1 : 0,7854 :: 256 : 201,0624 = \odot AC$ . For  $\frac{1}{2} D = AC$

Then  $201,0624 \times 4 = 804,2496$  the Superficies of the whole Sphere.

Or, because  $3,1416$  is four times  $0,7854$ , therefore it will always be  $1 : 3,1416 :: DD : 3,1416 DD$  the Superficies of the Sphere (as before) and it's equal to the Curve Superficies of a right Cylinder. whose Diameter and Height are each  $=$  the Axis of the Sphere.

For  $3,1416 D =$  the Periphery of the Cylinder's Base, and that multiply'd with  $D$  its Height, will be  $3,1416 DD$  the Curve Superficies of the Cylinder. by *Theor.* 12.

And if to this there be added the Areas of its two Bases, (or Ends) *viz.*  $1,5708 DD$ . Then it is evident, that the whole Superficies of the Cylinder will be to that of the Sphere, in the Proportion of 3 to 2.

## S C H O L I U M.

From the Method here used in proving the last *Theorem*, it will be easy to find the Curve Superficies of any Segment or Part of a Sphere, that is cut off by a right Line, or Plane; *viz.* such as the Segment  $aTm$  in the last Scheme, whose Curve Superficies is  $\odot aT$ , (as above.) Therefore (because  $\square ab + \square Tb = \square aT$ ) it will be  $\odot ab + \odot Tb =$  the Curve Superficies of that Segment.

But if the Axis  $TS$ . and Height  $Tb$  of the Segment are given, then it will be  $TS \times Tb = \square aT$  as in the third Step above. Which gives this Proportion or Theorem.

*Viz.*



Viz.  $\left\{ \begin{array}{l} \text{As the Axis of the Sphere : is to the whole Super-} \\ \text{ficies of the Sphere :: so is the Height of any Seg-} \\ \text{ment : to its Curve Superficies} \end{array} \right.$

To which, if there be added the Area of the Segment's Base, the Sum will be the Superficies of the whole Segment.

## THEOREM XVIII.

*Every Sphere is equal to two Thirds of its Circumscribing Cylinder.*

That is, of a Cylinder, whose Height and Diameter of its Base are each equal to the Axis of the Sphere.

### Demonstration.

According to the Work in the last Theorem, it appears, that  $\odot ab.$   $\odot ed.$   $\odot yf.$  &c. do constitute the Solidity of the Sphere, and that  $\square aT,$   $\square eT,$   $\square yT,$  &c. are a Series of Terms in *Arithmetical Progression*,  $\square AT$  being the greatest Term, and  $TC$  the Number of Terms. Therefore  $\odot AT \times \frac{1}{2} TC =$  the Sum of all the Series.

Per Lemma 2.

And because  $\square aT - \square Tb = \square ab.$   $\square eT - \square Td = \square ed.$   $\square yT - \square Tf = \square yd.$   $\square AT - \square TC = \square AC,$  &c. wherein  $\square Tb,$   $\square Td,$   $\square Tf$  &c. are Series of Squares. whose Roots  $Tb, Td, Tf,$  are in *Arithmetical Progression*;  $\square TC$  being the greatest Term, and  $TC$  the Number of Terms. Therefore  $\odot TC \times \frac{1}{3} TC =$  the Sum of all that Series. Per Lemma 3.

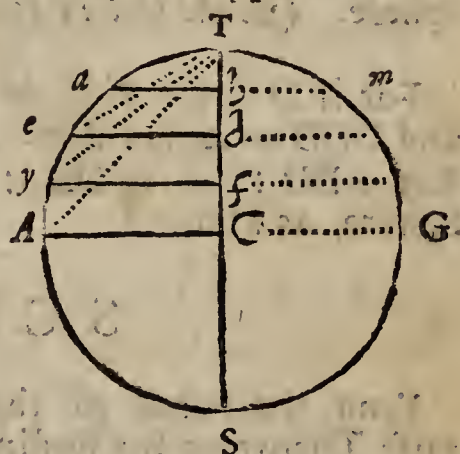
Consequently,  $\odot AT \times \frac{1}{2} TC : - \odot TC \times \frac{1}{3} TC =$  the Sum of the Series  $\odot ab.$   $\odot ed.$   $\odot yf.$  &c. which constitute the Solidity of the half Sphere  $ATG.$  Put  $D = 2TC$  the Axis of the Sphere. Then  $\frac{1}{2} D = \frac{1}{2} TC,$  and  $\frac{1}{3} D = \frac{1}{3} TC.$  And because  $\square AT = 2 \square TC;$  therefore  $\odot AT = 2 \odot TC = 1,5708 DD.$  And  $1,5708 DD \times \frac{1}{2} D = 0,3927 DDD.$

Again,  $\odot TC \times \frac{1}{3} TC = 0,7854 DD \times \frac{1}{3} D = 0,1309 DDD.$  Then  $0,3927 DDD - 0,1309 DDD = 0,2618 DDD$  the Solidity of the Semi-sphere  $ATG.$

Consequently.  $0,2618 DDD \times 2 = 0,5236 DDD$  will be the solid Content of the whole Sphere which is equal to  $\frac{2}{3}$  of the Cylinder, whose Diameter of its Base and Height  $= D.$

For  $0,7854 DDD =$  the Solidity of the Cylinder, by Theorem 11. But  $\frac{2}{3}$  of  $0,7854 DDD = 0,5236 DDD,$  as before.

Therefore, &c. as by Theor.



Example



*Example,* Suppose the Axis  $D = 16$ . then  $DDDD = 4096$ , and  $1:0,5236 :: 4096 : 2144,6656$  the solid Content of that Sphere.

### Corollaries.

1. Hence it appears, that the solid Content of every Sphere is equal to its Superficies multiply'd into one sixth Part of its Axis.

For its Superficies is  $3,1416 DD$ , by *Theorem 17*.

But  $3,1416 DD \times \frac{1}{6} D = 0,5236 DDD$  the solid Content, as before.

2. And hence it is also evident, that there is the like *Ratio* or *Habitude* between the Cube and its inscribed Sphere. as is between the Square and its inscrib'd Circle. And that is, As the Superficies of any Cube : is to the Superficies of its inscrib'd Sphere :: so is the solid Content of that Cube : to the solid Content of the Sphere. (*See the Circle's Proportion, Page 407.*)

For if  $D =$  the Side of the Cube, then  $6 DD =$  its Superficies, and  $DDD =$  its Solidity. And  $3,1416 DD =$  the Sphere's Superficies. But  $6 DD : 3,1416 DD :: DDD : 0,5236 DDD$  the Solidity of the Sphere, as above.

### SCHOLIUM.

From the Proof of this Theorem, it will be easy to deduce or raise Theorems for finding the solid Content of any Frustum or Segment of a Sphere; as  $aTm$  in the last Figure.

For we there suppose the Segment  $aTm$  to be constituted of an infinite Series of Circles, which have the same *Ratio* with all those Circles that constitute the *Semi-sphere*.

Therefore it follows, that  $\odot aT \times \frac{1}{2} Tb : - \odot bT \times \frac{1}{3} Tb$  will be will be the Sum of all the Circles intercepted between  $T$  and  $b$ . Consequently it will be Solidity of that Segment.

And because the  $\square ab + \square Tb = \square aT$ : Therefore,

$\odot ab + \odot Tb \times \frac{1}{2} Tb - \odot Tb \times \frac{1}{3} bT =$  the same Solidity.

Let  $c = ab$ , half the Segment's Base;  $h = Tb$ , its Height; and  $S =$  the Solidity of the Segment or Frustum.

Then  $\odot ab = 3,1416 cc$ . and  $\odot Tb = 3,1416 hh$ .

Consequently,  $\frac{3,1416 cch + 3,1416 hhh}{2} - \frac{3,1416 hhh}{3} = S$

which being reduced will become  $3 cch + hhh \times 0,5236 = S$ .

Or  $1,909855) 3 cch + hhh (= S$ . For  $0,5236) 10000 (1,909855$  which is one Theorem for finding the Frustum's Solidity.

*Note,*



*Note,* Here we suppose the Height of the Segment, and the Diameter of its Base to be given: But if the Axis of the Sphere, and the Height of the Segment be given. then putting  $D =$  the Sphere's Axis.  $h =$  the Segment's Height, and  $c$  as before, it will be  $D - h \times h = cc$ . Viz.  $Dh - hh = cc$ .

Therefore  $3 Dhh - 2 hhh = 3 cch + hhh$ .

Consequently,  $3 Dhh - 2 hhh \times 0.5236 = S$  the Frustrum's Solidity.

Or  $1.99985) 3 Dhh - 2 hhh (= S$ . As before.

Which is a second Theorem for finding the same Frustrum  $a T m$ .

And if it be requir'd to find the middle Part  $am NK$ , usually call'd the Middle Zone of a Sphere; then, because it is supposed that  $am = NK$ , or which is all one, that  $lC = CB$ ; therefore it is plain, that it twice the Segment  $a T m$  be taken from the Solidity of the whole Sphere, there will remain the Middle Zone  $am NK$ .

But because that Work is a little troublesome, I shall here shew how to raise a Theorem for doing it.



First, Because  $AC = yC = eC = aC = TC$ . Therefore it will be  $\square AC - \square Cf = \square yf$ .  $\square AC - \square Cd = \square ed$ .  $\square AC - \square Cb = \square ab$ , &c.

Here because  $\square AC$ .  $\square AC$ .  $\square AC$ , &c. are a Series of Equals, and  $Cb$  the Number of all the Terms; therefore  $\square AC \times Cb =$  the Sum of all the Series, by Lemma 1.

And  $\square Cf$ .  $\square Cd$ .  $\square Cb$ . &c. being a Series of Squares, whose Roots are in *Arithmetical Progression*. beginning at the Center or Point C, viz. 0, Cf, Cd, Cb, wherein the greatest Term is  $\square Cb$  and, Number of Terms is  $Cb$ . Ergo  $\square Cb \times \frac{1}{3} Cb =$  the Sum of all the Series, by Lemma 3.

Consequently, the  $\odot AC \times Cb : - \odot Cb \times \frac{1}{3} Cb =$  the Sum of all the Series  $\odot yf$ .  $\odot ed$ .  $\odot ab$ , which do constitute the Solidity of the half Zone  $am AG$ .

And because  $\square AC - \square Cb = \square ab$ . Ergo  $\odot AC - \odot ab = \odot Cb$ .

Consequently,  $\odot AC \times Cb : - \frac{\odot AC + \odot ab : \times CB}{3} = 2 \odot AC + \odot ab : \times \frac{1}{3} Cb$  will be the Solidity of the half Zone.

Put  $D = AG = 2 AC$ .  $x = am$ . and  $H = b$   $B = 2 Cb$ .

Then  $\odot AC = 0.7854 DD$ .  $\odot ab = 0.7854 xx$ . And if we turn the common Factor 0.7854 into the Divisor 1.27323, and then take

Triple of that Divisor, viz. 3,8197 (as before in the Frustums of Pyramids) the Result of the preceding Work will produce this following Theorem.

**Theorem IX.**  $\left\{ \frac{2DD + xx}{3,8197} : xH = \right\} \begin{matrix} \text{the middle Zone} \\ \text{amNK.} \end{matrix}$

## T H E O R E M XX.

*Spheres are in Proportion one to another, as the Cubes of their Diameters. (18. e. 12.)*

### Demonstration.

Suppose  $D =$  the Diameter or Axis of any Sphere, and  $d =$  the Diameter of another Sphere, either greater or lesser.

Then is  $0,5236 DDD =$  the Solidity of one Sphere, and  $0,5236 ddd =$  the Solidity of the other Sphere, by Theorem 18.

But  $DDD : ddd :: 0,5236 DDD : 0,5236 ddd$ .

Q. E. D.

## T H E O R E M XXI.

*The solid Content of every Spheroid is equal to two Thirds of its circumscribing Cylinder.*

### Demonstration.

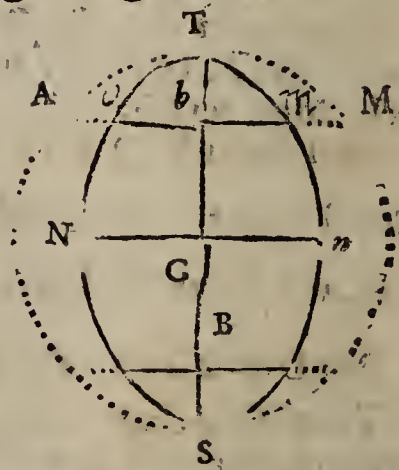
Suppose the Figure  $NTnSN$  in the annex'd Scheme. to represent a Spheroid. form'd by the Rotation of the Semi-Ellipsis  $TNS$  about its transverse Axis  $TS$ , (as by Definition 15.)

Let  $D = TS$  the Length of the Spheroid, and the Axis of its circumscribing Sphere. And  $d = Nn$  the Diameter of the greatest Circle of the Spheroid.

Then because  $\square TC : \square NC :: \square Ab : \square ab$ , by Step 3. in Theor. 7. Therefore it will be  $DD : dd :: \square Ab : \square ab :: \odot Ab : \odot ab$  &c.

But the Sum of an infinite Series of such Circles as  $\odot Ab$ , (whose Diameters are Chords) do constitute the Solidity of the Sphere, (as before at Theorem 18.) And the Sum of an infinite Series of such Circles, as  $\odot ab$ , (viz. whose Diameters are Ordinates of the Ellipsis) do constitute the Solidity of the Spheroid, by Definition 15.

Ergo  $DD : dd :: 0,5236 DDD : 0,5236 ddD$   
 $=$  the Solidity of the Spheroid, by Lem. 6.



But



But  $0,5236 ddD = \frac{2}{3}$  of the Cylinder, whose Diameter is  $=d$ , and Height  $=D$ , by Theorem II. Q. E. D.

Now from this Proportion between the Sphere and its inscrib'd Spheroid, it will be very easy to deduce Theorems for finding the solid Content, either of the Segment or middle Zone of any Spheroid, having the same Height with that of the Sphere.

For  $\left\{ \begin{array}{l} \text{As the Solidity of the whole Sphere: is to the Solidity of the whole Spheroid: so is any Part of the Sphere: to the like Part of the Spheroid, by Conv. to Lem. 6.} \end{array} \right.$

As for Instance, Suppose it were requir'd to find the middle Zone of any Spheroid.

Let  $D = TS$ , and  $d = Nn$ , as above; and  $H = bB$ .  $x = AM$ , as in Theorem 19. And let  $c = am$ .

Then  $\left\{ \frac{2DD + xx}{3,8197} \times H = \text{the middle Zone of the Sphere. And} \right.$

$0,5236 DDD : 0,5236 ddD :: \frac{2DD + xx}{3,8197} \times H : \frac{2dd \times H}{3,8197} + \frac{xx dd \times H}{3,8197 DD} =$   
the middle Zone of the Spheroid.

Again,  $DD : dd :: xx : cc$ . Therefore  $\frac{xxdd}{DD} = cc$ .

Consequently,  $\frac{xxdd}{DD} \times \frac{H}{3,8197} = \frac{cc}{3,8197} \times H$ . Which being taken instead of  $\frac{xxdd \times H}{3,8197 DD}$  there will arise this following

**Theorem XXII.**  $\left\{ \frac{2dd + cc}{3,8197} : \times H = \right.$  the middle Zone of the Spheroid, being the very same with Theorem 19.

*Note* In the same manner you may raise Theorems for finding the Segment of a Spheroid, cut off at either of its Ends, &c.

## THEOREM XXIII.

*The Area of every Parabola is equal to two Thirds of its circumscribing Parallelogram.*

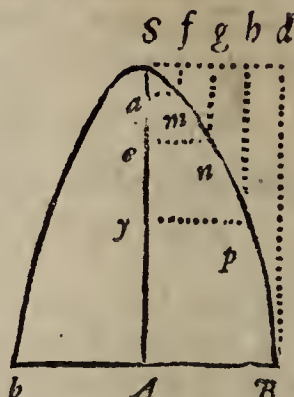
### Demonstration.

Let the Figure  $SAB$  represent half a Parabola; make  $DB$  parallel to the Axis  $SA$ , and  $Sl$  parallel to the Semi-Ordinate  $AB$ . And suppose  $Sl$  to be divided into an infinite Series of equidistant  
Points,

Points, as  $f, g, h$  &c. and from those Points imagine a Series of parallel Lines, *viz.*  $fm, gn, hp, \&c.$  to touch the Curve of the Parabola, and meet the Semi-Ordinates  $ma, ne, yp, \&c.$

Then according to the Property of the Parabola it will

$$\begin{array}{lcl}
 \text{be } \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. & \left| \begin{array}{l} SA : \square AB :: Sa : \square am \\ SA : \square AB :: Se : \square en \\ SA : \square AB :: Sy : \square yp, \&c. \end{array} \right. \\
 \text{But} & & Sa = fm . Se = gn . Sy = hp . SA = dB \\
 & & \text{Therefore alternately it will be} \\
 3, & 4 & \square AB : dB :: \square yp : hp \\
 2, & 5 & \square AB : dB :: \square en : gn \\
 1, & 6 & \square AB : dB :: \square am : fm
 \end{array}$$



In these Proportions  $\square am, \square en, \square yp, \&c.$  are a Series of Squares, whole Roots,  $Sf, Sg, Sh, \&c.$  are in *Arithmetical Progression*, beginning at the Point  $S$ . And because the Lines  $hp, gn, fm, \&c.$  have the same *Ratio* therefore they are as such a Series of Squares, wherein  $dB$  is the greatest Term, and  $Sl$  the Number of Terms.

Consequently,  $\frac{dB \times Sb}{3} =$  the Sum of all those Lines, by *Lemma 3*:

But  $SA \times AB = dB \times Sl$ . Therefore  $\frac{SA \times AB}{3} =$  the Sum of all that Series of Lines; but all those Lines do constitute the Area of the Semi-Parabola's Complement, *viz.* the Area of what half the Parabola  $SAB$  wants of completing or filling up the Parallelogram  $Sl AB$

Wherefore  $SA \times AB : -\frac{1}{3} SA \times AB = \frac{2 SA \times AB}{3}$  will be the Area of half the Parabola  $SAB$ .

Consequently,  $\frac{1}{3} SA \times bB$ , will be the Area of the whole Parabola  $bSB$ .

*Example.* Suppose the Base or greatest Ordinate of a Parabola to be  $bB = 24$  and its intercepted Diameter or Axis be  $SA = 33$ . Then  $2 SA \times bB = 66 \times 24 = 1584$ . and  $\frac{1}{3} 1584 = 528$  the Area of that Parabola.

## THEOREM XXIV.

*Every Parabolic Conoid is equal to one half of its circumscribing Cylinder.*

*Demon-*



Demonstration.

If any Semi-parabola (as  $BSA$ ) be turn'd or mov'd about its Axis, ( $SA$ ) it will form a solid Parabolic Conoid, constituted of an infinite Series of Circles, viz.  $\odot ba$ ,  $\odot fe$ ,  $\odot gy$ , &c. by Defin. 17.

Now, according to the Property of every Parabola, it will be,

$$SA:AB::AB:\frac{\square AB}{SA}=L, \text{ the Latus Rectum.}$$

$$\text{Then } \begin{cases} S_a \times L = \square ba \\ S_e \times L = \square fe \\ S_y \times L = \square gy, \text{ \&c.} \end{cases}$$

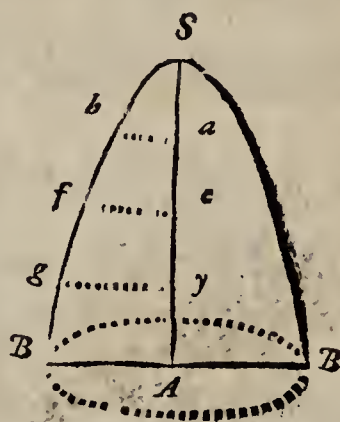
Here  $S_a \times L$ ,  $S_e \times L$ ,  $S_y \times L$ , &c.

are a Series of Terms in *Arithmetical Progression*. Therefore  $\square ba$ ,  $\square fe$ ,  $\square gy$ , &c. are alio a Series of Terms in the same Progression, beginning at the Point  $S$ , wherein  $\square AB$  is the greatest Term, and  $SA$  the Number of all the Terms. Therefore  $\square AB \times \frac{1}{2} SA =$  the Sum of all the Series, by *Lemma 2*.

Consequently,  $\odot AB \times \frac{1}{2} SA =$  the Sum of all the Series of  $\odot ba$ ,  $\odot fe$ ,  $\odot gy$ , &c. which do constitute the Solidity of the Conoid.

And putting  $D = 2AB$  and  $H = SA$ .

Then,  $0.7854 DD \times \frac{1}{2} H = 0.3927 DDH$  will be the solid Content of the Conoid; which is just half the Cylinder, whose Base  $= D$ , and Height  $= H$ . See *Theorem II*. Q. E. D.



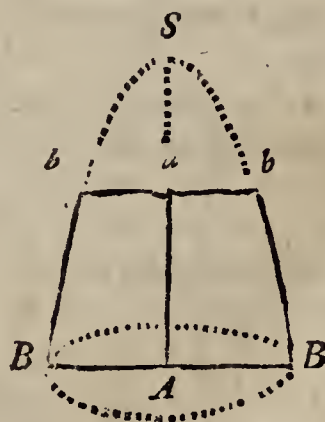
This being understood, it will be easy to raise a Theorem for finding the Lower Frustum of any Parabolic Conoid.

For supposing  $b = aA$  the Height of the Frustum, and  $p = sa$  the Height of the Part  $bsb$  cut off. Then  $b + p = SA$  the Height of the whole Conoid.

Consequently,  $\frac{\odot AB \times b + \odot AB \times p}{2} =$  the Solidity of the whole Conoid.

And  $\frac{\odot ba \times p}{2} =$  the Solidity of the Part cut off

$$\begin{array}{l|l|l} \text{Ergo} & 1 & \frac{\odot AB \times b + \odot AB \times p - \odot ba \times p}{2} = \\ & & \text{the Solidity of the Frustum.} \\ \text{But} & 2 & b + p : \square AB :: p : \square ba \\ \text{Conseq.} & 3 & b + p : \odot AB :: p : \odot ba \\ 3 & \therefore 4 & \odot AB \times p = \odot ba \times b + \odot ba \times p \end{array}$$



$$\begin{array}{r|l}
 4 - \odot ba \times p & 5 \quad \odot AB \times p : - \odot ba \times p = \odot ba \times b \\
 1 \times 2 & 6 \quad \odot AB \times b : + \odot AB \times p : - \odot ba \times p = 2F \\
 6 - 5 & 7 \quad \odot AB \times b = 2F : - \odot ba \times b \\
 7 + \odot ba \times b & 8 \quad \odot AB + b : + \odot ba \times b = 2F \\
 8 \div 2 & 9 \quad \frac{\odot AB + \odot ba}{2} : \times b = E \text{ the Frustum's Solidity}
 \end{array}$$

Let  $D = AB$  as before, and  $d = 2ba$  the Diameter of the Part cut off. Then we shall have this following

**Theorem XXV.**  $\{ 0,3927 DD + 0,3927 dd : \times b = \text{the Solidity of the Frustum requir'd.}$

Or  $\left\{ \frac{DD + dd}{2,5464} \times b \text{ the Frustum. For } ,3927 \right\} 1,0000 (= 2,5464$

And because  $2,5464 + \frac{2,5464}{2} = 3,8196$ . Therefore it may be made  $3,8196) DD + dd : \times \frac{1}{2} b (= \text{the same Frustum, \&c.}$

**Note.** The Reason why I have reduced this Theorem to have the same Divisor with those of the Frustums of Pyramids, &c. will best appear farther on, viz. when they all come to be apply'd to Practice in Gaging.

## THEOREM XXVI.

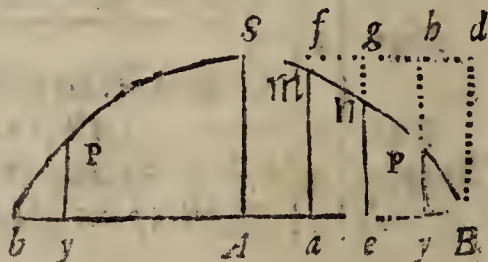
*Every Parabolic Spindle (or Pyramidoid) is equal to eight Fifteenths of its circumscribing Cylinder.*

### Demonstration.

If any Acute Parabola, as  $bSB$ . be turn'd or mov'd about its greatest Ordinate  $bAB$  it will form a Solid, call'd a Parabolic Spindle, constituted of an infinite Series of  $\odot ma$ ,  $\odot ne$ ,  $\odot py$ , &c. by Definition 18.

Let us suppose the Line  $Sd$  parallel to  $AB$ , &c. As at Theorem 23. Then it hath already been prov'd, that the Lines  $fm$ ,  $gn$ ,  $hp$ , &c. are a Series of Squares, whose Roots are in *Arithmetical Progression*. Consequently, their Squares, viz.  $\square fm$ ,  $\square gn$ ,  $\square hp$ , &c. will be a Series of Bi-quadrates, whose Roots will be in *Arithmetical Progression*; which being premised, we may proceed thus.

$$\text{First } \left\{ \begin{array}{l} 1 \quad SA - fm = ma \\ 2 \quad SA - gn = ne \\ 3 \quad SA - hp = py, \text{ \&c.} \end{array} \right.$$





$$\begin{array}{l} 1 \text{ } \odot \text{ } 2 \text{ } \left| \begin{array}{l} 4 \\ 5 \\ 6 \end{array} \right| \begin{array}{l} \square SA - 2SA \times fm + \square fm = \square ma \\ \square SA - 2SA \times gn + \square gn = \square ne \\ \square SA - 2SA \times hp + \square hp = \square py, \text{ \&c.} \end{array} \end{array}$$

1. In these Equations, the  $\square SA$ ,  $\square SA$ ,  $\square SA$  being a Series of Equals, and  $AB$  the Number of all the Terms; therefore it will be  $\square SA \times AB =$  the Sum of all the Series, by Lemma 1.

2. Because  $fm$ ,  $gn$ ,  $hp$ , &c. are as a Series of Squares, wherein  $SA$  is the greatest Term, and  $AB$  the Number of all the Terms. Therefore  $\frac{2SA \times SA \times AB}{3} = \frac{2\square SA \times AB}{3}$  will be the Sum of all that Series, by Lemma 3.

3. And the  $\square fm$ ,  $\square gn$ ,  $\square hp$ , &c. will be a Series of Terms in the Ratio of Biquadrates, as above,  $\square dB = \square SA$  being the greatest Term, and  $AB$  the Number of all the Terms; therefore it will be  $\frac{\square SA \times AB}{5} =$  the Sum of all the Series, by Lemma 5.

Wherefore it follows, that  $\square SA \times AB - \frac{2\square SA \times AB}{3} + \frac{\square SA \times AB}{5} =$  the Sum of all the Series of  $\square ma$ ,  $\square ne$ ,  $\square py$ , &c.

That is,  $\frac{8\square SA \times AB}{15} =$  the Sum of all the Series of  $\square ma$ ,  $\square ne$ ,  $\square hp$ ,  $\square dB$ , &c.

Consequently,  $\frac{8\odot SA \times AB}{15} =$  the Sum of all the Series of  $\odot ma$ ,  $\odot ne$ ,  $\odot py$ , &c, which do constitute the Solidity of half the Spindle, viz. of  $SAB$ .

Therefore, putting  $D = 2SA$ , and  $H = 2AB$  (viz.  $bAB$ ) it will be  $0,41888 DDH =$  the Solidity of the whole Parabolic Spindle  $bSB$ , being  $\frac{8}{15}$  of  $0,7854 DDH$  the Solidity of its circumscribing Cylinder. Q. E. D.

From hence we may also raise a Theorem for finding the Frustum  $SAPy$  of the last Figure.

For  $\odot SA$  being the greatest Term,  $\odot py$  the least Term, and  $Ay$  the Number of all the Terms and Circles included between  $A$  and  $y$ .

$$\begin{array}{l} \text{Therefore } \left| \begin{array}{l} 1 \\ 2 \end{array} \right| \begin{array}{l} \int \square SA - \frac{2SA \times hp}{3} + \frac{\square hp}{5} : \times Ay = z \text{ the Sum of all} \\ \text{the Series } \square SA, \square ma, \square gn, \square py. \\ 1 \times 3 \left| \begin{array}{l} 2 \\ 3 \end{array} \right| 3 \square SA - 2SA \times hp + \frac{3\square hp}{5} : \times Ay = 3z \end{array} \end{array}$$

$$2 \div A y \quad 3 \quad 3 \square SA - 2 SA \times hp + \frac{3 \square hp}{5} = \frac{3z}{A y}$$

But  $4 \quad \square SA - 2 SA \times hp = \square py - \square hp.$  By 6th Step.

$$3 - 4 \quad 5 \quad 2 \square SA + \frac{3 \square hp}{5} = \frac{3z}{A y} - \square py + \square hp$$

$$5 \pm \&c. \quad 6 \quad 2 \square SA + \square py - \frac{2}{5} \square hp = \frac{3z}{A y}$$

Conseq.  $7 \quad 2 \odot SA + \odot py - \frac{2}{5} \odot hp : \times \frac{1}{3} A y = z.$  the Sum of all the Series of  $\odot SA . \odot ma . \odot ne . \odot py$ , which do constitute the Solidity of the Frustum  $SApy$ . Therefore putting  $D = 2 SA$  as before,  $C = 2 py$ ,  $x = 2 hp$ , and  $H = Ay$ , it will be  $1,5708 DD + 0,7854 CC - 0,31416 xx : \times \frac{1}{3} H =$  the Frustum  $SApy$ . And if we make  $L = 2H$ ,

Then  $1,5708 DD + 0,7854 CC - 0,31416 xx : \times \frac{1}{3} L =$  the Double of that Frustum, being the Middle Zone. And by turning these Factors into one common Divisor, as in the Frustum of the Conoid at *Theorem 25. Page 430.* there will arise this following *Theorem.*

**Theorem XXVII,**  $\{ 3,8196 \} 2DD + CC - 0,4 xx : \times L (=$  the Middle Zone of a Parabolic Spindle.

It may be expected, that I should now proceed to shew how the Area of any Hyperbola, and the Contents of such Solids as may be form'd by the Rotation of that Figure about its Axis, &c. may be found. But because those Things cannot be exactly perform'd by any certain or settled *Theorems.* as these of the Circle, Ellipsis and Parabola have been, I have therefore omitted them, and refer the Reader to *Dr. Wallis's Algebra. Chap. 90, &c.* or to the *Philosoph. Transact.* Number 34. wherein he may find the Method of forming infinite Series relating to the Squaring of any Hyperbola, &c. which are too tedious to be fully explain'd and demonstrated in this small Tract, it being only intended as an Introduction, which I shall here conclude.



A N  
A P P E N D I X  
O F

# Practical Gaging.

**T**HE Art of Gaging is that Branch of the Mathematicks call'd *Stereometry*, or the *Measuring of Solids*, because the Capacities or Contents of all Sorts of Vessels used for Liquors, &c. are computed as tho' they were really Solid Bodies, which any one that hath made himself Master of the foregoing Parts of this Treatise, may easily understand, without any further Directions.

However, because it is not to be suppos'd, that every one who designs to undertake the Office or Employment of a Gager, hath made so great a Progress in Mathematical Learning, I have therefore presented the young Gager with this Appendix, wherein I have only inserted such Rules as are useful in Gaging, and have been already demonstrated in this Treatise. But herein, I presuppose that he hath acquir'd, (or if not, it is very requisite he should acquire) a competent Knowledge both in Arithmetick and Geometry. That is.

I. In Arithmetick he should understand the principal Rules very well, especially Multiplication and Division, both in whole Numbers and Decimal Parts; (which may be easily learn'd out of the 2d, 3d, and 5th Chapters of *Part I.*) that so he may be ready in computing the Contents of any Vessel, and casting up his Gages by the Pen only, *viz.* without the Help of those Lines of Numbers upon sliding Rules, so much applauded, and but too much practis'd, which at best do but help to guess at the Truth. I mean such Pocket Rules as are but nine Inches, or a Foot long, whose Radius of the double Line of Numbers is not six Inches; and therefore the Graduations or Divisions of those Lines are so very close, that they cannot be well distinguish'd. 'Tis true, when the Rules are made two or three Foot long. (I had one of six Foot) then they may be of some Use, especially in small Numbers; altho' even then, the Operations may be much better. (and almost as soon) done by the Pen; for indeed the chief Use of sliding Rules, is only in taking of Dimensions, and for that Purpose they are very convenient.

K k k

II. In



II. In Geometry the Gager should understand not only how to take Dimensions, (which is best learn'd by Practice) but also how to divide any irregular Figure or Superficies, as Brewers Backs, or Coolers, &c. into the easiest and fewest Regular Figures they will admit of, that so their Area's may be truly computed with the least Trouble. And this may be learn'd (with a little Care and Diligence, out of the 1st, 2d, and 5th Chapters of *Part III.* which the Gager should be well acquainted with.

Also he ought to have so much Skill in Solids, as to be able, even at Sight (but this must be acquir'd by Experience) to determine what Sort of Figure any Vessel is of, (*viz.* any Tun, or close Cask) or what Figure it may be best reduc'd to, so that its Dimensions may be truly taken, and the Content thereof computed with the least Error. I say, with the least Error, because it is very difficult, if not impossible, to do it exactly; for there is not any Tun or Cask, &c. so regularly made, as by the Rules of Art it is requir'd to be.

III. Besides the aforementioned, the young Gager must know, that all Dimensions useful in Gaging, are to be taken in Inches, and Decimal Parts of an Inch; and if they are taken in any other Measures, as Feet, or Yards, &c. those Measures must be reduc'd to Inches (see *Sect. 4. Page 42.*) because the Contents of all Sorts of Vessels taken Notice of in Gaging) are computed by the Standard Gallon of its kind, whose Content is known to be a certain Number of Cubic Inches. That is, the Beer or Ale Gallon contains 282, the Wine Gallon 231, and the Corn Gallon, 268,8 Cubic Inches. (See the five Tables, &c. in *Page 34, 35, 36,* which I here suppose the Gager to have learn'd perfectly by Heart.) Consequently, if either the Superficial or Solid Content of any Vessel, as Back, Tun, Cask, &c. be once computed in Cubic Inches, it will be easy to know how many Gallons, either of Ale, Wine, or Corn, that Vessel will hold.

*Note,* I have here said, the Superficial Content in Cubic Inches, which may seem to be very improper, according to the Definition given of a Superficies in *Page 279.* But you must know, that in the Business of Gaging all Superficies, or Area's, are always understood to be one Inch deep; otherwise it could not be said (as in the Gager's Language it is) that the Area of such a Back, or of such a Circle, &c. is so many Gallons.

These Things being very well understood, the young Gager will be fitly prepar'd to understand the following Problems, which are such as have (most of them) been already propos'd in the foregoing Parts of this Treatise, and only are here apply'd to Practice; and therefore I shall, for Brevity's sake, often refer to those Theorems and Problems.



**Sect. I.** *To find the Area of any Right-lin'd Superficies in Gallons.*

## P R O B L E M I.

*To find the Area of any Square Tun, Back, or Cooler, &c. either in Ale, Wine, or Corn Gallons.*

**Rule.** { *Multiply the given Length or Breadth ( being here equal ) into itself, and the Product will be the Area in Inches; then divide that Area by 282, or 231, or 268,8 and the Quotient will be the Area requir'd.*

*Example,* Suppose the Side of a Square Tun, Back, or Cooler, be 124,5 Inches, what will its Area be in Gallons?

First  $124,5 \times 124,5 = 15500,25$  the Area in Inches,

Then 282 ) 15500,25 ( 54,96, &c. the Area in Ale Gallons.

And 231 ) 15500,25 ( 67,10, &c. the Area in Wine Gallons.

Or 268,8 ) 15500,25 ( 57,66, &c. the Area in Corn Gallons.

But if any one would rather work by Multiplication than by Division, he may turn or change any Divisor into a Multiplier, if he divide Unity, or 1, by that Divisor, ( vide Problem III. Page 402.)

Thus 282 ) 1,000000 ( 0,003546 the Multiplier for Ale Gallons.

And 231 ) 1,000000 ( 0,004329 the Multiplier for W. Gallons.

Or 268,8 ) 1,000000 ( 0,003722 the Multiplier for Corn Gallons.

Consequently,  $15500,25 \times 0,003546 = 54,96$ , &c. the Area in Ale Gallons. As before; and so on for the rest.

## P R O B L E M II.

*To find the Area of any Tun, Back, or Cooler, in the Form of a Right-angled Parallelogram, in Ale Gallons, &c.*

See the Rule for finding its Area in Inches, as Prob. I. Page 339. Then either divide ( or multiply ) that Area, as above, and you will have the Area in Gallons.

*Example,* Suppose the Length of a Brewer's Tun, Back, or Cooler, be 217,5 Inches, and its Breadth 85,6 Inches, what will its Area be in Ale or Beer Gallons, &c.

First  $217,5 \times 85,6 = 18648$ . Then 282 ) 18648 ( 66,12, &c.

Or  $18648 \times 0,003546 = 66,12$ , &c. the Area requir'd, &c.



## P R O B L E M IV.

*To find the Area of any Triangular Tun, Back, or Cooler, in Ale Gallons, &c.*

See the Rule for finding its Area in Inches, at *Prob. 3. Page 340.* then divide (or multiply) that Area as before, and you will have the Area requir'd.

*Example,* If the Length of the Base of a Triangular Cooler be 86,4 Inches, and its Perpendicular Breadth be 57 Inches, what will its Area be in Ale Gallons?

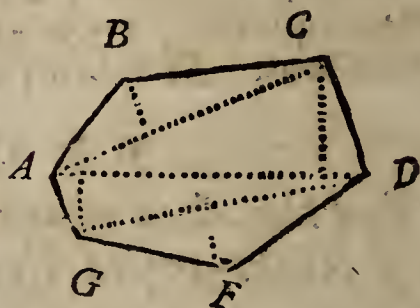
First,  $86,4 \times \frac{57}{2} = 2462,4$ . Then  $282 \div 2462,4 = 8,73$ , &c.

Or  $2462,4 \times 0,003546 = 8,73$ , &c. the Area in Ale Gallons.

Proceeding thus, you may easily find the Area of any Tun, Back, or Cooler, whether it be in the Form of a Rhombus, Rhomboides, Trapezium, or of any other Polygon, either Regular, or Irregular, in Ale or Beer Gallons, &c. if you first divide it into Triangles, and then find the Area's of those Triangles, (as in the 2d, 4th, 5th, and 6th Problems in Chapter 5. Part 3.) the Sum of those Area's being divided (or multiply'd) by its proper Divisor, (or Multiplier) as above, will give the Area requir'd.

Now the practical Way of dividing any Polygonous Tun, Back, &c. into Triangles, is by help of a chalk'd Line, such as the Carpenters use, and may be thus perform'd.

Suppose any Brewer's Tun, Back, or Cooler, in the Form of the annex'd Figure *ABCDEF*. Let one End of the chalk'd Line be fasten'd with a Nail (or otherwise) in any Corner or Angle of the Back, as at *A*, then straining it to the Angle at *C*, strike the Diagonal Line *AC*, upon the Bottom of the Back, and straining it again to the Angle *D*, strike another Diagonal Line, as *AD*, and so on for the Diagonal Line *GD*, &c. Then having mark'd out all the Diagonals, the Perpendiculars may be thus found:



Fasten (as before) one End of the chalk'd Line in the Angle *B*, and then by moving it to and fro upon the Stretch, find out the nearest Distance between the Angle at *B*. and the Diagonal Line *AC*; there strike a Line, and it will mark out the Perpendicular from *B* to the Line *AC*, and so on for the other Perpendiculars, which being all mark'd out upon the Bottom of the Back, measure them, and each Diagonal by a Line of Inches, &c. And then the Area of that Back may be computed, as directed above.

And



And here, by the way, it may be observ'd, That the Number of Triangles will always be less by two; and the Number of the Diagonals less by three, than the Number of the Sides of any Right-lin'd Figure that is so divided.

Having found (as above) the true Area of any Brewer's Back or Cooler, (which, according to the Laws of Excise, ought always to be fix'd or immovable) the next Thing will be to find out the true Dipping or Gaging Place in that Back, that so the true Quantity of Worts may be computed or cast up at any Depth; which may be thus done.

1. When the Bottom of the Back is cover'd all over (of any Depth) either with Wort or Liquor, (*viz.* Water) then dip it in eight or ten several Places (more or less according to the Largeness of the Back) as remote and equally distant one from another as you well can, noting down the wet Inches and Decimal Parts of every Dip.

2. Divide the Sum of all those Dips or wet Inches by the Number of Places you dipp'd in, and the Quotient will be the Mean Wet of all those Dips.

3. Lastly, find out such a Place by the Side of the Back (if you can) that just wets the same with that mean Dip, and make a Notch or Mark there for the true and constant dipping Place of that Back. Then if any Quantity of Worts (which do cover the whole Back) be dipp'd or gaged at that Place, and the wet Inches so taken be multiply'd into the Area of the Back in Gallons, the Product will show what Quantity (*viz.* how many Gallons) of Wort are in that Back at that Time, provided the Sides of the Back do stand at right Angles with its Bottom,

## Sect. 2. To find the Area of any Circular and Elliptical Superficies in Gallons.

1. I have demonstrated in Chapter 6. Part 3. and Theorem 3, 5, 6. Part 5. that the Periphery of the Circle, whose Diameter is Unity or 1. is 3,14159265, &c. (or for common Use 3,1416) And that its Area is 0,78539816, &c. (or 0,7854 *fere*)

2. Also, that the Peripheries of all Circles are in Proportion one to another, as the Diameters are; and their Area's are in Proportion to the Squares of their Diameters. That is,

As 1 : 3,1416 :: the Diameter of any Circle: To its Periphery.  
And 1 : 0,7854 :: the Square of the Diameter: To the Area.

Upon



Upon these two Proportions depends the Solution of all the common or practical Questions about a Circle. See Page 408 and 409.

### P R O B L E M IV.

*The Diameter of any Circle being given in Inches, to find the Periphery.*

**Rule.**  $\left\{ \begin{array}{l} \text{Multiply the given Diameter with 3,1416, and} \\ \text{the Product will be the Periphery requir'd. See} \\ \text{Prob. 1. Page 408.} \end{array} \right.$

*Example,* Suppose the Diameter of a Circle be 54,5 Inches, and it were requir'd to find its Periphery.

Then  $54,5 \times 3,1416 = 171,21$ , &c. Inches, is the Periphery requir'd.

The Converse of this is easy, viz. by having the Periphery given, to find the Diameter. See Prob. 3. Page 408.

### P R O B L E M V.

*The Diameter of any Circle being given, (in Inches) to find its Area in Gallons.*

**Rule.**  $\left\{ \begin{array}{l} \text{Multiply the Square of the propos'd Diameter into} \\ \text{0,7854, and the Product will be the Area in} \\ \text{Inches (see Prob. 2. Page 408.) that Area being} \\ \text{divided by 282 or 231, &c. the Quotient will be} \\ \text{the Area requir'd.} \end{array} \right.$

*Example,* Suppose the given Diameter be 54,5 Inches, as above. First  $54,5 \times 54,5 = 2970,25$ . And  $2970,25 \times 0,7854 = 2332,83$  the Area in Inches.

Then 282 ) 2332,83 ( 8,2724 the Area in Ale or Beer Gallons.

And 231 ) 2332,83 ( 10,0988 the Area in Wine Gallons.

Or 268,8 ) 2332,83 ( 8,6788 the Area in Corn Gallons.

But these Area's in Gallons may be much easier found, without knowing the Circle's Area in Inches as above, by having the Square of the Diameter of that Circle whose Area is one Gallon; which may be thus found, by Theorem 6. Page 407.  $0,785398 : 1 :: 282 : 359,05$  the Square of the Diameter of the Circle whose Area is 282 Cubic Inches, viz. one Ale Gallon.

And from this Proportion will arise these following Divisors; viz. 0,785398 ) 282,000000 (359,05 will be a Divisor for A. G. And 0,785398 ) 231,000000 (294,12 will be a Divisor for W. G. Or 0,785398 ) 268,800000 (342,24 will be a Divisor for C. G.

IF



If the Square of the Diameter of any Circle be divided by any one of these constant or fix'd Divisors, the Quotient will shew that Circle's Area in their respective Gallons. As for Instance in the last Circle, whose Square of its Diameter is 2970,25.

Then	359,05)	2970,25	( 8,2725	the Area in <i>A. G.</i>	}	As before.
And	294,12)	2970,25	(10,0988	the Area in <i>W. G.</i>		
Or	342,24)	2970,25	( 8,6788	the Area in <i>C. G.</i>		

Now these Divisors may be turn'd into Multipliers, by dividing Unity or 1. as in Page 435. Or rather by dividing the Area in Inches of that Circle whose Diameter is 1.

That is, 0,785398 by 282. Or by 231. &c.

Thus	282)	0,785398	(0,002785	the Multiplier for Ale Gal.
And	231)	0,785398	(0,003399	the Multiplier for Wine Gal.
Or	268,8)	0,785398	(0,002922	the Multiplier for Corn Gal.

These Multipliers are the respective Area's of a Circle whose Diameter is 1. And therefore, if the Square of the Diameter of any Circle be multiply'd with any of these Numbers, the Product will be that Circle's Area in Gallons of the same Name.

*Viz.*  $2970,25 \times 0,002785 = 8,2725$  the Area in Ale Gal. as above.  
 And  $2970,25 \times 0,003399 = 10,0988$  the Area in Wine Gallons, &c.

Thus you see, that if the Diameter of any Circle be given in Inches, there are three several Ways of finding its Area in Gallons, and all equally true; but that which is perform'd by the constant Divisors is most generally practis'd.

## P R O B L E M VI.

*The Transverse (or longest Diameter) and the Conjugate (or shortest Diameter) of any Elliptical Superficies being given, to find its Area in Gallons.*

**Rule.** { *Multiply the two Diameters, (viz. the Length and Breadth) together, and divide their Product by 359,05 for Ale Gallons, or 294,12 for Wine Gallons, &c. the Quotient will be the Area requir'd. See Theorem 7. Page 412.*

*Example.* Suppose the longest Diameter to be 73,5 Inches, and the shortest Diameter to be 51,6 Inches, what will the Area be in Ale Gallons?

First  $73,5 \times 51,6 = 3792,6$ . Then 359,05 ) 3792,6 ( 10,56 the Area in Ale Gallons. Or 294,12 ) 3792,6 ( 12,89 the Area in Wine Gallons, &c.

No,



*Note,* The two last Problems are of great Use in Gaging of Wort amongst Country Victuallers, who generally brew but short Lengths of Ale, (*viz.* perhaps between 20 and 60 Gallons at a Brewing) and cool their Worts in several small open Vessels or Tubs, whose Bases or Bottoms are either a Circle, or an Ellipsis, having their Sides but low, and are most commonly wider at the Top than at the Bottom.

Now a practical Way of computing the Quantity of Wort that is at any Time in one of those open Tubs, is briefly thus: When the Tub is dry, find the true Area of its Bottom, according to its Figure. (as above) and either mark that Area on the outside of the Tub, which was the Way I generally us'd to order, because the Victuallers did often lend their cooling Tubs one to another) or else number the Tub, and enter its Area (and its Number) into the Stock-Book; then when any of those Tubs hath Wort in it, take the Diameter of the Surface or Top of the Wort, and find that Area, adding it and the Bottom Area together. If either the half Sum of those two Area's be multiply'd with the Depth of the Wort, (taken as near the Middle of the Tub as you well can) or, if the Sum of those two Area's be multiply'd with half the Depth, (so taken) the Product will shew the Quantity of the Wort very near the Truth.

## P R O B L E M VII.

*The Diameter of any Circle, and the versed Sine (*viz.* the Height) of any Segment being given, to find the Area of that Segment in Gallons.*

In the 410th and 412th Pages, you have two Ways (and their Examples) of finding the Area of any Segment of a Circle in Inches; then if that Area in Inches be divided by 282, or 231, &c. the Quotient will be its Area in Gallons. But because the Area of any such Segment may be readily found in Gallons (without finding its Area in Inches) by help of a Table of Segments, whose Construction is laid down in the Problem, Page 411, &c. I have here inserted a Compendium of such a Table, which will serve very well for common Practice, not only to find the Area of any Segment of a Circle in Gallons; but also to find the Number of Gallons that are either drawn out, or remaining in any Cylicindric Vessel lying along; or of any close Cask (being first reduc'd to a Cylinder) its Axis lying parallel to the Horizon, usually call'd the Ullage of a Cask, as shall be shew'd further on.



A Table of the Segments of a Circle, whose Area is Unity or 1. The Diameter being divided by parallel Chord-Lines into 100 equal Parts.

V. S.	Segment.	V. S.	Segment	V. S.	Segment.	V. S.	Segment.
1	0,0017	26	0,2066	51	0,5127	76	0,8155
2	0,0048	27	0,2178	52	0,5255	77	0,8262
3	0,0087	28	0,2292	53	0,5382	78	0,8369
4	0,0134	29	0,2407	54	0,5509	79	0,8474
5	0,0187	30	0,2523	55	0,5635	80	0,8576
6	0,0245	31	0,2640	56	0,5762	81	0,8677
7	0,0308	32	0,2759	57	0,5888	82	0,8776
8	0,0375	33	0,2878	58	0,6014	83	0,8873
9	0,0446	34	0,2998	59	0,6140	84	0,8968
10	0,0520	35	0,3119	60	0,6265	85	0,9059
11	0,0598	36	0,3241	61	0,6389	86	0,9149
12	0,0680	37	0,3364	62	0,6514	87	0,9236
13	0,0764	38	0,3486	63	0,6636	88	0,9320
14	0,0851	39	0,3611	64	0,6759	89	0,9402
15	0,0941	40	0,3735	65	0,6881	90	0,9480
16	0,1032	41	0,3860	66	0,7002	91	0,9554
17	0,1127	42	0,3986	67	0,7122	92	0,9625
18	0,1224	43	0,4112	68	0,7241	93	0,9692
19	0,1323	44	0,4238	69	0,7360	94	0,9755
20	0,1424	45	0,4365	70	0,7477	95	0,9813
21	0,1526	46	0,4491	71	0,7593	96	0,9866
22	0,1631	47	0,4618	72	0,7708	97	0,9913
23	0,1738	48	0,4745	73	0,7822	98	0,9952
24	0,1845	49	0,4873	74	0,7934	99	0,9983
25	0,1955	50	0,5000	75	0,8045	100	1,0000

The Use of this Table of Segments depends upon the following Proportion:

*Viz.* { As the Diameter of any proposed Circle: is to 100: (the Diameter of the Tabular Circle) :: so is the Height of any Segment of the propos'd Circle: to a Versed-Sine in the Table.

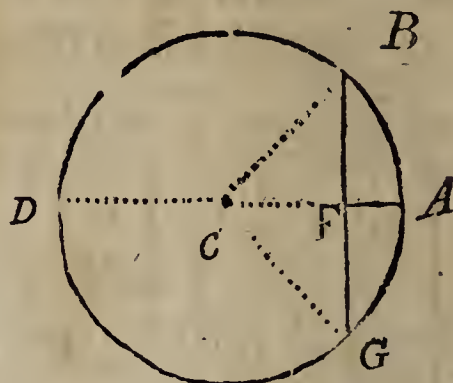
Then if the Tabular Segment, which stands against that Versed-Sine, be multiply'd into the Circle's Area, (either in Inches or Gallons) the Product will be the Area of the Segment requir'd, (of the same Name) *viz.* If the Circle's Area be Inches. the Segment will be Inches: If Gallons. the Segment will be Gallons.

*Example.* Let the Diameter of the given Circle be  $DA = 62,5$  Inches, and the Height of the Segment sought, be  $FA = 20$  Inches, what will its Area be in Ale-Gallons?

First, the Area of the whole Circle will be 10,8793 Ale-Gallons. By *Problem 5*. And the Proportion will stand thus,  $62,5 : 100 :: 20 : 32$  the Versed-Sine of the Table, whose Segment is 0,2759.

Then  $10,8793 \times 0,2759 = 3,0016$  Ale-Gallons, being the Area of the Segment  $BAGF$ , as was requir'd. The like may be done for Wine-Gallons, Corn-Gallons, or Inches.

And upon Occasion, the like Segments of any Ellipsis may be easily found. See the Proportions in the Corollaries to the 7th and 8th Theorems, *Page 412, &c.* to which I here for Brevity's sake refer the Reader.



*Sect. 3. To compute the Contents of such Vessels (viz. Tuns, &c.) as are in the Form of the following Solids.*

*Note,* Before the young Gager proceeds to these Computations, he should be well acquainted with such Solids as are defin'd in *Page 402 and 403*. And then he may easily understand what Sort of Figures are meant in the following *Problems*, without the Repetition of many Words.

### P R O B L E M VIII.

*To find the Content of any Prism, whose Sides are parallelograms; what Form soever its Base is of.*

That is, to compute the Content (in Gallons) of any Tun, &c. whose Sides are Parallelograms which stand upright, or at right Angles with its Bottom.

First, find its solid Content in Inches by *Theorem 9. Page 414*. Then divide that Content by 282. or 231. Or by 268,8 the Quotient will shew the Content in their respective Gallons, viz. in Ale, Wine, or Corn-Gallons.

Or else multiply the Content in Inches with 0,003546, (Or 0,004329, &c. (See the Multipliers, *Page 435*.) those Products will be the Content in their respective Gallons.

Or otherwise thus.

Find the true Area of the Tun's Base or Bottom, as directed in *Section 1, Page 435*. That Area being multiply'd with the Tun's Height (viz. Depth within) will produce the Content in Gallons, as before.

I take



I take the Work of this Problem to be so very easy, it needs no Example.

## PROBLEM IX.

*To find the Content of any Pyramid (in Gallons) whose Base is bounded with Right-lines.*

Every Pyramid is one third Part of its circumscribing Prism, by Theorem 10. Page 415. Therefore

If the Area of the Base of any Pyramid in Gallons, be multiply'd into one third of its perpendicular Height; or if one third of that Area be multiply'd with the whole Height, either of those Products will be the Content of the Pyramid in Gallons, &c.

But the Content of any Square Pyramid may be easily found in Gallons by this Rule.

**Rule.** { *Square the Side of its Base, and multiply that Square with the perpendicular Height; then divide that Product by 846 = 282 × 3 for Ale-Gallons, or by 693 = 231 × 3 for Wine-Gallons, or by 806,4 = 268,8 × 3 for Corn-Gallons, the Quotient will be the Content requir'd.*

Or if you multiply the said Product with 0.001182 for A. G.

Or with 0.001443 for W. G. Or lastly with 0.001241 for C. G. the Result will be the Content requir'd, As before.

## PROBLEM X.

*To find the Content (in Gallons) of the Frustum of any square Pyramid, cut off by a plane Parallel to its Base.*

First. Either by Theorem 15. Page 419. or Theorem 16. Page 420, find the propos'd Frustum's Solidity in Cubic Inches. Then divide that Content in Cubic Inches by 282. or 231, &c. and the Quotient will be the Content of the Frustum in their respective Gallons.

But from the aforesaid Theorem 15. there may be easily deduc'd the following general Rule for finding the Content of the like Frustum of any Pyramid what Form soever its Bases are of (supposing them to be parallel) whether they are alike or unlike.

**Rule.** { *First find the Area of each Base (viz. the Top and Bottom Area's of the propos'd Frustum) then find a geometrical Mean between those two Area's, (by Lemma 1. Page 83.) the Sum of those two Area's, and their Mean, being multiply'd into one Third of the Frustum's Height, will produce the Content requir'd.*

L I I 2

Example,



*Example.* Suppose a Tun in the Form of the lower Frustum of a Pyramid, whose Bases are equilateral Triangles. Let the Side of the Top be 42 Inches, the Side of the Bottom be 63,4 Inches, and its Height (*viz.* Depth ) be 33 Inches, what will the Content of that Tun be in Ale-Gallons?

First find the Area of each Base in Inches, by Problem 7. Page 343; then find what those Area's are in Ale-Gallons, by Problem 3. Page 436. Multiply those two Area's together; the Square Root of their Product will be the Mean Area, &c. as in this *Example*.

*Example.*  $\left\{ \begin{array}{l} \text{The Area of the Top is } 2,71 \\ \text{The Area of the Bottom is } 6,12 \\ \text{The Mean Area will be } 4,07 \end{array} \right\}$  Ale-Gallons.  
Their Sum is 12,90

Then  $12,9 \times \frac{33}{3} = 141,9$ . Or  $\frac{12,9}{3} \times 33 = 141,9$  the Content requir'd.

### P R O B L E M X I.

*To find the Content of any Right Cylinder in Gallons.*

That is, to compute the Content of any round Tun, &c. whose Diameters at Top, and Bottom are equal, and at Right-angles with its Sides.

The Content of such a Tun may be found by Theorem 11. Page 415. Or otherwise by the following Rule.

*Rule.*  $\left\{ \begin{array}{l} \text{Multiply the Square of the Diameter into the Height} \\ \text{and divide the Product by } 359,05 \text{ ( or multi-} \\ \text{ply with } 0,002785 \text{ ) \&c. as in Page 439. that} \\ \text{Quotient or Product will be the Content re-} \\ \text{quir'd.} \end{array} \right.$

*Exam.* Suppose the Diameter be 42,5 and the Height 31,5 Inches.

First  $42,5 \times 42,5 = 1806,25$  And  $1806,25 \times 31,5 = 56896,875$

Then  $359,05 \mid 56896,875$  (158,46 the Content in A. Gal. &c.

### P R O B L E M X II.

*To find the Content of any Cone or round Pyramid in Gallons.*

Because every Cone is one Third of its circumscribing Cylinder. (See Theorem 13. Page 416.) therefore its Content may be truly found by the following Rule.

*Rule.*  $\left\{ \begin{array}{l} \text{Multiply the Square of the Diameter of its Base,} \\ \text{into the perpendicular Height; then divide their} \\ \text{Product by } 1077,15 = 359,05 \times 3 \text{ for Ale-Gal-} \\ \text{lons, or by } 882,36 = 294,12 \times 3 \text{ for Wine-Gallons,} \\ \text{and the Quotient will be the Content requir'd.} \end{array} \right.$

Or



Or if the said Product be multiply'd with  $0,000928 = \frac{0,002785}{3}$  or with  $0,001133 = \frac{0,0034}{3}$  those Products will be the Content in their respective Gallons.

*Example,* Suppose the Diameter of the Base be 42,5. and the perpendicular Height be 31,5 Inches, what will the Content be in Ale-Gallons? As before.

First  $42,5 \times 42,5 = 1806,25$ . And  $1806,25 \times 31,5 = 56896,875$

Then  $1077,15 \mid 56896,875$  (52,82. Or  $56896,25 \times 0,000928 = 52,82$  the Content in Ale-Gallons. And so on for Wine or Corn-Gallons.

## P R O B L E M XIII.

*To find the Content of the Lower Frustum of any Cone, in Gallons.*

That is, to compute the Content of any round Tun, &c. whose Diameters at Top and Bottom are parallel but unequal.

The Content of such a Tun may be found by the Rule at Problem 10. but from Theorem 16. Page 420. it will be easy to deduce this following Rule.

**Rule.** { *To the triple Product of the Top and Bottom Diameters, add the Square of their Difference; Multiply that Sum into the Height (or Depth) then divide the last Product by 1077,15 for Ale-Gallons, or by 882,36 for Wine-Gallons, the Quotient will be the Content requir'd.*

*Example,* Suppose the Diameter at the Top be 52,4 Inches, the Diameter at the Bottom 44,6, and the Height 30 Inches.

First,  $52,4 \times 44,6 = 2337,04$ . And  $2337,04 \times 3 = 7011,12$  } Add  
 Also  $52,4 - 44,6 = 7,8$ . And  $7,8 \times 7,8 = 60,84$  }

The Height 30  $\times 7071,96 = 212158,8$

Then  $1077,15 \mid 212158,80$  (196,96 } the Content in A. Gal.  
 Or  $212158 \times 0,000928 = 196,96$  }

And so on for either Wine or Corn-Gallons, as Occasion requires. But if the Tun (or Vessel) be not truly circular, that is, if either its Top or Bottom (or both of them) be elliptical, whether they are alike, or unlike it matters not. the Content of such a Tun may be truly found by the General Rule at Problem 10.

## P R O B L E M XIV.

*The Axis or Diameter of any Sphere or Globe, being given (in Inches) to find its Content in Gallons.*

Every Sphere is two Thirds of its circumscribing Cylinder. by Theor. 18. Page 423. from whence and Theor. 20. Page 426. it is prov'd,



prov'd, that if the Cube of the Axis of any Sphere (taken in Inches be multiply'd into 0,5236. the Product will be the Content of that Sphere in Cubic Inches. Consequently, if that Content be divided by 282, or by 231, &c. the Quotient will be the Content in Gallons.

But those two Works of multiplying with 0,5236, and then dividing by 282, or by 231, &c. may be contracted into one.

Thus 282 ) 0,5236 ( 0,001856 will be a Multiplicat. for A. G.

And 231 ) 0,5236 ( 0,002266 will be a Multiplicat. for W. G.

Or 0,5236 ) 282 ( 538,57 will be a Divisor for Ale-Gal.

And 0,5236 ) 231 ( 441,17 will be a Divisor for Wine-Gal.

From hence arises this following Rule.

**Rule.** { *If the Cube of the Axis of any Sphere, be divided by 538,57 (or multiply'd with 0,001856) or divided by 441,17, (or else multiply'd with 0,002266) the Quotient (or Product) will be the Sphere's Content in their respective Gallons.*

*Example,* Suppose the Axis or Diameter of a Sphere or Globe, be 22 Inches, how many Ale-Gallons may it hold?

Then  $22 \times 22 \times 22 = 10648$ . And  $538,57 \mid 10648 (19,76$  A. G.

Or  $10648 \times 0,001856 = 19,76$  Ale-Gal. the Content requir'd.

And so, for either Wine. or Corn-Gallons as Occasion requires.

### PROBLEM XV.

*To find the Content of any Segment of a Sphere in Gallons.*

In the Scholium, Page 424, there are two Theorems for resolving of this Problem according to the Data.

1. If the Diameter of the Segment's Base, and its Height are given, the Content may be found by the first of those Theorems, which gives this Rule.

**Rule 1.** { *To the triple Square of Half the Diameter add the Square of the Height; then multiply that Sum into the Height, and divide the Product by 538,57 for A. G. or by 441,17 for W. G. &c. As above.*

2. But if the Axis of the Sphere. and the Height of the Segment are given, the Content may be found by the second of those Theorems.

**Rule 2.** { *From the triple Product of the Axis into the Height, subtract twice the Square of the Height; then multiply the Remainder into the Height, and divide that Product by 538,57, &c. As in the last Problem.* Either



Either of these Rules will produce the Content of the Segment in Gallons.

*Example.* Suppose the Diameter of the Segment's Base be 28 Inches, and its Height be 8 Inches, what may it contain in Ale-Gallons?

First,  $28 \div 2 = 14$ . Then (by Rule 1.)  $14 \times 14 \times 3 = 588$ .  
 And  $6 \times 6 = 36$ . Next  $588 \div 36 = 16.33$ . Again  $16.33 \times 6 = 97.98$ .  
 Lastly,  $588.57 \div 3744 = 0.157$  the Content requir'd.

*Note,* This Problem may be of Use in Gaging the Crowns of Brewers Coppers, &c.

**Sect. 4.** *The practical Method of Gaging any fix'd Tun or Copper, and making a Table to shew what it will hold at every Inch Deep, usually call'd Incking of a Tun, &c.*

First, you must know, that most (if not all) Brewers Tuns are so fix'd, as to lean a little for Conveniency of cleansing their Drink, which is usually call'd the Drip or Fall of the Tun. Now this Drip or Fall of any Tun, is the Hoof of such a Solid as that Tun is suppos'd to represent; and under that Consideration it may be found, as in Theorem 16. Page 420. But the practical (and indeed the best) Way, is to measure into the Tun (when it is dry) so much Liquor as will just cover its Bottom; for by that Means you do not only find the true Fall, but also a true horizontal or level Plane over the Bottom of the Tun; from which, if the Depth of the Tun (*viz.* the nearest Distance from the Top of the Tun to the Surface of the Liquor) be set off upon every one of its Sides, you will then have a true parallel Plane at the Top of the Tun to that of the Liquor.

Then if the Sides of the Tun are streight, from the Top to the Bottom, take as many Dimensions in the aforesaid two Planes, as are needful to find the true Area of each; and by those two Area's, and the aforesaid Depth, find so much of the Tun's Content (by the general Rule at Problem 10.) as is betwixt those two Planes.

Next, to Inch that Tun, divide the Difference between the Top and Bottom Area's by the aforesaid Depth, and the Quotient will be an Addend or fix'd Number; which being added to the lesser Area, the Sum will be the Area of the next Inch; and being added to that Area, their Sum will be the Area of the third Inch, and so on from Inch to Inch, until the Area of every single Inch be found, the Sum of those Area's (if the Work be true) will amount (or be equal) to the Content found, as above. And if  
the



the Tun's Drip or Fall be added to the Sum of all those Area's, that Sum will be the whole or full Content of that Tun.

Now from hence it must needs be easy to conceive, that if 1. 2. 3. or any Number of those Area's accounted from the Bottom, be added to the Fall, that Sum will shew the Quantity of Liquor or Drink that is in the Tun, to such a Number of wet Inches from the Bottom, as there were Area's added together.

Or if the Sum of any Number of those Area's (being accounted from the Top) be subtracted from the Tun's whole Content, the Remainder will shew what Quantity of Liquor or Drink is in the Tun, when there is such a Number of dry Inches from the Top as there were Area's subtracted.

This being well consider'd, it will be easy to make a Table, either to every wet or dry Inch of any regular Tun, (*viz.* whose Sides are strait from Top to Bottom) what Form soever its Bases are of; and whether it stand upon the greater, or lesser Bases.

But if the Sides of the Tun are irregular (*viz.* not strait from its Top to the Bottom) then the best and easiest Way will be to divide or part the Tun into several Frustums, each of 10 Inches deep; and finding the Content of every single Frustum, by taking the Diameters in the Middle of every one of those 10 Inches, (that is, the first Diameter, at 5 Inches from the Top; the second Diameter at 15 Inches from the Top, &c.) and multiplying their respective Area's with 10, (which is done by only removing the separating Comma's one Place forward to the Right-hand) if the Sum of all those Frustums be added to the Fall, (as before) that Sum will be the whole Content of the Tun.

*Note.* If you take the Height of the aforesaid 10 Inch Frustums in the Side of the Tun, you must allow for the Difference between the slant Height and the perpendicular Height in every Frustum.

Lastly, if from the whole Content of the Tun you subtract the mean Area of the first Frustum ten times, and from the Remainder subtract the mean Area of the second Frustum ten times, and from the last Remainder subtract the mean Area of the third Frustum. &c. until there remain nothing but the Fall or Hoof of the Tun, you will then by that Means have a Table that will shew what Quantity of Drink is in the Tun to any Number of dry Inches.

And this is also the Method of Gaging and Inching of Brewers Coppers, *viz.* by first measuring into the Copper so much Liquor as will just cover its Crown; and then dividing its perpendicular Height into Frustums, and its Sides into four equal Parts, that so the cross Diameters may be taken in the Middle of each



each Frustum : But if the Copper be much wider at the Top than at the Bottom, and its Sides spheroidal or arching, as generally all large Coppers are; then, instead of taking those mean Diameters in the Middle of every ten Inches, as above, you must take them in the Middle of every six Inches, and proceed on as before.

Now the Quantity of Liquor that would cover the Crown of the Copper, may be found without measuring it, as above. In order to that, I do suppose the Crown to be the Segment of a Sphere, and the lower Part of the Copper, wherein the Crown ariseth, to be the Frustum of a parabolic Conoid; then if the Diameter at the Top of the Crown, and its perpendicular Height are given, the Quantity of Liquor may be found by this following Rule.

**Rule.** { *From the Area of the Plane at the Top of the Crown, subtract  $1\frac{1}{3}$  of the Area of the Crown's Height; the Remainder being multiply'd into Half the Height of the Crown will produce the Quantity or Number of Gallons that will cover the Crown.*

This Rule is deduc'd from Scholium, Page 424, and Theorem 15. Page 430.

**Sect. 5.** *To compute the Content of any close Cask in Gallons, viz. of any Butt, Pipe, Hogshead, Barrel, &c.*

In order to perform this difficult Part of Gaging, the three following Dimensions of the propos'd Cask must be truly taken in Inches, and Decimal Parts of an Inch.

*Viz.* { *The Bulge or Bung Diameter within the Cask.  
Either of the Head Diameters, supposing them both equal.  
And the Length of the Cask within.*

*Note, In taking of these Dimensions, it must be carefully observ'd,*

1. That the Bung-hole be in the Middle of the Cask; also, that the Bung-staff, and the Staff over-against the Bung-hole, are both regular or even within.

2. That the Heads of the Cask are equal and truly circular; if so, the Distance between the Inside of the Chine to the Outside of its opposite Staff, will be the Head Diameter within the Cask, very near.

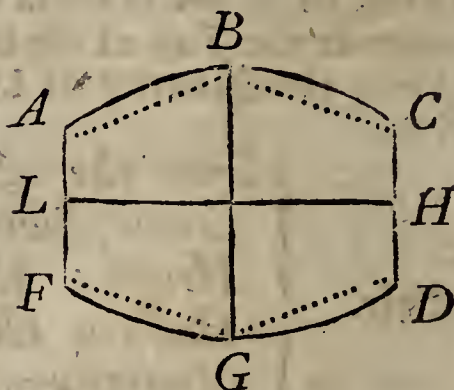
3. With a sliding Pair of Calipers (made on purpose for that Use) take the shortest Distance or Length between the Outsides of the two Heads; (supposing them even) from that Length subtract  $1\frac{1}{2}$  Inch (more, or less, according to the Largeness of the  
M m m Cask)



Cask) for the Thickness of the two Heads, the Remainder will be the Length of the Cask within.

Now, by these Dimensions, one would suppose the Content of the Cask were perfectly limited; but it will be easy to perceive, by the following Figure, that the Diameters (abovesaid) and the Length of one Cask may be equal to those of another, and yet one of those Casks may contain or hold several Gallons more than the other.

As for Instance, suppose the annexed Figure  $ABCDGF$ , to represent a Cask; then it is plain, that if the outward curved Lines  $ABC$ , and  $FGD$  are the Bounds or Staves of the Cask, it must needs hold more than if the inner strait or prick'd Lines were its Bounds or Staves; and yet the Bung Diameter  $BG$ , Head Diameter  $CD$  and  $AF$ , and the Length  $LH$  are the same in both those Casks.



Whence it plainly appears, that no one certain or general Rule can be prescrib'd to find the true Content of all Sorts of Casks, and therefore Gagers do usually suppose every Cask to be in the Form of some one of these following Solids.

- Viz.* { I. The middle Zone or Frustum of a Spheroid.  
 II. The middle Zone or Frustum of a Parabolic Spindle.  
 III. The lower Frustums of two equal Parabolic Conoids.  
 IV. The lower Frustums of two equal Cones.

Now the Way of guessing at the Cask's Form, and computing its Content according to that suppos'd Form, I shall here shew in their Order.

I. If the Staves of the Cask are very curved or arching (as the outward Lines of the last Figure) then the Cask is suppos'd to be in the Form of the middle Zone or Frustum of a Spheroid, whose Content may be computed, by *Theorem 22. Page 427.* which gives these two Rules.

**Rule I.** { To twice the Square of the Bung Diameter, add the Square of the Head Diameter; multiply that Sum into the Length, and divide the Product by 1077,15. viz.  $3,8197 \times 282$  for Ale-Gallons; and by 882,36. viz.  $3,8197 \times 231$  for Wine Gallons. Or thus

**Rule**



**Rule 2.** *To twice the Area of the Bung Circle, add the Area of the Head Circle; multiply their Sum into one Third of the Length, and the Product will be the Content in their respective Gallons.*

**Example 1.** Suppose a Cask in the Form of the middle Zone of a Spheroid, whose Bung Diameter is 31,5, Head Diameter 24,5, and its Length 42 Inches.

First,  $31,5 \times 31,5 \times 2 = 1984,5$ . And  $24,5 \times 24,5 = 600,25$

Again  $1984,5 + 600,25 = 2584,75$ . And  $2584,75 \times 42 = 108559,5$

Then  $1077,15 \mid 108559,5$  (100,78 the Content in Ale-Gallons.

And  $882,35 \mid 108559,5$  (123,03 the Content in W. Gallons.

Or thus. by the second Rule.

Bung Diameter, 31,5 twice its Circles Area is 5,5270

Head Diameter, 24 5 its Circles Area is 1,6718

The Length 42 divided by 3 is 14. 7,1988 = their Sum

Then  $7,1988 \times 14 = 100,78$  the Content in Ale-Gal. As before.

And so the Content in Wine-Gal. may be found.

**II.** If the Staves of the Cask are not quite so much curved or arching as was suppos'd before, the Cask is then taken for the middle Frustum of a Parabolic Spindle, and its Content is computed, as by *Theorem 27. Page 432.* which gives this Rule.

**Rule.** *To twice the Square of the Bung-Diameter, add the Square of the Head-Diameter; from their Difference subtract four Tenths of the Square of the Difference of the Diameters; multiply the Remainder into the Length, and divide the Product by 1077,15, &c. As above.*

**Example 2.** Suppose the Dimensions the same as before. Then

$31,5 \times 31,5 \times 2 : + 24,5 \times 24,5 = 2584,75$ . And  $31,5 - 24,5 = 7$

Again  $7 \times 7 \times 0,4 = 19,6$ . And  $2584,75 - 19,6 : \times 42 = 107736,3$

Then  $1077,15 \mid 107736,3$  (100,01 the Cont. in A. G. &c. for W. G.

**III.** When the Staves of the Cask are but very little curved or arching then 'tis suppos'd to be in the Form of the Frustums of two equal Parabolic Conoids, abutting or joyning together upon one common Base at the Bulge, and the Content may be found by *Theorem 25. Page 430.* which gives these Rules.

M m m 2

Rule



Thus. { To the Square of the Bung-Diameter add the Square of the Head-Diameter; multiply their Sum into the Length, and divide the Product by 718,08

(Viz.  $2,5464 \times 282$ ) for Ale Gallons; or by 588,22 (Viz.  $2,5464 \times 231$ ) for Wine Gallons. Or thus,

Rule 2. { To the Area of the Bung-Circle add the Area of the Head-Circle; multiply the Sum into Half the Length, and the Product will be the Content requir'd.

Example 3. With the same Dimensions as before. Then  $31,5 \times 31,5 : + 24,5 \times 24,5 = 1592,5$ . And  $1592,5 \times 42 = 66885$  And 718,08 ) 66885 (93,01 the Content in Ale-Gallons. Or 588,22 ) 66885 (113,7 the Content in Wine-Gallons.

IV. If the Staves of the Cask are strait from the Bulge to the Head, as the inner prick'd Lines in the last Figure, (if such a Cask can be made) it is then taken for the lower Frustrums of two equal Cones, abutting or joining together upon one common Base at the Bulge. And its Content may be computed as at Problem 13. Page 445. or by Theorem 15. Page 419. Thus,

Rule { To the Sum of the Squares of the Head and Bung-Diameters add their Product; then multiply that Sum into the Length, and divide the last Product by 1077,15. Or by 882,36. The Quotient will be the Content, &c.

Example 4. With the same Dimensions as before.

First,  $31,5 \times 31,5 : + 24,5 \times 24,5 : + 31,5 \times 24,5 = 2364,25$

And  $2364,25 \times 42 = 99298,5$  Then 1077,15 ) 99298,5 (92,18 the Content in Ale-Gallons. And so on for Wine-Gallons.

Thus you have the Methods of computing the true Contents of the four Solids, in whose Forms all Casks are suppos'd to be. And by the Examples it appears, that four such Casks as have their Dimensions all equal, and the same with those above-mentioned, their Contents will be as in the Margin.

From the Disproportion or Inequality of these Differences, it will be easy to conceive, that there may be several Casks whose Contents cannot be truly found, according to the aforesaid suppos'd Forms; and therefore, in order to rectify the said Inequalities, some Authors (that have written upon this Subject) have laid down

	Ale-Gallons.	Differ.
I.	100,78	
II.	100,01	0,77
III.	93,01	7,09
IV.	92,18	0,83



down Theorems of their own Invention; and yet call'd them by these Names) others have propos'd Tables for the same Purpose. But since it is so, that we can only guess at the Truth, the plainest and easiest Way is to be preferr'd in Practice; and that is, by finding such a mean Diameter as will reduce the propos'd Cask to a Cylinder.

Thus,

Rule.

*Multiply the Difference between the Head and Bung Diameters with 0,7 or with 0,65, or with 0,6 or with 0,55. according as the Staves of the Cask are more or less arching; add the Product to the Head Diameter, and the Sum will be the mean Diameter requir'd. Then find the Content as at Prob. 11. Page 444.*

Example. With the same Dimensions as before. Then the Bung Diameter less the Head Diam. is  $31,5 - 24,5 = 7$ . And

	M. D.	A. G.	Cont.	Dif.
24,5 + {	$7 \times 0,7 = 29,4$	its Area 2,4073	$42 = 101,10$	
	$7 \times 0,65 = 29,05$	2,3504	$42 = 98,71$	2,39
	$7 \times 0,6 = 28,7$	2,2941	$42 = 96,35$	2,36
	$7 \times 0,55 = 28,35$	2,2385	$42 = 94,03$	2,32

From these it may be observ'd, that the Difference between each Cask's Content is regular, and very near equal; which plainly shews, that there is not so much Room left for Error this Way of computing their Contents, as was by the aforesaid Forms.

Now the first of these four (*viz.* with 0,7) is very commonly used amongst Gagers for all Sorts of Casks; but I did never gage any Cask that would contain quite so much as that Rule did make it; and the Reason doth appear very plain from Theorem 22. Page 427. being compar'd with Theorem 19. Page 426. and the last Figure, *viz.* that no Cask (being regularly made) can hold more than the middle Frustum of a Spheroid. But I always found by Experience, that if the second and third of these Rules (*viz.* with 0,65 and 0,6) were duly apply'd, they would answer very near the Truth amongst the common Sort of Casks; and the fourth Rule (*viz.* with 0,55) will come pretty near the Truth in computing the Contents of Casks, whose Staves are almost strait betwixt the Head and the Bung, *viz.* such as Wine-Pipes, &c.

Sect. 6. To find what Quantity of Liquor is either drawn forth, or remaining in any spheroidal Cask, usually call'd the Ullage of a Cask; which hath two Cases.

Case 1. To find what Quantity of Liquor is in the Cask, when its Axis is perpendicular to the Horizon, *viz.* when it stands upright upon one of its Heads.

In



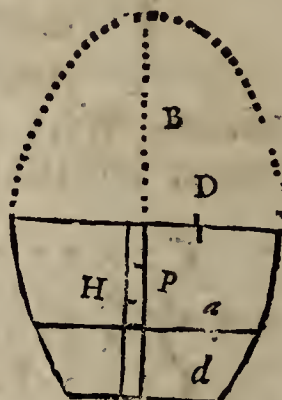
In order to perform this the easiest Way, it will be convenient to know how to calculate the Area of any Circle betwixt the Bung and Head, whose Distance from the Bung or Middle of the Cask is given. Now that may be done by this Proportion.

*As the Square of half the Length of the Cask: is to the Difference between the Bung and the Head Area's :: so is the Square of any Circle's Distance from the Bung: to the Difference between the Bung Area, and the Area of that Circle, viz. the Area of the Liquor's Surface.*

### Demonstration.

Let  $\begin{cases} H = \text{Half the Length of the Cask.} \\ D = \text{Half the Bung Diameter.} \\ d = \text{Half the Head Diameter.} \end{cases}$

And  $\begin{cases} P = \text{the Distance of any Circle from the Bung.} \\ a = \text{Half the Diameter of that Circle.} \end{cases}$



Then, according to the common Property of the Ellipsis, Page 368. it will be,

$BB : DD :: BB - HH : dd.$  And  $BB : DD :: BB - PP : aa.$

Ergo  $\begin{cases} \frac{DDHH}{DD - dd} = BB. \end{cases}$  And  $\begin{cases} \frac{DDPP}{DD - aa} = BB. \end{cases}$

Consequently,  $\begin{cases} \frac{DDHH}{DD - dd} = \frac{DDPP}{DD - aa} \end{cases}$

This Equation being brought out of the Fractions, will become  $DDHH - aaHH = DDPP - ddPP.$

Which gives this Analogy  $HH : DD - dd :: PP : DD - aa.$

Then  $DD - aa$  being subtracted from  $DD$ , will leave  $aa.$

But Circles Area's are in Proportion to the Squares of their Diameters, by Theorem 6. Page 407. Therefore, &c. Q. E. D.

Then from the Bung-Area subtract one third Part of the afore-said Difference, viz. between the Bung-Area, and the Area of the Liquor's Surface; multiply the Remainder with the Liquor's Distance from the Bung, and the Product will shew what Quantity of Liquor is either above or under Half the Content of the Cask.

Example. Let us suppose a Cask of the same Dimensions with that in the first Example, Page 451. and let it be requir'd to find what Quantity of Liquor is in it (of Ale-Measure) when there is but 9 Inches wet. Here Half the Length of the Cask is 21 Inches,



Inches, whose Square is 441, and the Liquor's Distance from the Bung is  $21 - 9 = 12$ . Its Square is 144. The Difference between the Bung and Head Area's is 1,0917 ( $= 2,7635 - 1,6718$ ) Then  $441 : 1,0917 :: 144 : 0,3564$ .

And  $2,7635 - 0,3564 = 2,4071$  the Area of the Liquor's Surface.

Again  $3 \times 0,3564 = 0,1188$ . And  $2,7635 - 0,1188 = 2,6447$

Then  $2,6447 \times 12 = 31,7364$ , what the Cask wants of being half-full. Consequently  $50,39 - 31,73 = 18,66$  will be the Quantity of Liquor in the Cask at 9 Inches wet in Ale-Gallons.

And if the Cask had wanted but 9 Inches of being full; then  $50,39 + 31,73 = 82,12$  would have been the Quantity of Liquor in the Cask.

*Note*, Because the two first Terms (*viz.* 441 and 1,0917) in the Proportion, are fix'd, *viz.* continue the same for any Distance, it will be very easy to calculate the Area's of all the Circles betwixt the Bung and Head to every Inch, and by that Means to make a Table that will shew what Quantity of Liquor is either drawn out, or remaining in the Cask, at any Depth.

*Case 2. To find what Quantity of Liquor is in any Cask, when its Axis is parallel to the Horizon, viz. when it lies along.*

There are Variety of Tables to be found in Books of Gaging for this Purpose; but I always observ'd, that the following Method of computing the Ullage, by a Table of the Segments of a Circle, came very near the Truth in all Sorts of Casks, which is thus perform'd:

1. By the Bung and Head Diameters, find such a mean Diameter as you judge will reduce the propos'd Cask to a Cylinder. by the Method laid down in Page 453. And then find its full Content, as in those Examples.

2. From the Bung Diameter subtract the mean Diameter, and halve their Difference, (*viz.* divide it by 2.)

3. From the wet-Inches of the propos'd Ullage, subtract the said half Difference, and call it *x*; then observe this Proportion.

*Viz.*  $\left\{ \begin{array}{l} \text{As the Mean Diameter : is to 100 ( the Diameter} \\ \text{of the Tabular Circle ) :: so is the last Differ-} \\ \text{ence. ( viz. } x \text{ ) : to a Versed-Sine in the Table.} \\ \text{( Page 441. } \end{array} \right.$

Then if the Tabular Segment which stands against that Versed-Sine, be multiply'd into the Content of the Cask, the Product will shew the Ullage, *viz.* what Quantity of Liquor is either in the Cask, or drawn forth.

*Example,*

*Example 1.* Let the Cask be that of the second Sort, in *Page 453.* viz. whose Bung Diameter is 31,5 Inches, mean Diameter 29,05, and Content 98,71 Ale-Gallons; and suppose there were 10,5 Inches wet in it, it is requir'd to find the wet, and dry Gallons?

Here  $31,5 - 29,05 = 2,45$  its half is 1,22. And  $10,5 - 1,22 = 9,28$   
Then  $29,05 : 100 :: 9,28 : 0,319 = V.$  Sine; its Segment is 0,2748  
And  $98,71 \times 0,2748 = 27,12$  the Number of wet-Gallons.

Again  $31,5 - 10,5 = 21$  the dry Inches; and  $21 - 1,22 = 19,78$   
Then  $29,05 : 100 :: 19,78 : 0,68$ ; its Segment is 0,7241  
And  $98,71 \times 0,7241 = 71,48$  the Number of dry-Gallons.

*Proof*  $71,48 + 27,12 = 98,6$  the Content of the Cask very near which plainly shews the Truth of this Method.

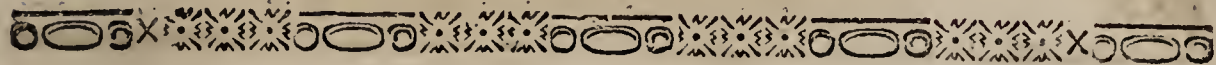
Thus far may suffice concerning Gaging of Backs or Coolers, Tuns, Coppers, and Casks, &c. To which I shall only add, That as the Contents of all Brewers Utensils are to be computed by the Ale-Gallons, so the Contents of all Distillers Utensils (viz. all their Wash-Backs, Stills, and Casks, &c.) must be computed by the Wine-Gallons.

And in gaging of Malt, (upon which there is now a Duty of four Shillings *per* Bushel) you must observe, That a Corn or Malt-Bushel doth contain 2150,42 Cubic Inches; (See *Page 42.*) and therefore in Gaging of Malt-Cisterns, or other Vessels, 2150,42 will be a constant or fix'd Divisor for finding the Area's of all Right-lin'd Figures in Bushels at one Inch deep, and 2738 will be a constant or fix'd Divisor for finding the Area's of Circular Figures.

I have omitted the Business of Gaging Mash-Tuns, and taking an Account of the Goods or Grains, in order to estimate what Quantity of Wort was produced from them, &c. because I could never find (by all my Observations) any Certainty therein; nor is it possible there should be any, by reason of the great Difference that is in Malt; (and its Grinding too) for the best Malt (well ground) will yield or produce the most Wort, and least Grains; on the contrary, bad Malt (being ill ground) yields the least Wort and most Grains.

N. B. For the Numbers proper to the *IRISH GALLON*, See the last Edition of *HAWNEY'S COMPLETE MEASURER*, Publish'd by S. FULLER, at the *Globe and Scales* in *Meath-Street*, *DUBLIN*.





A  
SUPPLEMENT

Not in any of the former EDITIONS of this

B O O K.

Containing the

H I S T O R Y  
O F

Logarithms.

W I T H

Several Easy M E T H O D S of Constructing the *Tables*  
of the Logarithms and Sines, &c. Also the Demon-  
stration of the Axioms and Doctrine of *Plane*

TRIGONOMETRY.

Extracted from the

*Philosophical* T R A N S A C T I O N S and the W O R K S  
of Dr. K E I L, R O N A Y N E, W A R D, &c.

---

*Cuncta Trigonus habet, satagit quæ docta Mathesis,  
Ille aperit clausum quicquid Olympus habet.*

---



# THE P R E F A C E.

**T**HE Mathematicks formerly received considerable Advantages; first, by the Introduction of the Indian Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematicks. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for other Phænomena of Nature; and lastly, the Usurer computes the Interest of his Money.

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom taking in the whole Doctrine, have indeed wrote learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouch'd. My Design therefore in the following Tract, is to supply what seemed still wanting, viz. to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.

The wonderful Invention of Logarithms we owe to the Lord Neper, who was the first that constructed and published a  
Canon



*Canon thereof, at Edinburgh, in the Year 1614. This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet Neper is universally allow'd the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

*The same Lord Neper afterwards invented another and more commodious Form of Logarithms, which he afterwards communicated to Mr. Henry Briggs, Professor of Geometry at Oxford, who was hereby introduced as a Sharer in the completing thereof: But the Lord Neper dying, the whole Business remaining was devolved upon Mr. Briggs, who, with prodigious Application, and an uncommon Dexterity, compass'd a Logarithmic Canon, agreeable to that new Form for the first twenty Chiliads of Numbers, (or from 1 to 20000) and for eleven other Chiliads, viz. from 90000 to 101000. For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publish'd at London in the Year 1624.*

*Adrian Vlacq published again this Canon at Goudæ in Holland in the Year 1628, with the intermediate Chiliads before omitted, filled up according to Brigg's Prescriptions; but these Tables are not so useful as Brigg's, because the Logarithms are continued but to 10 Places of Figures.*

*Mr. Briggs also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to 15 Places of Figures, and has subjoined to them the natural Sines, Tangents, and Secants, to 15 Places of Figures. The Logarithms of the Sines and Tangents are called Artificial Sines and Tangents, but the Sines and Tangents themselves are called Natural. These Tables, together with their Construction and Use, were publish'd after Brigg's Death, at London, in the Year 1633, by Heny Gellibrand, and by him called Trigonometria Britannica.*

*Since then, there have been published, in several Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, consist of but seven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 to 100000, which may be sufficient for most Uses.*

*The best Disposition of these Tables, in my Opinion, is that, first thought of by Nathaniel Roe, of Suffolk; and with some Alterations for the Better, followed by Sherwin in his Mathematical-Tables publish'd at London in 1705; wherein are the Logarithms from 1 to 101000 consisting of 7 Places of Figures. To which are subjoined the Differences and proportional Parts, by Means of which may be found easily the Logarithms of Numbers to 10000000, observing at the same Time that these Logarithms consist only of 7 Places of Figures. Here are also the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with some other Tables of Use in practical Mathematicks.*

---



# THE CONSTRUCTION OF Logarithms.

**T**HESE most excellent and useful Numbers were first invented by the famous and never to be forgotten Lord *Neper*, Baron of *Merchiston* in *Scotland*, aforesaid, (*Ann.* 1614.) who ingeniously contrived to perform Multiplication and Division of Natural Numbers, by only adding or subtracting certain Artificial Numbers, which he called *Logarithms*, and the Extraction of Roots by dividing the Log. by 2 for the Square: by 3 for the Cube: by 4 for the Biquadrat, &c.

This Invention of his (no doubt) proceeded from a mature Consideration of the Coherence that is betwixt Numbers in Geometrical Proportion and those in Arithmetical Progression.

As in these following:

*Viz.*  $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64 \cdot 128, \text{ \&c. Geometrical,} \\ 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7, \text{ \&c. Arithmetical.} \end{array} \right.$

It is very perceptible, that as the Numbers in the Geometrical Proportionals are produced by *Multiplication* or *Division*, those in the Arithmetical Progression are produced by *Addition* or *Subtraction*: As doth appear in this Example;

*Viz.*  $\left\{ \begin{array}{l} 4 \times 32 = 128 \\ 2 + 5 = 7 \end{array} \right\} \text{ or } \left\{ \begin{array}{l} 128 \div 32 = 4 \text{ Geometr.} \\ 7 - 5 = 2 \text{ Arithmet.} \end{array} \right.$

Again,  $\left\{ \begin{array}{l} 1 \cdot 10 \cdot 100 \cdot 1000 \cdot 10000 \cdot 100000, \text{ \&c. Geometr.} \\ 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5, \text{ \&c. Arithmet.} \end{array} \right.$

The same Coherence is betwixt these Latter, as was between the two first Ranks.

*Viz.*  $\left\{ \begin{array}{l} 1000 \times 10 = 10000 \\ 3 + 1 = 4 \end{array} \right\} \text{ or } \left\{ \begin{array}{l} 100000 \div 1000 = 100 \text{ Geometr.} \\ 5 - 3 = 2 \text{ Arithmet.} \end{array} \right.$

Either of these Examples do sufficiently shew the Reason and very Ground of Logarithms.

And

## 462 Construction of Logarithms.

And from the Latter of these it was, that the Prime Logarithms or Characteristics were first assigned.

As in this Table :

Natural Num.	Logarithms.
I	0,0000000
10	1,0000000
100	2,0000000
1000	3,0000000
10000	4,0000000
100000	5,0000000

Having laid this Foundation, the next Work was to find out the Logarithms of the intermediate Numbers situated betwixt 1 and 10, viz. of 2, 3, 4, 5, 6, 7, &c. and of those betwixt 10 and 100, viz. of 11, 12, 13, 14, 15, &c. and so on for the rest. This was a Work of some Difficulty, and very laborious.

The first Step in order thereunto (as I conceive) was to find out a Rank of continual Means betwixt 10 and 1, so as that the last (and least thereof) might be a mixed Number less than 2, and so near 1, as to have such a Number of Cyphers before the significant Figures thereof, as was intended the Places of Logarithms in the Table should consist of. Which Means are to be found, by extracting the Square Root of 10 (having first annexed a competent Number of Cyphers thereunto;) then extracting the Root of that Root, and so by a continued Extraction of Root out of Root, until there be a Root so qualify'd as before-mentioned: Which to make a Table to seven Places in the Logarithms, will require twenty-five several Extractions, the last of which will produce this Number, 1,00000006862238.

The next Step was to find out a Number betwixt (1) and (0) in Arithmetical Progression, that might truly correspond with the Mean before found (betwixt 10 and 1) such a Number must consequently be its Logarithm. And this may be found by a continual bisection (or halving) of 1, so often as was the Number of the foregoing Extractions. (to wit, twenty-five) the last of which Bisections will produce 0,000000029802322, &c. the true Logarithm of 1,00000006862238.

For as 1,00000006862238 by twenty-five continued Involution (viz. first into it self, then that Product into it self, and so on successively) will produce 10; so will 0,00000002980232 by the like Number of Doublings and Redoublings, produce 1.

This Mean (or Number) and its Logarithm being thus found, it will follow by Proportion,



# Construction of Logarithms. 463

*As the Significant Figures of this Mean : are to the Significant Figures of its Logarithms :: So are the Significant Figures of any Mean, betwixt any given Number and 1 : (having seven Cyphers before such Figures, as this hath) To the Significant Figures of its Logarithm. To which must be prefixed seven Cyphers to complete it. After which, being doubled, and redoubled according to the Number of Extractions required to produce its corresponding Mean, will at last discover the true Logarithm of the given Number. For the clearing of this, take an Example.*

Suppose it were required to find the Logarithm of the Number 2, to seven Places. First, by a continued Extraction of Root out of Root, beginning at 2, find such a Mean, or Root as before, betwixt 2 and 1, as will have seven Cyphers before its significant Figures; which after twenty-three several Extractions, will be this Number 1,00000008262958. Then according to the foregoing Proportions, it will be

$$6862238 : 2980232 :: 8262958 : 3588557$$

To which prefix seven Cyphers, as before directed, then will 1,00000008269958 have for its Logarithm, ,00000003588557; which being doubled and redoubled, as abovesaid, will produce 0,30102997958658 the true Logarithm of 2; which being contracted to seven Places, according to the first Design, and agreeable to the seven Places of Cyphers, then it will become 0,3010299. But in all the Tables that I have seen, the Logarithm of 2 is 0,3010300: I conceive the Reason is, because the remaining Figures 7958658 come so near Unity of the last Place in the retained Figures.

And by the same Method that this Logarithm of 2 is made, may the Logarithm of any other Number be found. But when once the Logarithms of a few of the prime Numbers, viz. of 3. 7. 11. 13. &c. (that is, of such Numbers as cannot be produced by the multiplying of two Integer Factors) are obtained, the rest may be easily composed by Addition and Subtraction only:

$$\text{For as } 3 \times 2 = 6 \text{ So Log. of } 3 + \text{Log. of } 2 = \text{Log. of } 6.$$

$$\text{And as } 10 \div 2 = 5 \text{ So Log. of } 10 - \text{Log. of } 2 = \text{Log. of } 5$$

The like of all Numbers that have aliquot Parts, (that is, such Integer Numbers as may be divided by Integers.) And indeed the Logarithms of several of the prime Numbers may also be obtained by Addition or Subtraction, as might easily be shewed, and is not difficult to conceive by any one, who but duly considers the Nature and Design of Logarithms; &c. of which I shall forbear saying any thing in this Place, and keep to my first Design herein, which was to give a brief Account of the ingenious Author's Method as I conceive it, of making the same: who undoubtedly found it a very difficult Work, by Reason there is required so many several Extractions of Roots out of Roots, which must needs



## 464 Construction of Logarithms.

needs render it both troublesom and laborious. Then to propose a different Method of raising the Logarithms of such prime Numbers before mentioned, which require the Extraction of Roots to obtain their respective Means, with one tenth Part of the Trouble and Time required by the foregoing Method. And not only so, but more exact; for by our present Method of converging Series, the Root of any Power, how high soever it be, is easily found at one single Extraction; and thereby the Errors which would arise by extracting a *Surd Root* out of a *Surd Root*, especially when often repeated, are avoided; and consequently such a Mean as may be required betwixt any Number and Unity, is thereby more exactly found.

Now how this may be performed, I here intend to shew, as briefly as I can. In order thereunto, take this as a Model.

Let  $a$  = the Root, or Mean required betwixt any Number and Unity :

$$\text{Then } \begin{cases} a^2 = \square a & a^4 = \square a^2 & a^8 = \square a^4 \\ a^{16} = \square a^8 & a^{32} = \square a^{16} & a^{64} = \square a^{32} \\ a^{128} = \square a^{64} & a^{256} = \square a^{128} & a^{512} = \square a^{256} \end{cases}$$

And so on successively with the Indices in Geometrical Progression, until the Power of  $a$  be made equal to such a Term in that Progression, as that the Root, or Value of  $a$  may have betwixt Unity and its significant Figures, so many Cyphers; as are the intended Number of Places in the Logarithms.

For Instance, Let it be required to find the Mean between 10 and 1; then the Power of  $a$  must be  $a^{33554432} = 10$ , this Index 33554432 being the 25th Term in Geometrical Progression, which may be thus determined.

Let 1, the Characteristic or Logarithm of 10, be divided by such a Term in Geometrical Progression, as will cause such a Number of Cyphers to be before the significant Figures in the Quotient, as are required to be before the Figures of the Root  $a$ ; suppose 7, as before.

Then  $1 \div 33554432 = ,00000002980232$ , &c. which is the true Arithmetical Mean (as before found, by a continual bisecting of 1) correspondent to that signify'd by  $a$ . And therefore the Value of  $a$  found by extracting the respective Root of  $10 = a^{33554432}$  will be the Mean required;

*Viz.* 1.00000006862238 whose Log. is ,00000002980232

These being found, are the Foundation of the rest, as before.

Then suppose it be required to find the Logarithm of any of the Prime Numbers; if you please, that of 2. In order thereunto, let  $a$  = the Root or Mean sought betwixt 2 and 1, as before; then must  $a$  be continually involved, as by the above Model, until its Index be equal to the greatest Term in Geometrical Progression, whole



# Construction of Logarithms. 465

whose Number of Places of Figures are to be equal to the Number of required Cyphers before  $a$ , to wit 7. According to which, the Power of  $a$  will be  $a^{8388608} = 2$  (this 8388608 being the 23d Term in Geometrical Progression) consequently the respective Root of  $2 = a^{8388608}$  will be the Mean required.

*Example.*

Let  $r + e = a$

Then will  $r^{8388608} + 8388608 r^{8388607} e$

$+ 35184367894528 r^{8388606} ee = a^{8388608} = 2$

Suppose  $r = 1$

Then  $1 + 8388608e + 35184367894528ee = 2$

That is  $8388608e + 35184367894528ee = 1$

Each Part being divided by the Co-efficient found prefixed to  $ee$ , viz. 351843, &c. then it will become

$,00000023e + ee = ,00000000000000284 = D$

Consequently  $\left\{ \frac{D}{,00000023 + e} = e \right.$

.....

$,00000000000000284 = D$

$,00000023$	$248$	$(,000000008 = e$
$+ e = ,00000008$	<hr/>	
	$36$	

Divisor  $,00000031$

First  $r = 1,$

$+ e = ,00000008$

New  $r = 1,00000008$

Which being duly involved, in the same Order as the Model denotes, and multiplied into the respective Co-efficients, will then produce these Numbers,

Viz.  $1,9563638967 + 16411168e + 68833416066289ee = 2$

Then  $16411168e + 68833416066289ee = ,0436361033$

And  $,0000002384e + ee = ,0000000000000063393 = D$

Consequently  $\left\{ \frac{D}{,0000002384 + e} = e \right.$

.....

$,0000000000000063393 = D$

$,0000002384$	$480$	$(,00000000263 = e$
$+ e = ,000000026$	<hr/>	

Divisor $,000000240$	$15393$
	$14460$
	<hr/>

Divisor $,0000002410$	$9330$
	$7230$
	<hr/>

O o o

Last

# 466. Construction of Logarithms.

$$\begin{array}{r} \text{Last } r = 1,000000008 \\ + e = ,00000000263 \\ \hline \end{array}$$

$$\text{New } r = 1,000000008263$$

I take only  $1,00000000268 = r$ ; the which being involved, and ordered as before will produce these following Numbers, viz.

$$1,999503684867 + 16773028e + 70351267454084ee = 2$$

$$\text{Then } 16773028e + 70351267454084ee = ,000496315133$$

$$,0000002384186e + ee = 0,0000000000000000000705481443 = D$$

$$\text{Consequently } \left\{ \frac{D}{,0000002384186 + e} = e \right.$$

$$,0000000000000000000705481443 = D$$

$$\begin{array}{r} ,0000002384186 \\ + e = ,00000000000295 \end{array} \quad \begin{array}{r} 47686 (,0000000000295 = e \\ \hline 2286214 \\ 2146023 \end{array}$$

$$\text{Divisor } ,00000023843 \quad \begin{array}{r} 2286214 \\ 2146023 \end{array}$$

$$\text{Divisor } ,000000238447 \quad \begin{array}{r} 14019143 \\ 11922405 \end{array}$$

$$\text{Divisor } ,0000002384481^* \quad \begin{array}{r} *20967380 \\ 19075848 \end{array}$$

\* Here I defist forming a new Divisor, and make use of the Abridgment.

$$\begin{array}{r} 1891532 \\ 1669136 \end{array}$$

$$\text{Last } r = 1,00000000826 \quad \begin{array}{r} 222396 \\ 214596 \end{array}$$

$$+ e = ,0000000000295879$$

$$a = 1,00000000826295879$$

This Value of  $a = 1,00000000826295879$  is the Geometrical Mean betwixt 2 and 1, as was required; (agreeable to that before found, by twenty-three several Extractions.) And by this Method of proceeding, may be found the Mean betwixt 10 and 1, viz.  $1,000000006862238$ , or betwixt any other of the (before mentioned) Prime Numbers and Unity, as might easily be shewed. But for Brevity Sake, I shall omit giving more Examples thereof, this one being sufficient (especially to the Ingenious) if well considered, and but once understood, to shew the Nature of, and Manner how, to proceed upon the like Occasion, of finding any proposed Mean. The next Thing will be to find the Logarithm of the Number



Number from whence such Mean was produced, which may be thus performed.

First, find its corresponding Arithmetical Mean, or Logarithm, by Proportion, (as in *Pag.* 462.) Then multiply that corresponding Mean, so found, into the Index Number of such Power as the Geometrical Mean was produced from; that Product will be the Logarithm of the given Number (without a continued Doubling and Redoubling, as before.) For the clearing of this, let it be required to complete the Logarithm of 2.

Having first found 1,00000006862238, the proper Geometrical Mean betwixt 10 and 1; also its corresponding Logarithm, 00000002980232 (as before directed) with them and the Mean betwixt 2 and 1, last found, viz. 1.0000000826295879; make use of the above-mentioned Proportion, (as in *Pag.* 463.) viz.

$$6862238 : 2980232 :: 826295879 : 358855729$$

To which prefix seven Cyphers to complete it (as before.) Then it will become 0000000358855729. This Number being multiplied into the Power of *a* (what that is, see *Pag.* 465.) will produce the Logarithm of 2.

$$\text{viz. } 0000000358855729 \times 8388608 = 0,30103000391352$$

But according to the first Design, it is required to have but seven Places, viz. 0,301300; which is the true Logarithm of 2 without any Defect.

Thus I have presented you with a new and expeditious Method of making Logarithms; which if they were required to fourteen or fifteen Places (I can modestly say) they might then be made with one twentieth Part of the Time and Trouble required by the first Method.

---

# M E T H O D III.

## A N E W

# T A B L E

## O F

# LOGARITHMS.

*Composed by Mr. LONG.*

*Finding the Logarithm by Division only, and the Natural Number belonging to a Logarithm, by Multiplication only.*

O o o 2

Log.

# 468 Construction of Logarithms.

Log.	Nat. Num.	Log.	Nat. Num.
0,9	7.943282347	0,00009	1.000207254
0,8	6.309573445	0,00008	1.000184224
0,7	5.011872336	0,00007	1.000161194
0,6	3.981071706	0,00006	1.000138165
0,5	3.162277660	0,00005	1.000115136
0,4	2.511886432	0,00004	1.000092106
0,3	1.995262315	0,00003	1.000069080
0,2	1.584893193	0,00002	1.000046053
0,1	1.258925412	0,00001	1.000023026
0,09	1.230268771	0,000009	1.000020724
0,08	1.202264435	0,000008	1.000018421
0,07	1.174897555	0,000007	1.000016118
0,06	1.148153621	0,000006	1.000013816
0,05	1.122018454	0,000005	1.000011513
0,04	1.096478196	0,000004	1.000009210
0,03	1.071519305	0,000003	1.000006908
0,02	1.047128548	0,000002	1.000004605
0,01	1.023292992	0,000001	1.000002302
0,009	1.020939484	0,0000009	1.000002072
0,008	1.018591388	0,0000008	1.000001842
0,007	1.016248694	0,0000007	1.000001611
0,006	1.013911386	0,0000006	1.000001381
0,005	1.011579454	0,0000005	1.000001151
0,004	1.009252886	0,0000004	1.000000921
0,003	1.006931669	0,0000003	1.000000690
0,002	1.004615794	0,0000002	1.000000460
0,001	1.002305238	0,0000001	1.000000230
0,0009	1.002074475	0,00000009	1.000000207
0,0008	1.001843766	0,00000008	1.000000184
0,0007	1.001613109	0,00000007	1.000000161
0,0006	1.001382506	0,00000006	1.000000138
0,0005	1.001151956	0,00000005	1.000000115
0,0004	1.000921459	0,00000004	1.000000092
0,0003	1.000691015	0,00000003	1.000000069
0,0002	1.000460623	0,00000002	1.000000046
0,0001	1.000230285	0,00000001	1.000000023

This



# Construction of Logarithms. 469

This Table I sometimes make use of for finding the Logarithm of any Number propos'd, and *vice versa*. Suppose I had Occasion to find the Logarithm of 2000. I look in the first Class of my Table (the whole Table consists of 8 Classes) for the next less to 2, which is 1.995262315, and against it is 3, which consequently is the first Figure of the Logarithm sought. Again, dividing the Number propos'd 2, by 1.995262315 the Number found in the Table, the Quotient is 1.002374467; which being look'd for in the second Class of the Table, and finding neither its Equal, nor a Lesser, I add 0 to the Part of the Logarithm before found, and look for the said Quotient 1.002374467 in the third Class, where the next less is 1.002305238, and against it is 1, to be added to the Part of the Logarithm already found; and dividing the Quotient 1.002374467, by 1.002305238. last found in the Table, the Quotient is 1.000069070; which being sought in the fourth Class gives 0, but being sought in the fifth Class gives 2, to be added to the Part of the Logarithm already found; and dividing the last Quotient by the Number last found in the Table, viz. 1.000046053, the Quotient is 1.000023015, which being sought in the sixth Class, gives 9 to the Part of the Logarithm already found: And dividing the last Quotient by the new Divisor, viz. 1.000002072, the Quotient is 1.000000219, which being greater than 1.000000115; shews that the Logarithm already found, viz. 3.3010299 is less than the Truth by more than half an Unit; wherefore adding 1, you have Briggs's Logarithm of 2000, viz. 3.3010300.

If any Logarithm be given, suppose 3.3010300, throw away the Characteristic, then over against these Figures 3...0...1...0...0, you have in their respective Classes 1.995262315.....0.....1.002305238.....0.....1.000069080.....0...0 which multiplied continually into one another, the Product is 2.000000019966, which by reason the Characteristic is 3, becomes 2.000,000019966, &c. that is, 2000. the Natural Number desired. I shall not mention the Method by which this Table is fram'd, because you will easily see that from the Use of it.

It is obvious to the intelligent Reader, that these Classes of Numbers are no other than so many Scales of mean Proportionals: In the first Class, between 1 and 10; so that the last Number thereof, viz. 1.258925412 is the tenth Root of 10, and the rest in order ascending are the Powers thereof. So in the second Class, the last Number 1.023292992 is the hundredth Root of 10, and the rest in the same Manner are Powers thereof. So 1.002305238 in the third Class, is the tenth Root of the last of the second, and the rest its Powers, &c. Or, which is all one, each Number in the preceding Class, is the tenth Power of the corresponding Number in the next following Class: Whence 'tis plain, that to construct

---

## 470 Construction of Logarithms.

---

Construct these Tables requires no more than one Extraction of the fifth or sixth Root for each Class, the rest of the Work being done by the common Rules of Arithmetick.

---

### M E T H O D IV.

Their Construction, according to the common Rules, given by many Extractions of Roots, is tedious; the best Way yet known is this which follows.

#### *To make a Table of Logarithms.*

*First*, Put for the Logarithm of 1, a Cypher for the Index, and a competent Number of Cyphers for the Logarithm, according to the Number of Places you would have your Logarithms consist of; for 10 an Unit with the same Number of Cyphers; for 100, 2, with as many Cyphers; for 1000, 3, with as many Cyphers, &c.

*Secondly*, Find the Difference between some two Logarithms above 1000 or rather above 10000, that differ by Unity; thus. Multiply the two Numbers together, and that Product you must multiply again by 43429448190325183896 \* which last Product divided by the Arithmetical Mean between both Numbers, the Quotient is the Difference sought.

Suppose we would find the Difference between the Log. 10000, and 10001, the Product of these two Numbers is 1.00010000, which multiplied by 4343 produceth 43434343; this divided by 10000.5, quotes 4343. Now if to the Logarithm of 10000, which is 4.00000000, you add the Difference before found, to wit, 434, the Sum 4.0000434 is the true Logarithm of 10001 to 7 Places.

*Thirdly*. Having thus found the Difference of any two Logarithms Differing by Unity, and consequently some of the Logarithms by dividing the Difference found by the Arithmetical Mean, between any two Numbers Differing by Unity, you shall have the Difference of the Logarithm of those two Numbers.

Thus to find the Difference betwixt the Logarithm of 274, and 275; divide 4343 the Difference of the Logarithm of 10000, and 10001 by 274.5 the Quotient 15821, is the Difference sought.

*Fourthly* Having by this Means found a few of the prime Logarithms, the rest are made by Addition and Subtraction, and having made

---

\* Which is the Subtangent of the Curve expressing Brigg's Logarithms. See Keil's Trig. Pag. 135. 140, &c.



# Construction of Logarithms. 471

made the Canon upward, above 1000 to 10000, by Consequence it is made for all inferior Numbers.

The prime Numbers to which Logarithms must be found, in first Place, are these, 2 . 3 . 7 . 11 . 13 . 17 . 19 . 23 . 29 . 31 . 37 . 41 . 43 . 47 . 53 . 59 . 61 . 67 . 71 . 73 . 79 . 89 . 97, &c. or the same Numbers with Cyphers.

But since it was very tedious and laborious, to find the Logarithms of the Prime-Numbers, and not easy to compute Logarithms by Interpolation, by first, second and third, &c. Differences, therefore the great Men, Sir *Isaac Newton*, *Mercator* *Gregory Wallis*, and lastly, *Dr. Halley*, have published infinite converging Series by which the Logarithms of Numbers to any Number of Places, may be had more expeditiously and truer: Concerning which Series, *Dr. Halley* has written a learned Tract, in the *Philosophical Transactions*, wherein he has demonstrated those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found easily and expeditiously, the Logarithms of large Numbers.

Let  $z$  be an odd Number, whose Logarithm is sought; then shall the Number  $z - 1$  and  $z + 1$  be even, and accordingly their Logarithms and the Difference of the Logarithms will be had, which let be called  $y$ : Therefore, also the Logarithm of a Number which is a Geometrical-Mean between  $z - 1$  and  $z + 1$  will be given, viz. equal to the half Sum of the Logarithms. Now the Series

$$y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9} \&c. \text{ shall be}$$

equal to the Logarithm of the Ratio, which the Geometrical-Mean between the Numbers  $z - 1$  and  $z + 1$ , has to the Arithmetical-Mean, viz. to the Number  $z$ .

If the Number exceeds 1000, the first Term of the Series  $\frac{y}{4z}$  is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But if  $z$  be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2, with the Index 4 prefix'd to it; and the Difference of the Logarithms of 20000 and 20002, is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, viz. 0.00004342727687. And if this Difference be divided by 42, or 80004, the Quo-

Quotient  $\frac{y}{42}$  shall be — — — — — 0.00000 0000542813

And if the Logarithm of the Geometrical-Mean be added to the Quotient, the Sum will be the Logarithm of .20001. Wherefore it

4.30105 1709302416

is manifest, that to have the Logarithm to

4.30105 1709845230

14 Places of Figures, there is no Necessity of continuing out the Quotient beyond 6 Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq's* Tables, the two first Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000, are to be found by this Way, the Labour of doing them, will mostly consist in setting down the Numbers. *Note.* This Series is easily deduced from that found out by Dr. *Halley*; and those who have a Mind to be inform'd more in this Matter, let them consult his abovenam'd Treatise.

### Mr. *WARD's* Easy Method of making the Canon of Sines, Tangents, &c.

**FIRST**, Let me premise two Things, that the Periphery of a Circle, whose Radius is Unity or 1, is 6.283185, &c. and that the Natural Sine of one Minute doth *so insensibly* differ from the Length of the Arch of one Minute, *that* it may be taken for the same.

Consequently,  $\left\{ \begin{array}{l} \text{As the Periphery in Minutes : Is to the} \\ \text{Periphery in equal Parts of the Radius ::} \\ \text{So is one Minute : To the Parts agree-} \\ \text{ing to that Minute.} \end{array} \right.$

That is, 21600' : 6.283185 :: 1' : 0.000290888 = the Natural Sine of one Minute; which agrees with the largest Table of Sines I ever saw.

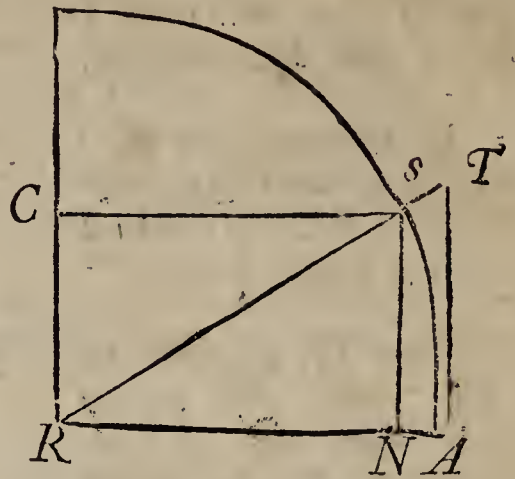
Having thus got the Sine of one Minute, its Co-Sine may be thus found :

Suppose



Suppose  $RA = RS$  the Radius of any Circle,  $SN =$  the Sine of the Arch  $SA$ . Then  $RN = CS$  is the Co-Sine of that Arch. But  $\square RS = \square SN = \square RN$  consequently  $\sqrt{\square RS - \square SN} = RN$ .

That is, From the Square of the Radius, subtract the Square of the Sine of  $1'$ , the Square Root of the Remainder will be the Co-Sine of  $1'$ , per *Chap. 9. Prop. 1.* In Numbers, the Sine of  $1'$  is 000290885, its Square is 0,000000084612; and  $1 - 0,000000084612 = 0,999999915388$ , the Square Root thereof is 0,99999995 = the Co-Sine required.



The Sine and Co-Sine of one Minute being thus obtain'd, all the rest of the Sines in the Quadrant may be gradually calculated by Mr. Michael Dary's Sinical Proportions; which I shall here insert, to the same Effect as they are in his Miscellanies; and then explain and demonstrate the Truth of those Proportions.

If a Rank of Arches be equi-different;

Then { *As the Sine of any Arch in that Rank : Is to the Sum of the Sines of any two Arches equally remote from it on each Side :: So is the Sine of any other Arch in the said Rank : To the Sum of the Sines of two Arches next it on each Side; having the same common Distance.*

Immediately after these Proportions, he lays down the following Equations:

Three Arches equi-different, being propos'd; if (saith he) you put  $Z =$  the Sine of the greater Extreme,  $y =$  the Sine of the lesser Extreme;  $M =$  the Sine of the Mean;  $m =$  the Co-Sine thereof;  $D$  the Sine of the common Difference;  $d =$  the Co-sine thereof; and  $R =$  the Radius.

1. Then  $Z + y = \frac{2Md}{R}$ .      2. Then  $Z - y = \frac{2mD}{R}$ .

3. Then  $Zy = MM - DD$ .      4. Then  $\frac{Z}{y} = \frac{Md + mD}{Md - mD}$ .

From the foregoing, it is evident, (saith he) that if two Thirds; viz. either the former or latter 60 Degrees, or the former 30 Degr. and the latter 30 Degr. of the Quadrant be completed with Sines; the remaining Part of the Quadrant may be completed by Addition, or Subtraction only.

P p p

Thus





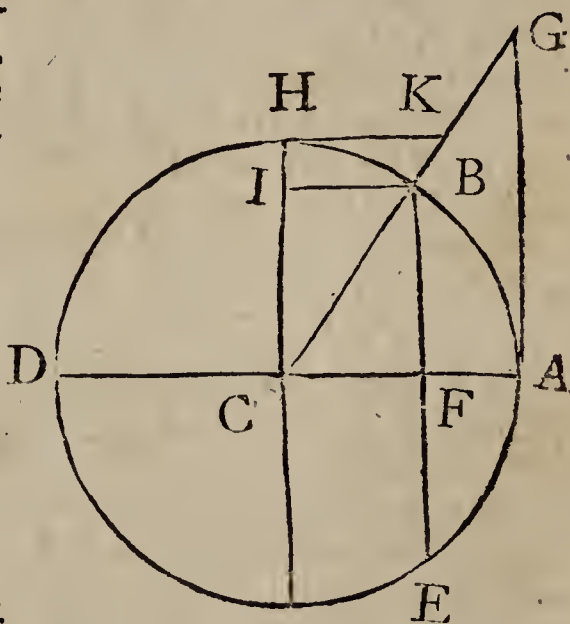
Thus you have an easy Way of making the Canon of Sines; which being once done, the Tangents and Secants may be found by the following.

That is, by *the first Scheme of this Problem*,  
 $RN : SN :: RA : TA$ . And  $RN : RS :: RA : RT =$  the Secant of  
 that Arch.

## DEFINITIONS.

2. A Chord or Subtense is a straight Line, connecting the Extremities of an Arch; as BE is the Chord of the Arches BAE, BDE.

P p p z



#### 4. The

4. The Difference of an Arch from a Quadrant, whether it be greater or less, is called its Complement; so HB is the Complement of the Arches BA, BD; BI is the Sine of that Complement, and therefore it is called the Co-sine, or Sine-Complement of the Arches BA. BD.

5. The Secant of an Arch is a strait Line drawn from the Center thro' one End of the Arch till it meet with the Tangent, which is a strait Line touching the Circle at the Extremity of that Diameter which cuts the other End of the Arch; so CG is the Secant, and AG the Tangent of the Arches BA. BD: And CK is the Co-secant, and HK the Co-tangent of the said Arches.

6. A Versed Sine is the Segment of the Diameter intercepted between the Arch and its Sine: Thus FA is the Versed Sine of the Arch BA; and FD of the Arch BD.

7. Whatever Number of Degrees an Arch wants of a Semi-Circle is called its Supplement.

8. That Part of the Radius which is betwixt the Center and Sine is equal to the Co-sine; thus CF is = IB.

9. If an Arch be greater or less than a Quadrant, the Sum or Difference of the Radius and Co-sine is equal to the Versed Sine.

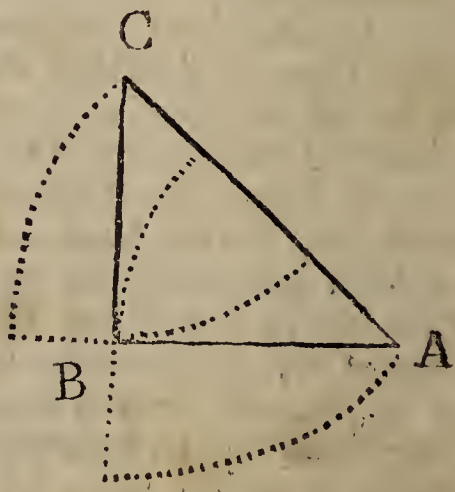
In a Triangle are six Parts, *viz.* three Sides and three Angles: Any three of which being given, except the three Angles of a Plane Triangle, the other three may be found either Mechanically, by the Help of a Scale of equal Parts and Line of Chords, or by an Arithmetical Calculation, if, supposing the Radius divided into any Number of equal Parts, we know how many of those equal Parts are in the Sine, Tangent, or Secant of any Arch propos'd: The Art of inferring which is called *Trigonometry*, and it is either Plane or Spherical.

*Plane Trigonometry* is solv'd by the Help of four fundamental Propositions, call'd *Axioms*.

#### *Axiom I.*

In a Right-angled Triangle ABC, if one Leg of the Right-angle, as AB or CB, be made the Radius of a Circle, then shall the other Leg CB or AB be the Tangent of the Angle opposite to it, and the Hypotenuse AC (or Side opposite to the Right-angle) its Secant (by Definition 5.)

But if the Hypotenuse AC be made the Radius of a Circle, then will the Legs (or Sides including the Right-angle) to wit CB and AB be the Sines of the Angles opposite (by Definition 3.)



Upon



Upon this *Axiom* depends the Solution of the seven Cases of Right-angled Plane Triangles.

*Note*, That the three Angles of a Plane Triangle make two Right-Angles. or 180 Degrees, by 32. I *Eucl.*

For the more easy making the Proportions for the Solution of Right-angled Triangles, observe, that as different Sides are made Radius, so the other Sides acquire different Names, which Names are either Sines, Tangents, or Secants, and are to be taken out of your Table.

To find a Side, any Side may be made Radius : Then say, As the Name of the Side given. is to the Name of the Side required ; so is the Side given, to the Side required.

But to find an Angle, one of the given Sides must be made Radius ; then, As the Side made Radius, is to the other Side ; so is the Name of the first Side (which is Radius) to the Name of the second Side ; which fourth Proportional must be found among the Sines or Tangents, &c. to be determin'd by the Side made Radius, and against it is the Angle required.

The Proportions for the Solution of seven Cases of Plane Right-angled Triangles.

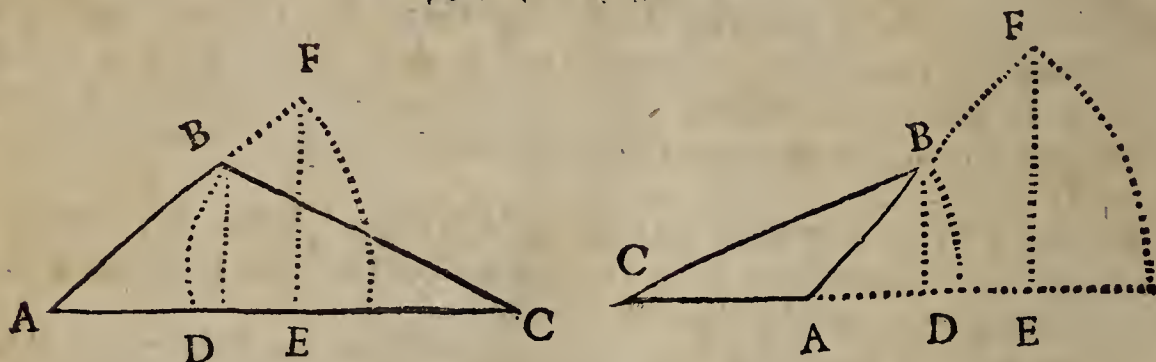
[See the next foregoing Fig.]

Given	Reqd.	Proportions.	Rad.	Case.
A B A and C	B C	Cofi. A : Si. A :: AB : BC. R : Tan. A :: AB : BC. Co-t. A : R :: AB : BC.	AC AB BC	1
A B A and C	A C	Cofi. A : R :: AB : AC. R : Sec. A :: AB : AC. Tan. A : Cofe. A :: AB : AC.	AC AB BC	2
A B B C	A and C	AB : BC :: R : Tan. A. Complement is C. BC : AB :: R : Tan. C. Complement is A.	AB BC	3
A B B C	A C	AB : BC :: R : Tan. A ; Then Cofi. A : R :: AB : AC. or $\sqrt{\square AB + \square BC} = AC$ (per 47. I. <i>Eucl.</i>	AB AC	4
A B A C	A and C	AC : BC :: R : Cofi. A. AB : AC :: R : Secant A.	AC AB	5
A B A C	B C	AC : AB :: R : Cofi. A ; Then R : Tan. A :: AB : BC, or $\sqrt{\square AC - \square AB} = BC$ .	AC AB	6
A C A and C	A B	R : Cofi. A :: AC : AB. Sec. A : R :: AC : AB. Cof. A : Cot. A :: AC : AB	AC AB BC	7

## Axiom II.

In any Triangle the Sides are proportional to the Sines of the opposite Angles.

## Demonstration.



Produce the lesser Side of the Triangle ABC, to wit AB to F, making  $AF = BC$ : Let fall the Perpendiculars BD, FE, upon the Side CA produc'd, if need be; then will FE be the Sine of the Angle A, and BD the Sine of the Angle C, to the Radius  $BC = AF$ .

Now the Triangles ABD and AFE, having the  $\angle A$  common to them both, and the  $\angle D = \angle E =$  to a Right-angle, are similar; wherefore (by 4.6 *Eucl. Elem.*)  $AF (BC) : AB :: FE : BD$ ; viz.  $:: \text{Si. A} : \text{Si. C.}$  Q. E. D.

Otherwise thus;  
By Ax. I.  $AB : R :: BD : \text{Si. A}$ , and  $BC : R :: BD : \text{Si. C}$ ;

Therefore  $AB \times \text{Si. A} (= R \times BD) = BC \times \text{Si. C}$ ;

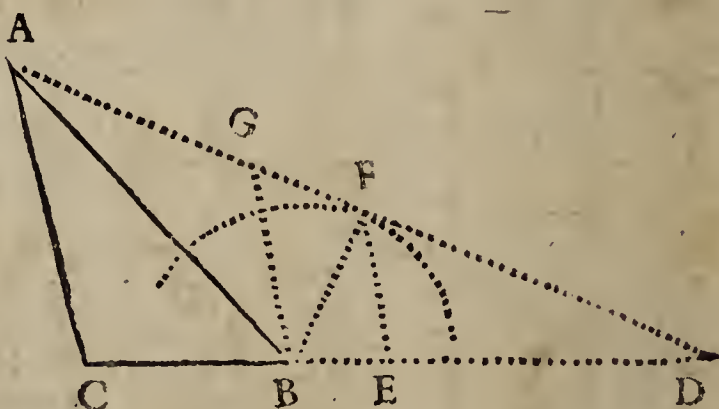
Wherefore  $AB : BC :: \text{Si. C} : \text{Si. A.}$  Q. E. D.

## Axiom III.

The Sum of the Legs of any Angle of a Plane Triangle, is to their Difference, as the Tangent of half the Sum of the Angles opposite to those Legs, is to the Tangent of half their Difference.

## Demonstration.

In the Triangle ABC produce CB, the lesser Leg of the Angle B, till BD becomes  $= BA$ , the greater Leg, and then bisect CD in E; join AD and bisect it also in F; draw BF, which (by 8.1 *Eucl. El.*) will be perpen. to AD; and draw EF, which (by 2.6 *Eucl. Elem.*) will be parallel to AC.



Then will the Angle  $ABF = FBD = \frac{1}{2} ABD$ , which external Angle ABD is (by 32.1 *Eucl. Elem.*)  $= BAC + C$ , that is to Sum of the opposite Angles required.

Draw then BG parallel to CA, so will the Angle GBA be (by 29.1 *Eucl. Elem.*) equal to its Alternate one BAC; and if from half the Sum



Sum of the opposite Angles you take the lesser Angle, *i. e.* If from  $\angle ABF$  you take the  $\angle GBA$ , there will remain  $\angle GBF =$  half the Difference of the opposite Angles : And so also, if from  $CE$  half the Sum of the Legs, you take  $CB$  the lesser Leg, there will remain  $BE$  equal to half the Difference of the Legs. And then, since the  $\triangle ABF$  is Right-angled, if  $BF$  be made Radius,  $AF$  will be the Tangent of  $\angle ABF$  (*i. e.* the Tangent of half the Sum of the opposite Angles); and in the little  $\triangle GBF$ ,  $FG$  will be the Tangent of the  $\angle GBF$  (*i. e.* the Tangent of half the Difference of the opposite Angles): But the Segments of the Legs of any Triangle cut by Lines parallel to the Base, being (by *Schol. to 2. 6 Eucl. El.*) proportional;  $EC : EB :: FA : FG$ ; that is in Words, half the Sum of the Legs, is to half their Difference, as the Tangent of half the Sum of the opposite Angles, is to the Tangent of half their Difference: But Wholes are as their Halves; wherefore the Sum of the Legs, is to their Difference, as the Tangent of half the Sum of the Angles opposite, is to the Tangent of half their Difference. *Q. E. D.*

## Axiom IV.

The Base, or greatest Side of any Plane Triangle is to the Sum of the Legs, as the Difference of the Legs, is to the Difference of the Segments of the Base made by a Perpendicular let fall from the Angle opposite to the Base.

## Demonstration.

From the  $\angle B$ , on the Base  $AC$ , of the  $\triangle ABC$ , let fall the Perpendicular  $BD$ ; on  $B$  as a Center, with the greater Leg  $BC$ , as a Radius, describe the Circle  $BxCyZ$ ; and produce  $AB$  to  $x$  and  $y$ , and  $CA$  to  $Z$ . Then,

By the 35. 3 *Eucl. Elem.*  $Ay \times Ax$  is  $= AC \times AZ$ ; viz. :  $BC - BA : x : BC + BA : = AC \times : DC - DA :$

Therefore  $AC : BC + BA :: BC - BA : DC - DA$ . *Q. E. D.*

Otherwise,

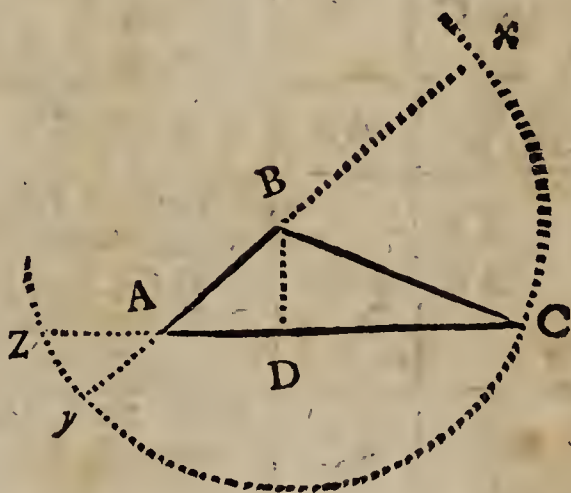
Let the Difference of the Squares of the Sides  $BC$  and  $AB$  be taken and divided by the Base  $AC$ , the Quotient shall be the Difference of the Segments of the Base aforesaid :

Or,

Square all the 3 Sides, and deduct the Square of one of the less Sides out of the Sum of the other two Squares, divide half the Remainder by the longest Side, the Quotient is the Alternate Segment of the Base.

The Proportions for the Solution of the six Cases of Plane oblique Triangles. [See the last Fig.]

*Given*





Given.	Reqd.	Proportions.	Ax.	Case.
AB BC and C	A	$AB : BC :: \text{Si. } C : \text{Si. } A.$	2	I

N. B. 1<sup>st</sup>, If the given Angle be Obtuse, the other 2 Angles then are each Acute.  
 2<sup>dly</sup>, If the Side opposite to the given Angle is longer than the Side opposite to the Angle sought, then is the Angle sought Acute, but if shorter, then is the said Angle doubtful, and may be either Acute or Obtuse, because both the Sine and its Complement to two Right Angles are the same: Wherefore to be certain, of what Quality the Angle opposite to the greatest Side is. Take the Sum and Difference of the greatest Side and Middle (or least) and their Logarithms; if the half of them be equal to the Logarithm of the third Side, the Angle opposite to the greatest Side is a Right Angle, but if the Logarithm of the third Side be greater than the half it is Acute, if less, it is Obtuse: Or, without Logarithms, multiply the said Sum by the Difference abovesaid, and extract the Square-Root,

which if  $\left\{ \begin{array}{l} \text{Equal to} \\ \text{Greater than} \\ \text{Less than} \end{array} \right\}$  the third Side, then is the greatest Angle  $\left\{ \begin{array}{l} \text{Right} \\ \text{Obtuse} \\ \text{Acute} \end{array} \right\}$

AB BC and C	A C	$AB : BC :: \text{Si. } C : \text{Si. } A.$ Hence, by Subtraction, the $\angle B$ will be known. $\text{Si. } A : \text{Si. } B :: BC : AC.$	2	2
-------------------	-----	--	---	---

A, C and BC	AB	$\text{Si. } A : \text{Si. } C :: BC : AB.$	2	3
----------------	----	---	---	---

B AB BC	A and C	$BC + AB : BC - AB :: \text{Tan. } \frac{1}{2} \text{ Sum of the } \angle\text{s opposite} : \text{Tan. } \frac{1}{2} \text{ Difference of the } \angle\text{s opposite}.$ Then $\frac{1}{2} \text{ Sum} + \frac{1}{2} \text{ Difference} = \text{greater } \angle A;$ and $\frac{1}{2} \text{ Sum} - \frac{1}{2} \text{ Difference} = \text{lesser } \angle C.$	3	4
---------------	---------	--	---	---

B AB BC	AC	First, find the Angles by the last; then $\text{Si. } C : \text{Si. } B :: AB : AC.$	3 2	5
---------------	----	--	--------	---

AB BC AC	A B C	$AC : BC + BA :: BC - BA : DC - DA.$ Then $\frac{1}{2} AC + \frac{1}{2} DC - \frac{1}{2} DA = DC.$ And $\frac{1}{2} AC - \frac{1}{2} DC - \frac{1}{2} DA = DA.$ Then $AB : AD :: R : \text{Cos. } A.$ And $CB : DC :: R : \text{Cos. } C.$ And $180^\circ - \angle A - \angle C = \angle B.$	4  I I	6
----------------	-------------	---	-----------------	---

Or more readily at one Operation.

From half the Sum of the Sides subduct each particular Side, and let the Sum of the Logarithm of the half Sum and Difference of the Side subtending the enquired Angle be subducted from the Sum of the Log. of the other Difference and the doubled Radius, half the Remainder shall be the Log. of the Tangent of half the enquired Angle.

Agreeable to this Axiom in Gellibrand's Trig. Britannica. p. 46.

As the Rectangle of half the Sum of the Sides and the Difference between that half Sum and the Side opposite to the Angle required, is to the Rectangle of the other two Remainders; so is the Square of Radius, to the Square of the Tangent of half the Angle sought.

Ex Angulis latera, vel ex lateribus Angulos & mixtim in Triangulis tam planis quam Sphaericis assequi, Summa Gloria Mathematici est: Sic enim Caelum & Terras & Maria felici & admirando calculo Mensurat.

Fran. Vieta

F I N I S.

















